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Evaluation of some integrals involving classical polynomials of Hermite and Legendre using Laplace transform method and hypergeometric approach

Evaluación de algunas integrales que involucran los polinomios clásicos de Hermite y Legendre, usando el método de transformadas de Laplace y el enfoque hipergeométrico

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Abstract

In this paper we have described some novel integrals associated with different higher order polynomials such as classical Hermite's polynomials and classical Legendre's polynomials. The following integrals

$$\int_{-\infty}^{+\infty} x^n \exp(-x^2) H_{n-2k}(x)dx\ ,\int_{-\infty}^{+\infty} x^k \exp(-x^2) H_n(x)dx\ ,$$

$$\int_0^{\infty} t^n \exp(-t^2) H_n(xt)dt \quad \text{and} \quad \int_x^{\infty} t^{n+1} \exp(-t^2) P_n\left(\frac{x}{t}\right)dt$$

are evaluated using hypergeometric approach and Laplace transform technique, which is a different approach from the approaches given by the other authors in the field of special functions.

Key words and phrases: Gauss's summation theorem; classical Legendre's polynomials of first kind; classical Hermite's polynomials; generalized hypergeometric function; Laplace transformation.

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Resumen

En este artículo hemos descrito algunas integrales novedosas asociadas con diferentes polinomios de orden superior, tales como los polinomios clásicos de Hermite y los polinomios clásicos de Legendre. Las siguientes integrales

$$\int_{-\infty}^{+\infty} x^n \exp(-x^2) H_{n-2k}(x)dx \ , \int_{-\infty}^{+\infty} x^k \exp(-x^2) H_n(x)dx \ ,$$

$$\int_0^{\infty} t^n \exp(-t^2) H_n(xt)dt \quad \text{and} \quad \int_x^{\infty} t^{n+1} \exp(-t^2) P_n\left(\frac{x}{t}\right)dt$$

Las siguientes integrales se evalúan utilizando el enfoque hipergeométrico y la técnica de transformada de Laplace, que es un enfoque diferente de los enfoques dados por los otros autores en el campo de funciones especiales.

Palabras y frases clave: Teorema de la sumas de Gauss; polinomios clásicos de Legendre de primera clase; polinomios clásicos de Hermite; función hipergeométrica generalizada; Transformada de Laplace.

1 Introduction, definitions and preliminaries

The theory of integrals involving the classical orthogonal polynomials (Laguerre, Hermite, Legendre, Bessel, Tchebychev and as in Askey-scheme) has been developed by many authors (see, [1, 2, 3, 4, 5, 7, 6, 8, 9, 10]) and other special cases therein.

Motivated essentially, we aim at presenting four integral formulae associated with classical polynomials of Hermite and Legendre using Laplace transform method and hypergeometric approach. On specializing the parameters, the evaluated integrals may be reduced to almost elementary integrals and as special cases appearing in applied mathematics, engineering and physical sciences.

The widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & \left(\nu = n \in \mathbb{N}; \lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \end{cases}, \tag{1.1}$$

it is being understood *conventionally* that $(0)_0 = 1$ and assumed *tacitly* that the Γ quotient exists.

The *generalized hypergeometric function* ${}_pF_q$ with p numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and q denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$, is defined by

$${}_pF_q\left[\begin{array}{l} \alpha_1, \alpha_2, \ldots, \alpha_p; \\ \beta_1, \beta_2, \ldots, \beta_q; \end{array} z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_n}{\prod_{j=1}^{q} (\beta_j)_n} \frac{z^n}{n!}, \tag{1.2}$$

$\big(p, q \in \mathbb{N}_0;\ p \leqq q+1\ ;\ p \leqq q$ and $|z| < \infty;$

$p = q+1$ and $|z| < 1;\ p = q+1, |z| = 1$ and $\mathrm{Re}(\omega) > 0;\ p = q+1, |z| = 1, z \neq 1$ *and* $0 \geq \mathrm{Re}(\omega) > -1\big)$,
where

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

$$\big(\alpha_j \in \mathbb{C}\ (j = 1, 2, \ldots, p); \beta_j \in \mathbb{C} \backslash \mathbb{Z}_0^-\,(j = 1, 2, \ldots, q)\big).$$

Laplace transform of $t^{\alpha-1}$:

$$\int_0^\infty e^{-st} t^{\alpha-1} \mathrm{d}t = \frac{\Gamma(\alpha)}{s^\alpha}, \tag{1.3}$$

where $\mathrm{Re}(s) > 0$, $0 < \mathrm{Re}(\alpha) < \infty$ or $\mathrm{Re}(s) = 0$, $0 < \mathrm{Re}(\alpha) < 1$.

Guass's summation theorem: The Gauss's summation theorem plays a vital role in the proof of many interesting results and some physical problems [11, p.49(Th.18)]

$${}_2F_1\left[\begin{array}{rr} a, b; & \\ c; & 1 \end{array}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{1.4}$$

where $c \neq 0, -1, -2, -3...$ and $\mathrm{Re}(c-a-b) > 0$.

Special cases of Gauss's summation theorem: By using the Gauss's summation theorem, it is easy to prove (see, [11, p.69(Q.N.4)])

$${}_2F_1\left[\begin{array}{rr} \frac{-n}{2}, \frac{-n+1}{2}; & \\ b+\frac{1}{2}; & 1 \end{array}\right] = \frac{2^n (b)_n}{(2b)_n}, \tag{1.5}$$

where $b + \frac{1}{2} \neq 0, -1, -2, -3...$ and $n = 0, 1, 2, 3, ...$

$${}_2F_1\left[\begin{array}{rr} -n, b; & \\ c; & 1 \end{array}\right] = \frac{(c-b)_n}{(c)_n}, \tag{1.6}$$

where $c \neq 0, -1, -2, -3...$ and $n = 0, 1, 2, 3, ...$

Classical Hermite's polynomials: Here, we are interested to use hypergeometric form of classical Hermite's polynomials (see, [11, p.191])

$$H_n(x) = (2x)^n {}_2F_0\left[\begin{array}{rr} \frac{-n}{2}, \frac{-n+1}{2}; & \\ \overline{}; & \frac{-1}{x^2} \end{array}\right], \tag{1.7}$$

where $n = 0, 1, 2, 3, ...$

Legendre's polynomials of first kind: For the sake of convenient hypergeometric approach, we shall use two hypergeometric forms of Legendre's polynomials of first kind (see, [11, p.166 (Eq.4) and p.167 (Eq.7)])

$$P_n(x) = (x)^n {}_2F_1\left[\begin{array}{rr} \frac{-n}{2}, \frac{-n+1}{2}; & \\ 1; & \frac{x^2-1}{x^2} \end{array}\right], \tag{1.8}$$

$$P_n(x) = \frac{\left(\frac{1}{2}\right)_n (2x)^n}{n!} {}_2F_1\left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2}; \\ \frac{1}{2} - n; \end{array} \frac{1}{x^2}\right], \tag{1.9}$$

where $n = 0, 1, 2, 3, ...$

$$\{Gamma\ function\ of\ non-positive\ integers\}^{-1} = 0, \tag{1.10}$$

$$\{Factorial\ of\ negative\ integers\}^{-1} = 0. \tag{1.11}$$

2 First integral

Consider the following integral:

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} x^n \exp(-x^2) H_{n-2k}(x) dx \\ &= \int_{-\infty}^{+\infty} x^{2k+m} \exp(-x^2) H_m(x) dx, \end{aligned} \tag{2.1}$$

where $n = m + 2k$.
Applying the definition of classical Hermite's polynomials (1.7) in the equation (2.1), we get

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} x^{2k+m} \exp(-x^2)(2x)^m {}_2F_0\left[\begin{array}{c} \frac{-m}{2}, \frac{-m+1}{2}; \\ \overline{\hspace{3em}}; \end{array} \frac{-1}{x^2}\right] dx \\ &= 2^{m+1} \sum_{p=0}^{[\frac{m}{2}]} \frac{(\frac{-m}{2})_p (\frac{-m+1}{2})_p (-1)^p}{p!} \int_0^{\infty} \exp(-x^2) x^{2(k+m-p)} dx. \end{aligned} \tag{2.2}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the equation (2.2), we get

$$\begin{aligned} I_1 &= 2^m \sqrt{\pi} \left(\frac{1}{2}\right)_{k+m} \sum_{p=0}^{[\frac{m}{2}]} \frac{(\frac{-m}{2})_p (\frac{-m+1}{2})_p}{(\frac{1}{2} - k - m)_p p!} \\ &= 2^m \sqrt{\pi} \left(\frac{1}{2}\right)_{k+m} {}_2F_1\left[\begin{array}{c} \frac{-m}{2}, \frac{-m+1}{2}; \\ \frac{1}{2} - k - m; \end{array} 1\right]. \end{aligned} \tag{2.3}$$

Now applying the Gauss's summation theorem (1.5) in the equation (2.3), we get

$$I_1 = \frac{2^{-2k} \sqrt{\pi} (m + 2k)!}{k!}. \tag{2.4}$$

Now replacing m by $n - 2k$, we get

$$\int_{-\infty}^{+\infty} x^n \exp(-x^2) H_{n-2k}(x) dx = \frac{2^{-2k} \sqrt{\pi} n!}{k!}. \tag{2.5}$$

Particular case: When $k = 0$ in the equation (2.5), we get

$$\int_{-\infty}^{+\infty} x^n \exp(-x^2) H_n(x) dx = n!\sqrt{\pi}. \tag{2.6}$$

Using the expansion of x^n in a series of Hermite's polynomials, above integral of equation (2.6) was evaluated by Rainville [11].

3 Second integral

Consider the second integral:

$$I_2 = \int_{-\infty}^{+\infty} x^k \exp(-x^2) H_n(x) dx,$$

where $k = 0, 1, 2, ..., (n-1)$.

Case I: (i) When $n = 2m$ and $k = 2m-1$ in the integral I_2, we get

$$I_3 = \int_{-\infty}^{+\infty} x^{2m-1} \exp(-x^2) H_{2m}(x) dx \tag{3.1}$$

$$= 2^{2m} \int_{-\infty}^{+\infty} x^{4m-1} \exp(-x^2) {}_2F_0 \left[\begin{array}{cc} -m, -m+\frac{1}{2}; & \frac{-1}{x^2} \\ \overline{\qquad\qquad}; & \end{array} \right] dx$$

$$= 2^{2m} \sum_{p=0}^{m} \frac{(-m)_p(-m+\frac{1}{2})_p(-1)^p}{p!} \int_{-\infty}^{+\infty} x^{4m-2p-1} \exp(-x^2) dx.$$

Since the integrand is an odd function in x, so applying the property of definite integral, we get

$$\int_{-\infty}^{+\infty} x^{2m-1} \exp(-x^2) H_{2m}(x) dx = 0. \tag{3.2}$$

Case I: (ii) When $n = 2m$ and $k = 2m-2$ in the integral I_2, we get

$$I_4 = \int_{-\infty}^{+\infty} x^{2m-2} \exp(-x^2) H_{2m}(x) dx \tag{3.3}$$

$$= 2^{2m} \int_{-\infty}^{+\infty} x^{4m-2} \exp(-x^2) {}_2F_0 \left[\begin{array}{cc} -m, -m+\frac{1}{2}; & \frac{-1}{x^2} \\ \overline{\qquad\qquad}; & \end{array} \right] dx$$

$$= 2^{2m+1} \sum_{p=0}^{m} \frac{(-m)_p(-m+\frac{1}{2})_p(-1)^p}{p!} \int_{0}^{\infty} x^{4m-2p-2} \exp(-x^2) dx. \tag{3.4}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the equation (3.4), we get

$$I_4 = 2^{2m} \Gamma\left(2m - \frac{1}{2}\right) \sum_{p=0}^{m} \frac{(-m)_p(-m+\frac{1}{2})_p}{(\frac{3}{2}-2m)_p p!}$$

$$= 2^{2m} \Gamma\left(2m - \frac{1}{2}\right) {}_2F_1 \left[\begin{array}{cc} -m, -m+\frac{1}{2}; & 1 \\ \frac{3}{2}-2m; & \end{array} \right]. \tag{3.5}$$

Applying the Gauss's summation theorem (1.6) in the equation (3.5), we get

$$\begin{aligned} I_4 &= 2^{2m}\Gamma\left(2m-\frac{1}{2}\right)\frac{(1-m)_m}{\left(\frac{3}{2}-2m\right)_m} \\ &= 2^{2m}\Gamma\left(2m-\frac{1}{2}\right)\frac{\Gamma(\frac{3}{2}-2m)}{\Gamma(1-m)\Gamma(\frac{3}{2}-m)} \\ &= \frac{2^{2m}\pi}{\Gamma(1-m)\Gamma(\frac{3}{2}-m)}. \end{aligned} \tag{3.6}$$

When m=1,2,3,... and due to presence of $\Gamma(1-m)$ in denominator of the equation (3.6), we get

$$\int_{-\infty}^{+\infty} x^{2m-2}\exp(-x^2)H_{2m}(x)dx = 0. \tag{3.7}$$

Similarly, when $n = 2m$ and $k = 2m-3, 2m-4, ...2, 1, 0$, corresponding integrals obtained from the integral I_2, will be zero.

Case II: (i) When $n = 2m+1$ and $k = 2m$ in the integral I_2, we get

$$I_5 = \int_{-\infty}^{+\infty} x^{2m}\exp(-x^2)H_{2m+1}(x)dx \tag{3.8}$$

$$\begin{aligned} &= 2^{2m+1}\int_{-\infty}^{+\infty} x^{4m+1}\exp(-x^2)\,{}_2F_0\left[\begin{array}{c} -m, -m-\frac{1}{2}; \\ \overline{};\end{array} \frac{-1}{x^2}\right]dx \\ &= 2^{2m+1}\sum_{p=0}^{m}\frac{(-m)_p(-m-\frac{1}{2})_p(-1)^p}{p!}\int_{-\infty}^{+\infty} x^{4m-2p+1}\exp(-x^2)dx. \end{aligned}$$

Since the integrand is an odd function in x, so applying the property of definite integral, we get

$$\int_{-\infty}^{+\infty} x^{2m}\exp(-x^2)H_{2m+1}(x)dx = 0. \tag{3.9}$$

Case II: (ii) When $n = 2m+1$ and $k = 2m-1$ in the integral I_2, we get

$$I_6 = \int_{-\infty}^{+\infty} x^{2m-1}\exp(-x^2)H_{2m+1}(x)dx \tag{3.10}$$

$$\begin{aligned} &= 2^{2m+1}\int_{-\infty}^{+\infty} x^{4m}\exp(-x^2)\,{}_2F_0\left[\begin{array}{c} -m, -m-\frac{1}{2}; \\ \overline{};\end{array} \frac{-1}{x^2}\right]dx \\ &= 2^{2m+2}\sum_{p=0}^{m}\frac{(-m)_p(-m-\frac{1}{2})_p(-1)^p}{p!}\int_{0}^{\infty} x^{4m-2p}\exp(-x^2)dx. \end{aligned} \tag{3.11}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the equation (3.11), we get

$$\begin{aligned} I_6 &= 2^{2m+1}\sqrt{\pi}\left(\frac{1}{2}\right)_{2m}\sum_{p=0}^{m}\frac{(-m)_p(-m-\frac{1}{2})_p}{(\frac{1}{2}-2m)_p p!} \\ &= 2^{2m+1}\sqrt{\pi}\left(\frac{1}{2}\right)_{2m}\ {}_2F_1\left[\begin{matrix} -m, -m-\frac{1}{2}; \\ \frac{1}{2}-2m; \end{matrix}\ 1\right]. \end{aligned} \tag{3.12}$$

Applying the Gauss's summation theorem (1.6) in the equation (3.12), we get

$$\begin{aligned} I_6 &= 2^{2m+1}\sqrt{\pi}\left(\frac{1}{2}\right)_{2m}\frac{(1-m)_m}{(\frac{1}{2}-2m)_m} \\ &= 2^{2m+1}\sqrt{\pi}\left(\frac{1}{2}\right)_{2m}\frac{\Gamma(\frac{1}{2}-2m)}{\Gamma(\frac{1}{2}-m)\Gamma(1-m)} \\ &= \frac{2^{2m+1}\pi}{\Gamma(\frac{1}{2}-m)\Gamma(1-m)}. \end{aligned} \tag{3.13}$$

When $m = 1, 2, 3, ...$ and due to presence of $\Gamma(1-m)$ in denominator of (3.13), we get

$$\int_{-\infty}^{+\infty} x^{2m-1}\exp(-x^2)H_{2m+1}(x)dx = 0. \tag{3.14}$$

Similarly, when $n = 2m+1$ and $k = 2m-2, 2m-3, ..., 2, 1, 0$, corresponding integrals obtained from I_2, will be zero.
The unification of above integrals can be written as:

$$\int_{-\infty}^{+\infty} x^k\exp(-x^2)H_n(x)dx = 0, \tag{3.15}$$

where $k = 0, 1, 2, 3, ..., (n-1)$.
Above integral (3.15) was derived with the help of Hermite's differential equation and orthogonal property of Hermite's polynomials.
From the equation (2.6) and the equation (3.15), we get

$$\int_{-\infty}^{+\infty} x^k\exp(-x^2)H_n(x)dx = \begin{cases} 0, & k = 0, 1, 2, 3, \ldots, n-1 \\ n!\sqrt{\pi}, & k = n. \end{cases} \tag{3.16}$$

4 Curzon's integral

Consider the third integral:

$$I_7 = \int_0^{\infty} t^n\exp(-t^2)H_n(xt)dt \tag{4.1}$$

$$\begin{aligned} &= (2x)^n \int_0^\infty t^{2n} \exp(-t^2)\ {}_2F_0 \left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2}; \\ \overline{}; \end{array} \frac{-1}{(tx)^2} \right] dt \\ &= (2x)^n \sum_{p=0}^{[\frac{n}{2}]} \frac{(\frac{-n}{2})_p (\frac{-n+1}{2})_p (-1)^p (x)^{-2p}}{p!} \int_0^\infty t^{2n-2p} \exp(-t^2) dt. \end{aligned} \tag{4.2}$$

Using suitable substitution and applying the definition of Laplace transform (1.3) in the equation (4.2), we get

$$\begin{aligned} I_7 &= (2)^{n-1} x^n \sum_{p=0}^{[\frac{n}{2}]} \frac{(\frac{-n}{2})_p (\frac{-n+1}{2})_p (-1)^p (x)^{-2p}}{p!} \Gamma\left(n - p + \frac{1}{2}\right) \\ &= (2)^{n-1} x^n \sqrt{\pi} \left(\frac{1}{2}\right)_n \sum_{p=0}^{[\frac{n}{2}]} \frac{(\frac{-n}{2})_p (\frac{-n+1}{2})_p (x)^{-2p}}{(\frac{1}{2} - n)_p p!} \\ &= (2)^{n-1} x^n \sqrt{\pi} \left(\frac{1}{2}\right)_n \ {}_2F_1 \left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2}; \\ \frac{1}{2} - n; \end{array} \frac{1}{x^2} \right]. \end{aligned} \tag{4.3}$$

Applying the definition of Legendre's polynomials (1.9) in the equation (4.3), we get

$$\int_0^\infty t^n \exp(-t^2) H_n(xt) dt = \frac{\sqrt{\pi}}{2} n! P_n(x). \tag{4.4}$$

Curzon [6] and [11, p.191(eq.4), p.199(Q.N.4)] evaluated the integral I_7, by using a different approach.

5 Rainville's integral

Consider the fourth integral:

$$I_8 = \int_x^\infty t^{n+1} \exp(-t^2) P_n\left(\frac{x}{t}\right) dt. \tag{5.1}$$

Applying the definition of Legendre's polynomials (1.8) in the equation (5.1), we get

$$\begin{aligned} I_8 &= \int_x^\infty t^{n+1} \exp(-t^2) \left(\frac{x}{t}\right)^n \ {}_2F_1 \left[\begin{array}{c} \frac{-n}{2}, \frac{-n+1}{2}; \\ 1; \end{array} \frac{x^2 - t^2}{x^2} \right] dt \\ &= x^n \exp(-x^2) \sum_{p=0}^{[\frac{n}{2}]} \frac{(\frac{-n}{2})_p (\frac{-n+1}{2})_p (x)^{-2p}}{(1)_p p!} \int_x^\infty \exp(-t^2 + x^2) t (x^2 - t^2)^p dt \\ &= \frac{1}{2} x^n \exp(-x^2) \sum_{p=0}^{[\frac{n}{2}]} \frac{(\frac{-n}{2})_p (\frac{-n+1}{2})_p (-1)^p (x)^{-2p}}{p! p!} \int_0^\infty \exp(-T) T^p dT, \end{aligned} \tag{5.2}$$

where $(t^2 - x^2) = T$.
Using suitable substitution and applying the definition of Laplace transform (1.3) in the equation

(5.2), we get

$$\begin{aligned} I_8 &= \frac{1}{2}x^n\exp(-x^2)\sum_{p=0}^{[\frac{n}{2}]}\frac{(\frac{-n}{2})_p(\frac{-n+1}{2})_p(-1)^p(x)^{-2p}}{p!} \\ &= \frac{1}{2}x^n\exp(-x^2)\,{}_2F_0\left[\begin{array}{ll} \frac{-n}{2}, \frac{-n+1}{2}; & \frac{-1}{x^2} \\ \overline{\qquad\quad}; & \end{array}\right]. \end{aligned} \tag{5.3}$$

Applying the definition of classical Hermite's polynomials (1.7) in the equation (5.3), we get

$$\int_x^\infty t^{n+1}\exp(-t^2)P_n\left(\frac{x}{t}\right)dt = 2^{-n-1}\exp(-x^2)H_n(x). \tag{5.4}$$

Rainville [10, p.271] and [11, p.191(eq.5), p.199(Q.N.6)] evaluated the integral I_8, by using a different approach.

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