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Topological Solitons I: Sigma Models

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Abstract

We study classical topological solitons in nonlinear sigma models.

Key words: Soliton; skyrmion; topology.

Solitones Topológicos I: Modelos Sigma

Resumen

Estudiamos solitones topológicos clásicos en modelos sigma no lineales.

Palabras clave: Solitón; skyrmión; topología.

1. Introduction

Nonlinear science has developed strongly over the past 40 years, touching upon every discipline in both the natural and social sciences. Nonlinear systems appear in mathematics, physics; chemistry, biology, astronomy, metereology, engineering, economics and many more (1).

Within the nonlinear phenomena we find the concept of 'soliton'. It has got some working definitions all amounting to the picture of a travelling wave of semi-permanent shape. A soliton is a non-singular solution of a non-linear field equation whose energy density has the form of a lump localised in space. Although solitons arise from nonlinear wave-like equations they have particle-like properties, hence the suffix *on* to covey a corpuscular picture to the *soli*tary wave. Solitons exist as density waves in spiral galaxies, as lumps in the ocean, in plasmas, molecular systems, protein dynamics, laser pulses propagating in solids, liquid crystals, elementary particles and nuclear physics (2, 3).

According to whether the solitonic field equations can be solved or not, solitons are said to be integrable or nonintegrable. The former are generally found only in one dimension; their dynamics is quite restricted with the lumps moving undistorted in shape and, in the event of a collision, scattering off merely undergoing a phase shift. In higher dimensions solitons enjoy a richer dynamics but now we are in the realm of nonintegrable models, where analytical solutions are practically limited to static configurations and Lorentz transformations thereof. The time evolution in these models is studied with the help of numerical simulations and other approximation techniques. A trait of nonintegrable solitons is that they carry a conserved quantity of topological nature, the *topological charge* -hence the designation 'topological solitons'. Entities of this kind exhibit interesting stability and scattering properties, including soliton annihilation which can occur when lumps with opposite topological charges (one positive, one negative) collide. For nuclear/particle physics such dynamics is of great relevance.

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The topological charge Q may be interpreted in a natural way if we imagine the soliton as a subatomic particle carrying Q as a constant of motion. Amongst the most successful models which have made use of this appealing idea is the (3+1)-dimensional (3 space, 1 time) *Skyrme model* of hadron physics (4, 5, 6). It deals with an effective theory of pions and how to derive baryons and their interactions within such theory. Its soliton solutions, the skyrmions, are thought of as classical protons and neutrons with the topological charge being the baryon number. This model leads to results which are in good qualitative agreement with experimental results of nuclear physics (7, 8). It is interesting to highlight that solitons in the Skyrme model appear directly, by construction, whereas solitons within the framework of grand unified theories, for instance, come about as an offshoot: they emerge as domain walls, cosmic strings and monopoles through the Kibble mechanism.

At first the Skyrme model received short shrift, partly because of the advent of quantum chromodynamics (QCD). Nevertheless, the Skyrme model acquired popularity towards the end of the 70s due to speculations that it might serve as a link between QCD and the familiar old theory (valid in the low-energy regime) where the interbaryonic forces occur via the exchange of Yukawa π mesons. In this low-energy regime QCD encounters the difficulty of having no small parameter to describe the dynamics of quarks and gluons. But in the limit as the number of colours N tends to infinity, QCD reduces to a theory of effective mesons with interactions of order 1/N. Amazingly, in such limiting case baryons may be regarded as solitons of an effective meson theory without any further referente to their quark content (9). In 1983 the model was boosted further when it was shown that its lagrangian, supplemented with a Wess-Zumino term, reproduces the quantum numbers of baryons in QCD (10). Remarkably, this latter result comes from simply eliminating a certain discrete symmetry y of the skyrmion lagrangian which is not a symmetry of QCD.

The Skyrme system is just one example of a large family of non-linear models known as *chiral* or *sigma models* (11, 12) introduced in the 1960s to describe β-decay and strong interactions where topology played no role (13).

In two spatial dimensions sigma models bear several properties in common with (3+1)-dimensional Yang-Mills gauge field theories of particle physics, namely, conformal invariance, spontaneous symmetry breaking, asymptotic freedom and the existente of soliton solutions. Obtaining information about the quantum field theory of gauge systems in three-space, starting from classical solutions of the corresponding more tractable equations of low-dimensional analogues, is one of the ideas behind the study of chiral models. These models are interesting by themselves: known as harmonic maps, they represent a rich industry of research in pure mathematics (14).

Our work is split in two papers.In the next section of the present paper we review in some detail the theory of solitons, including Derrick's theorem and topological considerations. In section 3, 4 and 5 we work out typical examples in one, two and three spatial dimensions, respectively. In the companion paper we will focus on two specific versions of the *CP1* sigma model (original and Skyrme) in two spatial dimensions and consider its stability and scattering properties.

2. Soliton theory

Due to the scarcity of analytic soliton solutions of our non-linear systems which, in addition, must be relativistically invariant, in soliton theory we analyse static finite-energy configurations and try to obtain as much information as possible without explicitly solving the field equations. In so doing, topological techniques, a virial-like theorem and an ingenious completing-the-square procedure (subsection 3.1) are of enormous utility.

2.1. Derrick's theorem

Consider the class of Lorentz-invariant non-linear scalar field theories in a Minkowski space in $(D+1)$ dimensions (D space, one time): $x_\mu x^\mu = \left(x^0\right)^2 - \sum_{i=1}^{D}\left(x^i\right)^2$, $\mu = 0,1,2,\ldots,D$. And consider those systems described by a lagrangian density of the standard relativistic form

$$\mathcal{L} = C\sum_{a=1}^{n}\sum_{\mu=1}^{D}\partial_\mu\phi_a\partial^\mu\phi_a - U\left(\vec{\phi}\right)$$
$$= C\left(\partial_\mu\vec{\phi}_a\right)\cdot\left(\partial_\mu\vec{\phi}_a\right) - U\left(\vec{\phi}_a\right), \qquad [1]$$

where $\vec{\phi} \equiv \left\{\phi_a\left(x^\mu\right); a = 1,2,\ldots,n\right\}$ denotes a vector in the internal space of the fields and the number C is an adjustable constant. The function U is non-negative and vanishes only at its absolute minima -set to naught without loss of generality.

We are concerned with the possible existence of non-singular solutions whose energy density at a given time is finite in some finite region of space, and falls to zero at spatial infinity sufficiently fast as to be integrable. Such localised energy density has a distinctive lump-like profile usually able to propagate without much change in shape. The energy lump itself, or its corresponding field solution, is known as a soliton.

The static energy or potential energy is read-off from [1] as

$$V\left(\vec{\phi}\right) = C\int\left(\partial_\kappa\vec{\phi}\right)\cdot\left(\partial_\kappa\vec{\phi}\right)d^Dx + \int U\left(\vec{\phi}\right)d^Dx$$
$$= V_1\left(\vec{\phi}\right) + V_2\left(\vec{\phi}\right),\ \ k = 1,\ldots,D, \qquad [2]$$

in obvious notation. A static solution of the model [1] is an extremum condition $\delta V = 0$ for [2]. Taking $C = 1/2$ we get

$$\nabla^2\vec{\phi} - \frac{d}{d\vec{\phi}}U = \vec{0}. \qquad [3]$$

Now let $\vec{\phi}(x)$ be a solution to [3] and consider the one-parameter family of configurations obtained by re-scaling $\vec{x} \mapsto \gamma\vec{x}$

$$\vec{\phi}_\gamma(\vec{x}) = \vec{\phi}_1(\gamma\vec{x}) \qquad [4]$$

With the help of equation [2] we obtain

$$V\left[\vec{\phi}_\gamma(\vec{x})\right] = \gamma^{2-D}V_1\left[\vec{\phi}_1(\vec{x})\right] + \gamma^{-D}V_2\left[\vec{\phi}_1(\vec{x})\right], \qquad [5]$$

wherefrom

$$\frac{d}{d\gamma}V\left[\vec{\phi}_\gamma(\vec{x})\right] = (2-D)\gamma^{1-D}V_1\left[\vec{\phi}_1(\vec{x})\right] - D\gamma^{-1-D}V_2\left[\vec{\phi}_1(\vec{x})\right] \qquad [6]$$

Since $\vec{\phi}_1(\vec{x})$ is a local extremal of V, it must in particular produce $\frac{d}{d\gamma}V\left[\vec{\phi}_\gamma(\vec{x})\right]_{\gamma=1} = 0$, i.e.,

$$(2-D)V_1\left[\vec{\phi}_1(\vec{x})\right] = DV_2\left[\vec{\phi}_1(\vec{x})\right]. \qquad [7]$$

Inasmuch as both V_1 and V_2 are non-negative, equation [7] precludes the existente of non-trivial static solutions for the class of models (1) when $D \geq 3$ (time-dependent solutions are not precluded). This is the content of the so-called Derrick's theorem (17, 18). It allows one to tell solely from the form of the lagrangian and the dimensionality of space whether a given theory may possess solitons. If we are seeking solitons in $D>2$ it is necessary to somehow modify the lagrangian [1].

Research has therefore been carried out for different types of non-linear equations with various possible values of D. We are going to examine some of there models below, but first let us acquaint ourselves with how topology steps into the soliton scene.

2.2. Topological considerations

One of the basic tasks of topology is to learn how to discern non homeomorphic figures. With this aim one introduces a class of invariant quantities which do not change with homeomorphic transformations of a given figure. The study of topological spaces is connected with the resolution of questions like: Can one describe a class of invariants of a given manifold? Does there exist a set of integral invariants, fully characterising a given manifold? Integral invariants are in their own way 'quantum numbers' of a manifold (a similar problem is envisaged in physics, namely, to characterise a particle having given its special parameters, *v.gr.*, spin, charge, mass). Among such tasks is the classification of n-dimensional surfaces, compact, connected, orientable and 2-dimensional for example, as those we shall encounter later.

The integral degrees of freedom of the soliton field give rise to an integral space whose manifold (the field solutions) can define a non-trivial mapping onto the manifold of the 'physical' *D*-dimensional space. Each mapping can be characterised by an integral number which is a conserved quantity -associated with the topology of the solutions as outlined above and with nothing to do with Noether's theorem.

This type of maps is the subject of homotopy theory. Consider two maps f and g from a manifold $\mathcal{N}$ to a manifold $\mathcal{M}$: f$; \mathcal{N} \mapsto \mathcal{M}$, g$; \mathcal{N} \mapsto \mathcal{M}$. These mappings are homotopic if they can be continuously deformed one into the other: $\mathcal{F}: \mathcal{N} \times [1,0] \mapsto \mathcal{M}$, with the continuous connecting map $\mathcal{F}$ satisfying $\mathcal{F}(x,0) = f(x)$, $\mathcal{F}(x,1) = g(x)$. That is, as the continuous variable t in $\mathcal{F}(x,t)$ varies continuously from 0 to 1 in the integral $[0,1]$, the function $f(x)$ is deformed continuously into $g(x)$.

Homotopy is an equivalence relation that partitions the manifold of continuous maps from $\mathcal{N}$ to $\mathcal{M}$ into equivalent classes $[f]$. A map from one homotopy sector cannot be continuously deformed into another sector. Homotopy classes are topological invariants of the pair of spaces above, since they are unchanged under homeomorphism of $\mathcal{N}$ or $\mathcal{M}$. This must be so, for homeomorphism is a continuous map itself. We can think of classical time evolution as a homotopy between initial and final state field configurations, and visualise $[f]$ as the class of fields conserved as time elapses.

A classification of topological spaces may be achieved by selecting a standard 'test body' $\mathcal{N}$ and permitting $\mathcal{M}$ to vary through the family of target spaces under study. The sphere S_2 defined by

$$\sum_{k=1}^{n+1} (x_k)^2 = constant,$$

is a usual choice for $\mathcal{N}$. Here S_0 corresponds to just two points ($x_1 = \pm$ *constant*), S_1 is a circle or a ring, S_2 is a sphere and so on. Another interesting choice for $\mathcal{N}$ is the two-torus T_2.

Homotopy classes can be endowed with a group structure via the operation $[f+g] = [f] + [g]$. By $\pi_n(\mathcal{M})$ we denote the homotopy group associated with the maps $S_n \mapsto \mathcal{M}$. These groups are generalisations of the first homotopy group or fundamental group $\pi_1(\mathcal{M})$: it consists of the set of classes of closed paths on $\mathcal{M}$ which are not homotopic to one another. Now, a closed path on $\mathcal{M}$ can be represented as the image of a fixed circle $\mathcal{N}$= S_1. The associated fundamental group $\pi_1(\mathcal{M})$ is then the set of nonhomotopic maps $S_1 \mapsto \mathcal{M}$ By replacing the circle by the n-sphere we obtain the higher groups $\pi_1(\mathcal{M})$. As an illustration may serve the fundamental group $\pi_1(S_2) = 0$, which says that on a spherical surface all closed paths are homotopic and can be shrunk to a point (simple connectedness). For the two-torus we have $\pi_1(T_2) = Z \otimes Z$, signifying that there exists an infinite number of closed paths which are not homotopic to one another. An arbitrary closed path on T_2 is homotopic to a path passing r times along the parallel of the torus and s times along its meridian, and it is labelled by the pair of integers (r, s). Note that a path with $r = s = 0$ is contractable to a point. The classes $\pi_1(T_2)$ are relevant, for instance, in characterising general ring-vortex-coipfigurations in both Higgs and sigma models (19) and in the periodic CP^1 model (20).

In the standard case when the target manifold is also a sphere, it can be proven that (21)

$$\pi_n(S_n) = Z,$$
$$\pi_n(S_m) = 0,\ \ n < m$$
$$\pi_n(S_1) = 0,\ n > 1. \qquad [8]$$

The last two expressions indicate that the homotopy groups involved are trivial: all maps can be deformed one into the other. The interesting case when the group of homotopy classes is isomorphic to the group of integers Z means that each homotopy sector can be labelled by an integer, the topological charge or Brouwer degree of the map. A theory with non-trivial topology is said to be stable in the sense that no configuration can evolve out of its original topological class.

The scenario for the expressions [8] often comes about in the sigma models from demanding that the energy of the fields involved be finite at spatial infinity, the localised fields playing the role of the homotopy maps. When D>1 the fields must tend to the same value at spatial infinity, regardless of direction. Whence, the spatial degrees of freedom of the fields may be regarded as a one-point compactification $\Re_D \cup \{\infty\} \cong S_D$, leading to the maps

$$S_D \mapsto S_m. \qquad [9]$$

The homotopy classification is valid for any localised static field configuration (the set of which spans the so-called configuration space). The same classification holds for localised solutions all right (moduli space) as they are subsets of finite-energy configurations.

In connection with the O(3) model in (2+1) dimensions with standard and periodic boundary conditions we shall study in the companion paper the cases $S_2 \mapsto S_2$ and $T_2 \to S_2$, respectively.

In any case, the topological index Q can be computed through

$$Q = (constant)\int_{\mathcal{N}}(\phi^* w), \qquad [10]$$

where $\phi^* w$ is a suitable volume-form on $\mathcal{N}$. The mapping $\phi^*: \mathcal{M} \mapsto \mathcal{N}$ is the pull-back map induced by $\phi: \mathcal{N} \mapsto \mathcal{M}$. The constant in [10] normalises Q to an integer.

A large number of soliton-bearing models can be conveniently considered in the context where the target manifold has the structure of a coset space (22). The idea is to find a continuous group G of symmetries acting on the manifold $\mathcal{M}$ in such a way that, given a point $p \in \mathcal{M}$ the action of G over p produces the whole of $\mathcal{M}$. This transitivity property is technically stated as $\forall p_1, p_2 \in \mathcal{M}, \exists\, g \in G |\, gp_1 = p_2$. Given this, a homomorphism between $\mathcal{M}$ and G (or some related group) could probably be established. However, note that the said procedure will yield $\mathcal{M}$ more than once in general, the aim being to obtain it only once. The gist of the matter then lies on the question: When do two elements $g_1, g_2 \in G$ yield the same point p of $\mathcal{M}$? Observing that $g_1 p = g_2 p \to g_2^{-1} g_1 p = p$, we realise that the answer is: When $g_2^{-1} g_1$ leaves p unaltered, *i.e.*, when $h = g_2^{-1} g_1 \in H(p)$, the isotropy group of p: $H(p) = \{h \in G | hp = p\}$. But $h = g_2^{-1} g_1 \to g_2 h = g_1$, meaning that two elements of G operate on p to produce the same point of $\mathcal{M}$ iff they belong to the same left coset of G with respect to H(p). Now we recall from group theory that G may be partitioned into disjoint cosets with the characteristic -suitable for our objective- that every element of G belongs to one *and only one* left coset of G with respect to H(p). This guarantees that $\mathcal{M}$ will be obtained only once when acted upon by the coset space $G/H(p)$. The identification we desire is then

$$\mathcal{M} = G / H(p) = \{ gH(p) | g \in G \} \quad [11]$$

description independent of the choice of p if, as usual in physics, $\mathcal{M}$ is homogeneous. The manifold $\mathcal{M}$ can now be seen to adopt a variety of forms. Notably (see (11) and references therein):

- Grassmannian sigma-models in $2mn$ dimensions:

$$\mathcal{M} = \frac{SU(m+n)}{SU(m) \times SU(n) \times U(1)} = \mathcal{G}_{m.n} \quad [12]$$

They require mn ($2mn$) complex (real) fields. -The case $\mathcal{G}_{1.n}$ is known as the complex projective space CP^n:

$$\mathcal{M} = \frac{SU(n+1)}{SU(n) \times U(1)} = CP^n \quad [13]$$

- Sn or O(n) sigma-models: The fields take values on the sphere S_{n-1} the acting symmetry group being $SO(n - 1)$. Given a point p of the target manifold, the rotations that leave it invariant are those about the direction of p itself; so its isotropy group is $SO(n - 1)$. We then have

$$\mathcal{M} = \frac{SO(n)}{SO(n-1)} = S_{n-1} \quad [14]$$

Other than giving a systematic classification of important solitonic models, the coset description [11] permits the calculation of the associated homotopy groups in a relatively easy fashion. For example, using the result $\pi_2(G/H) = \pi_1(H)$, valid when G is both connected $[\pi_0(G) = 0]$ and simply connected $[\pi_1(G) = 0]$, we obtain from [13]

$$\pi_2(CP^n) = \pi_1(SU(n) \times U(1)) = \pi_1(SU(n)) \otimes \pi_1(U(1)) = \pi_1(U(1)) = Z[U(1) = S_1] \quad [15]$$

a special case of [8]. In particular, since CP^1 is isometric to S_2, the above result for n = 1 applies to O(3) as well. These two specific models are essentially the same. As a generalisation for arbitrary n, however, CP^n is more appropriate than is $O(n)$ by virtue of continuing to give topological soliton solutions for arbitrary n in the plane. This is not difficult to infer: in two spatial dimensions the case $O(n > 3)$ produces, from expression [9] and [14], $S_2 \mapsto S_{n>2}$. Whereupon [8] tells us that the associated homotopy group is the trivial $\pi_2(S_{n>2}) = 0$, which cannot accommodate topological objects. On the other hand, the non-trivial CP^n result [15] holds for all n.

3. Solitons in one dimension

The simplest models governed by [1] involve one single real scalar field dwelling in a line. An interesting example is the so-called ϕ^4 theory (23-25), which plays an important role in gauge theories. It corresponds to a Higgs-like function of the form

$$U(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 \quad [16]$$

where λ, m are positive constants.

The static equation of motion for this system readily follows from inserting [16] into [3]. The resulting equation is solved by the 'kink'

$$\phi(x) = \pm \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{mx}{\sqrt{2}}\right) \quad [17]$$

Finite-energy solutions must obey the boundary conditions: $\lim_{x \to \pm\infty} \phi(x) \to \pm \frac{m}{\sqrt{\lambda}}$ the minima of the potential energy.

The kink provides an example of spontaneous symmetry breaking: its lagrangian is invariant under reflections $\phi \to -\phi$ (the internal degree of freedom of the system) whereas the two fundamental states $\pm m / \sqrt{\lambda}$ are not; rather, they are transformed finto one another under reflections. Other symmetries of [17] are parity $x \to -x$ and space translations $x \to x + x_0$.

The homotopic maps for this model are the correspondence between the two vacuum states (S_0) and the points at infinity (a 0-sphere as well). We have four topological classes, *v.gr.*, the kink sector, the anti-kink sector, and the two vacua. These sectors are characterised by the pair of indices

$$[\phi(-\infty), \phi(\infty)]: \left[-\frac{m}{\sqrt{\lambda}}, \frac{m}{\sqrt{\lambda}}\right], \left[\frac{m}{\sqrt{\lambda}}, -\frac{m}{\sqrt{\lambda}}\right], \left[\frac{m}{\sqrt{\lambda}}, \frac{m}{\sqrt{\lambda}}\right], \left[-\frac{m}{\sqrt{\lambda}}, -\frac{m}{\sqrt{\lambda}}\right] \quad [18]$$

The topological index can be defined as the 'charge'

$$Q = \int_{-\infty}^{\infty} k_0(x) dx, \quad k_0(x) = \frac{\sqrt{\lambda}}{2m} \frac{d\phi}{dx} \quad [19]$$

of the conserved 'current'

$$k^{\mu} = \frac{\sqrt{\lambda}}{2m} \in_{\mu\nu} \partial^{\nu} \phi(x), \quad \mu, \nu = 0, 1, \quad [20]$$

where $\in_{\mu\nu}$ is the Levi-Civita pseudotensor. We see that the topological charge is ± 1 for the kink and zero for the minima $\pm m / \sqrt{\lambda}$. The system possesses topological stability, in the sense that a kink will not decay into either of the minima because it is not homotopic to any of them. Also note that [19] is (*constant*) $\int d\phi$, equation [10] with $\phi^* w = d\phi$ a one-form.

Even though we might not be able to explicitly calculate the evolution of the system, of what happens after, say, a kink and an anti-kink (the solution with the minus sign in (17) collide, we know that the resulting field configuration will always be within one the four homotopy sectors [18]. For instance, an anti-kink coming from the far left and an kink approaching from the far right belong to the Q= -1+1=0 class, and there will they remain after the impact.

As it actually happens, explicit solutions of the time-dependent ϕ^4 model are not available. Its dynamics, studied through numerical simulations, indicate that the kinks do not retain their shapes under collisions. Also, they seem to repel each other when started off at rest, a characteristic present as well in (2+1) dimensional skyrmions.

The particle-like nature of [17] can be further substantiated by deriving an Einsteinian mass-energy formula between static and moving kinks. Since the model is Lorentz-invariant, travelling solutions can be obtained by Lorentz-boosting [17]

$$\phi_{\upsilon} = \pm \frac{m}{\sqrt{\lambda}} \tanh\left(\frac{m}{\sqrt{2}} \frac{x - \upsilon t}{\sqrt{1 - \upsilon^2}}\right), \; -1 < \upsilon < 1 \quad [21]$$

(We emphasise that this solution is not what we mean by an explicit time-dependent object derived from the full equation of motion, moving independently from other solutions). Now, from equations [2] and [17] we get

$$V(\phi) = \frac{m^4}{\sqrt{\lambda}} \int_{-\infty}^{\infty} \frac{1}{\cosh^4\left(\frac{m}{\sqrt{2}} x\right)} dx = \frac{2\sqrt{2} m^3}{3\lambda} \quad [22]$$

The energy V_{υ} for [21] is related to $V(\phi)$ by the mass-energy formula $V_{\upsilon}(\phi) = V(\phi) / \sqrt{1 - \upsilon^2}$. A schematic plot of te integrand in [22] gives a lump of matter positioned around *x= 0*, able to cruise along unscathed upon boosting.

The ϕ^4 model also illustrates what we mentioned in the previous chapter about solitons only steming from equations that possess a special, fine balance among their terms. If, instead of [16], we take the look-alike $\left(\phi^2 + \alpha\phi^4\right)^2$ say, then no soliton solutions are produced.

Also worthy of remark is the non-perturbative character of the kink: since it is singular when $\lambda \to 0$, a perturbation expansion in λ is no longer feasible; the quantum theory of solitons resorts to a semi-classical expansion that quantises around the classical solutions.

Amongst other important models in $D=1$ appear the KdV (28), the $O(3)$ [14] (26, 27), and sine-Gordon (29) systems. They are fully-integrable and have several interesting properties, *v.gr.*, possession of an infinite number of conserved quantities, presence of inverse scattering transform and Backlund tranformations.

3.1. Bogomolny technique

Let us now illustrate a useful procedure (30) for constructing static solutions. By completing squares, the static energy for unidimensional systems can be cast into

$$V(\phi) = \frac{1}{2}\int_{-\infty}^{\infty}\left[\partial_x\phi \pm \sqrt{2U(\phi)}\right]^2 dx \mp \int_{-\infty}^{\infty}\partial_x\phi\sqrt{2U(\phi)}\,dx$$
$$= \frac{1}{2}\int_{-\infty}^{\infty}\left[\partial_x\phi \pm \sqrt{2U(\phi)}\right]^2 dx \mp \int_{\phi(-\infty)}^{\phi\infty}\sqrt{2U(\phi)}\,d\phi. \quad [23]$$

Wherefore the inequality (referred to as the Bogomolny bound)

$$V(\phi) \geq \left|\int_{\phi(-\infty)}^{\phi(\infty)}\sqrt{2U(\phi)}\,d\phi\right|, \quad [24]$$

which imposes a lower limit to the energy of any static configuration in a given homo topy sector Q. The condition for equality minimises V and occurs iff

$$\partial_x\phi \pm \sqrt{2U(\phi)} = 0 \quad [25]$$

expression that is often called the Bogomolny equation. It is of first order, easier to solve than its parent second order equation. Upon inserting the quartic function [16] into [25] the field [17] readily follows.

Solutions of the Bogomolny equation automatically satisfy the original second order equation, but the reverse is not generally true. But for the kink model the double implication does hold. The kink, the anti-kink and the fundamental states 'saturate' the bound [24] and all other Q-sectors are empty. This feature occurs in all Poincare-invariant soliton systems in one dimension (31).

Finally, note that from [6] one derives:

$$\frac{d^2}{d\gamma^2}V\left[\vec{\phi}_\gamma(\vec{x})\right] = (2-D)(1-D)\gamma^{-D}V_1\left[\vec{\phi}_1(\vec{x})\right]$$
$$+D(D+1)\gamma^{-2-D}V_2\left[\vec{\phi}_1(\vec{x})\right]. \quad [26]$$

Taking into account that for $D = 1$ equation [7] gives $V_1\left[\vec{\phi}_1(\vec{x})\right] = V_2\left[\vec{\phi}_1(\vec{x})\right] > 0$, equation [26] for $D = 1$ gives

$$\frac{d^2}{d\gamma^2}V\left[\vec{\phi}_\gamma(x)\right]\Big|_{\gamma=1} = 2V_2\left[\vec{\phi}_1(x)\right] > 0 \quad [27]$$

Therefore, $\gamma = 1$ corresponds to a minimum of the potential energy and hence a soliton in $D = 1$ is stable. Its finely-balanced scaling behaviour is brought forth by equation [5]:

$$V\left[\phi_\gamma(x)\right] = \gamma V_1\left[\phi_1(x)\right] + \frac{1}{\gamma}V_2\left[\phi_1(x)\right]. \quad [28]$$

As we shall see in the next section, the situation is entirely different in two spatial dimensions.

4. Solitons in two dimension

Derrick's theorem for planar systems entails $V_2\left(\vec{\phi}_1\right) = 0$, in which case the lagrangian [1] reduces to

$$\mathcal{L} = C\left(\partial_\mu\vec{\phi}\right)\cdot\left(\partial_\mu\vec{\phi}\right), \quad \mu = 0,1,2. \quad [29]$$

An illustration is provided by the O(4) chiral model. It consists of a real vector

$$\vec{\phi} = \left(\phi_0,\phi_1,\phi_2,\phi_3\right) \quad [30]$$

restricted to take values on the 3-sphere S_3:

$$\vec{\phi}\cdot\vec{\phi} = \phi_0^2 + \phi_\kappa\phi_\kappa = 1 \quad [31]$$

summation over k=1,2,3 understood. The model is clearly invariant under the O(4) rotation group in internal space. The equation of motion that stems from [29]-[31] is

$$\partial^{\mu}\partial_{\mu}\vec{\phi} + \left(\partial_{\mu}\vec{\phi}\partial^{\mu}\vec{\phi}\right)\vec{\phi} = \vec{0} \qquad [32]$$

It is customary to take as the basic field the SU(2) quaternion

$$J = \begin{bmatrix} \phi_0 + i\phi_3 & \phi_2 + i\phi_1 \\ -\phi_2 + i\phi_1 & -\phi_0 + i\phi_3 \end{bmatrix} = \phi_0\tau_0 + i\,\tau_{\kappa}\phi_k \qquad [33]$$

where τ_0 is the 2x2 identity matrix and τ_k are the Pauli matrices. Laborious but straighforward manipulation yields

$$\partial_{\mu}J\,\partial^{\mu}J^{-1} = \begin{bmatrix} \left(\partial_{\mu}\vec{\phi}\right)\cdot\left(\partial^{\mu}\vec{\phi}\right) & 0 \\ 0 & \left(\partial_{\mu}\vec{\phi}\right)\cdot\left(\partial^{\mu}\vec{\phi}\right) \end{bmatrix},$$

in terms of which the lagrangian density [29] becomes

$$\mathcal{L} = \frac{C}{2}Tr\left(\partial_{\mu}J\,\partial^{\mu}J^{-1}\right), \qquad [34]$$

with Tr denoting the trace of the matrix.

Written in this form the invariance of the model under the so-called $SU(2)\ x\ SU(2)$ chiral transformations is manifest. Since the chiral group and the four-dimensional rotations have the same Lie algebra, the O(4) model is equivalently referred to as SU(2) chiral. Worthy of note is that upon-expanding [34] around the vacuum τ_0 one obtains a lagrangian of the Klein-Gordon type - an effective meson model; recall our earlier remark about skyrmions springing from an effective field theory of pions. Couched in quantum terminology, the pions are represented by the fluctuations of the field J around τ_0. The lagrangian [34] is the starting point of the Skyrme model.

With regards to the homotopy of fhe chiral model in two spatial dimensions first note that finiteness of the energy compactifies the plane into the unit 2-sphere as per [9]. Since the internal manifold is a 3-sphere we then have a trivial homotopy $\left[\pi_2\left(S_3\right) = 0\right]$ wherein no topological extended objects can be accommodated.

Now, the only localised solutions to [34] are those corresponding to J being antihermitian $\left[\phi_0 = 0\right]$, the O(3) subspace of O(4) [32]. In this case topological solitons do arise because $\pi_2\left(S_2\right) = Z$. Consequently, one frequently focuses on the O(3) model rather than O(4).

An interesting modification of the chiral system is the Ward model, where we have time-dependent lumps which do not lie in general in an O(3) subspace. This model is integrable but at the expense of destroying the relativistic invariance of the pure chiral scheme. Both trivial and non-trivial scattering have been observed in the Ward model [33, 34].

An important example of a soliton in two spatial dimensions is the vortex in the abelian Higgs model. Vortices illustrate the mechanism for obtaining dual-strings from gauge theories [35] and, upon suitable change of semantics, the vortex system turns into the Ginzburg-Landau model [36] in the statistical mechanics of a superconductor placed in a magnetic field. Here the magnetic flux is quantised by the topological charge.

A prototype presentation of the vortex lagrangian is (note the quartic kink-like potential):

$$\mathcal{L}_{vortex} = -\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta} + \left(D_{\alpha}\phi\right)^{*}\left(D^{\alpha}\phi\right) - \frac{\eta}{2}\left(|\phi|^2 - \frac{m^2}{\eta}\right)^2, \qquad [35]$$

where ϕ is a complex scalar field, $F_{\alpha\beta}$ is the familiar electromagnetic tensor, D_{α} is the covariant derivative and m, η are constants. The 't-Hooft-Polyakov monopole model is a non-abelian extension of [35].

The procedure followed to obtain [27] can also be applied here. One finds

$$\frac{d^2}{d\gamma^2} V\left[\vec{\phi}_\gamma(\vec{x})\right]\Big|_{\gamma=1} = 6V_2\left[\vec{\phi}_1(\vec{x})\right] = 0 \qquad [36]$$

unveiling the presence of zero modes. From [5] we further obtain

$$V\left[\vec{\phi}_\gamma(\vec{x})\right] = V_1\left[\vec{\phi}_1(\vec{x})\right], \qquad [37]$$

confirming the scale-free nature of bidimensional sigma models. Wherefore, planar solitons have no preferred scale and at the expense of no energy at all they can alter their size under small perturbations. In this sense they are unstable. In particular, such instability occurs in the planar O(3) model, but it is corrected in its Skyrme version.

Historically interesting is the fact that in the 1960s the quantum version of [29] interpreted ϕ_0 as the creation operator of a σ-particle and $\vec{\phi}$ designated a pion operator. The name 'sigma' was thus coined for most models with structure similar to [34]. The notation in terms of sigma and pion fields is still widely used.

5. Solitons in three dimensions

According to Derrick's theorem nontrivial static solitons in three or more spatial dimensions cannot exist for models based upon a lagrangian [1]. Adopting a more general standpoint one can circumvent such limitation, though. For instance, one can permit the interaction of the scalar field $\vec{\phi}$ with gauge fields, idea that leads to monopole theories. Or one can stick to scalar fields only and add extra terms to [1] which is the procedure implemented by Skyrme (6) he added an extra term to the O(4) model in four-dimensional space-time. The Skyrme lagrangian is given by

$$\mathcal{L}_{skyrme} = C_1\left(\partial_\mu\vec{\phi}\right)\cdot\left(\partial^\mu\vec{\phi}\right) - C_2\Big[\left(\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi}\right)^2 + \left(\partial_\mu\vec{\phi}\cdot\partial_\nu\vec{\phi}\right)\left(\partial^\mu\vec{\phi}\cdot\partial^\nu\vec{\phi}\right)\Big],\ \mu,\nu = 0,1,2,3, \qquad [38]$$

where the real vector $\vec{\phi}$ is of the form [30]. The constants C_j are free parameters which in principle can be calculated from QCD; in practice their values are fitted by phenomenological considerations.

In chiral notation the above lagrangian is usually written as

$$\mathcal{L}_{skyrme} = -\frac{F_\pi^2}{16} Tr\left(R_\mu R^\mu\right) + \frac{1}{32e^2} Tr\left(\left[R_\mu, R_\nu\right]\left[R^\mu, R^\nu\right]\right),\ R_\mu = \left(\partial_\mu J\right)J^\dagger \qquad [39]$$

where the SU(2) quaternion J is the 3-D analogue of [33]

$$J = \sigma\left(x^\mu\right)\tau_0 + i\,\vec{\tau}\cdot\vec{\pi}\left(x^\mu\right),\ \ \vec{\pi} = \left(\pi_1, \pi_2, \pi_3\right) \qquad [40]$$

The unitarity of J is guaranteed by the ordinary ligature on the fields:

$$\sigma^2 + \vec{\pi}^2 = 1 \qquad [41]$$

The routine finite-energy analysis exacts that localised lumps must tend to an absolute minimum of the integrand of the potential (using a particular choice of the parameters in [39]).

$$V_{skyrme} = -\int\left\{\frac{1}{2} Tr\left(R_j R_j\right) + \frac{1}{32} Tr\left(\left[R_j, R_k\right]\left[R_j, R_k\right]\right)\right\} d^3x,\ \ j,k = 1,2,3, \qquad [42]$$

at spatial infinity. Electing the $2\ x\ 2$ identity matrix as the vacuum, the finite-energy argument translates into

$$\lim_{|\vec{x}|\to\infty} J(\vec{x}) = \tau_0, \qquad [43]$$

effectively compactifying $\Re_3$ to a three-sphere. At any given time, finite-energy fields are maps $J: S_3 \mapsto S_3$ whose associated homotopy classification is dictated by $\pi_3\left(S_3\right) = Z$.

The topological index for this model is interpreted as the baryon number

$$Q_{skyrme} = \int B^0 d^3x, \\ = \int \frac{1}{24\pi^2} \int \in^{0\,jkl} Tr\left(R_j R_k R_l\right) d^3 x, \quad [44]$$

of the topological current (compare with [20])

$$B^{\mu} = \frac{1}{24\pi^2} \in^{\mu\nu\lambda\gamma} Tr\left(R_\nu R_\lambda R_\gamma\right). \quad [45]$$

Completing the square in [42] we get

$$-\frac{1}{2}\int Tr \sum_j \left\{R_j \pm \frac{1}{4} \in_{jkl} \left[R_k, R_l\right]\right\}^2 \\ d^3x \mp 12\pi^2 Q_{skyrme}, \quad [46]$$

the Bogomolny bound in the present case being

$$V_{skyrme} \geq 12\pi^2 \left|Q_{skyrme}\right|. \quad [47]$$

The equality in the above expression occurs iff

$$R_j \pm \frac{1}{4} \in_{jkl} \left[R_k, R_l\right] = 0 \quad [48]$$

for which no non-trivial analytic solutions have been found. Its simplest numerical solution corresponds to a quaternion of the form

$$J(\vec{x}) = \cos\left[f\left(|\vec{x}|\right)\right] + i\,\frac{\sin\left[f\left(|\vec{x}|\right)\right]}{|\vec{x}|}\,\vec{x}\cdot\vec{\tau}, \quad [49]$$

where the profile function $f\left(|\vec{x}|\right)$ is subject to $f(0) = \pi$ and $f(\infty) = 0$. It sets the skyrmion energy to the value $1.232\ x\ 12\pi^2$ which exceeds the minimal energy in [47] (37). Some scholars (38, 39) have been able to produce a value of V_{skyrme} closer to the minimal value by using instanton holonomies to generate skyrmion fields. So, the approximate solution [49] is a local minimum rather than an absolute one.

The first application of skyrmions in nuclear physics was the extraction of a nucleon-nucleon interaction energy of separated $Q = 1$ lumps (40, 41), idea later extended to Q = 2. The deuteron for instance, being the simplest nucleus, has been described as a quantised two-skyrmion by a number of people, using very particular approximations (42-45). As mentioned in the introduction, the results extracted from the Skyrme model are in qualitative accord with reality (7, 8). Approximate skyrmions on a cubic lattice belonging to Q=3,4,5,6 have been reported in (46). And more recently, high-technology multi-skyrmion scattering has been investigated using an economical approximation based on a solution of the sine-Gordon type (47).

The progress that has been made in deriving multi-configurations in three spatial dimensions bodes well for the longevity of the model, but still the multi-skyrmion problem is very hard to attack. Analytical solutions even for the simplest single-soliton case are not available.

Consequently, one is naturally led to investigate simpler models which still possess key features of the four dimensional ones. Through such low-dimensional analogues one hopes that a better understanding of the underlying mechanism of soliton dynamics will be attained, thenceforth assisting in the analysis of the more realistic, but more involved, (3+1) case. Skyrme himself used a (1+1) dimensional model (sine-Gordon) as calistenics to his (3+1) invention.

Finally, we present a Derrick-like argument in three spatial dimensions: Under dilations $\vec{x} \to \gamma\vec{x}$ the potential [42] goes to

$$V\left[J\left(\gamma\vec{x}\right)\right] = \gamma^{2-D} V_1\left[J(\vec{x})\right] + \gamma^{4-D} V_{sk}\left[J(\vec{x})\right], \quad [50]$$

where V_1, V_{sk} denote the first and second terms is the right-hand-side of [42]. Equation [50] is the analogue of equations [4]-[5]. Diferentiating we get

$$\frac{d}{d\gamma} V\left[J\left(\gamma\vec{x}\right)\right] = (2-D)\gamma^{1-D} V_1\left[J(\vec{x})\right] \\ +(4-D)\gamma^{3-D} V_{sk}\left[J(\vec{x})\right] \quad [51]$$

Setting the left hand side equal to zero for $\gamma = 1$ there follows

$$(4-D)V_{sk}(J) = (D-2)V_1(J), \qquad [52]$$

according to which the existence of solitons in $D = 3$ is now licit. Note also that plugging the value $D = 3$ into (50) we find

$$V\left[J(\gamma\vec{x})\right] = \gamma^{-1}V_1[J(\vec{x})] + \gamma V_{sk}[J(\vec{x})], \qquad [53]$$

characteristic of a stable lump if we recall the kink result [28].

The whys and wherefores of the additional Skyrme term in le lagrangian are clearly to stabilise the solitons. In the pure chiral limit $\left[V_{sk} = 0\right]$, equation [53] says that for any configuration J the energy can always be decreased by dilations $\gamma > 1$. In the limit as the latter goes to infinity the size of the lump collapses to zero. But a non-zero Skyrme term gives a minimal value of the potential energy equal to

$$V_{minimal} = 2\sqrt{V_{sk}V_1}\,. \qquad [54]$$

There are other important examples of solitons in three spatial dimensions, including Yang-Mills instantons, monopoles and dyons, this latter objects being carriers of both magnetic and electric charge.

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