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A note on the magnetic spherical pendulum

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Abstract

The magnetic spherical pendulum is a mechanical system consisting of a pendulum whereof the bob is electrically charged, moving under the influence of gravitation and the magnetic field induced by a magnetic monopole deposited at the origin. Physically not directly realizable, it turns out to be equivalent to a reduction of the Lagrange top. This work is essentially the log book of our attempts at understanding the simplest contemporary approaches to the magnetic spherical pendulum.

Key words: magnetic spherical pendulum, classical mechanics, magnetic monopole.

Una nota sobre el péndulo esférico magnético

Resumen

El péndulo esférico magnético es un sistema mecánico consistente de un péndulo con una masa pendular la cual está eléctricamente cargada, moviéndose bajo la influencia de la gravedad y un campo magnético inducido por un monopolo colocado en el origen de las coordenadas. Físicamente no es posible realizar el sistema debido a no haberse encontrado monopolos magnéticos en la naturaleza. El trabajo a continuación es un intento por generalizar y entender un simple sistema mecánico como lo es el péndulo esférico.

Palabras clave: péndulo esférico magnético, mecánica clásica, monopolo magnético.

1. Introduction

In classical mechanics, the term spherical pendulum refers to the configuration consisting of a massive particle suspended by a rigid weightless rod from a fixed point and subject to constant uniform gravitational field (1, 2). Equivalently the spherical pendulum may be regarded as a particle constrained to move on a sphere under constant uniform gravitational field. This is an old and established problem of classical mechanics which has been subjected to very detailed treatment in numerous textbooks (3-5). It is known that a full quantitive solution of the governing equations requires the use of elliptic functions. Many other generalizations of the problem have been proposed to study more general cases and come up with solutions some physical or not (6).

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An interesting if rather artificial modification of the problem ensues if the massive particle is also understood to be electrically charged (7-9) and a magnetic monopole is placed at the center of the sphere on which the particle in question is constrained to move. In this case the particle is subject not only to the gravitation and the forces of constraint but also to the Lorentz force that acts on charged particles that move in magnetic fields. Clearly rather far fetched from a scientific point of view, the problem has been invented for the purpose of illustrating the use of certain mathematical artifacts.

2. The system

For the standard spherical pendulum, the horizontal coordinates $x(t)$, $y(t)$ could written, with the adoption of a coordinate system and a source of suitable time, in the manner

$$x(t) = a\cos\left(\frac{g}{l}t\right), \; y(t) = b\sin\left(\frac{g}{l}t\right), \quad [1]$$

where g is a constant whose units are of angular velocity, and the above expression represents a periodic movement whose trajectory is an ellipse of semi-axes a and b.

For the spherical pendulum one finds the motions equations:

$$T = mg\cos\theta + ml(\dot{\theta} + \dot{\phi}^2 \sin^2\theta) \quad [2]$$

$$\ddot{\theta} - \dot{\phi}^2 \sin\theta\cos\theta + \omega_0 \sin\theta = 0 \quad [3]$$

$$\ddot{\phi}\sin\theta + 2\dot{\theta}\dot{\phi}\cos\theta = 0 \quad [4]$$

where $\omega_0 = g / l$ is the natural frequency of the system. The first of these equations [2] defines the tension on the rope with the constrain $r = l$. Alternatively, the equations [3] and [4] can be obtained directly with a Lagrangian formalism in spherical coordinates.

In particular we are interested in how the equations [2], [3] and [4], are modified in the presence of a magnetic monopole with coupling μ located in origin, with an electrical charge q in the particle, that is, in the pendulum. The resulting magnetic induction is for monopole given by:

$$B := \mu\frac{\mathbf{r}}{\|\mathbf{r}\|^3} = \frac{\mu}{l^3}\mathbf{r}. \quad [5]$$

Of course, under Lorentz's force,

$$\mathbf{F_B} = q\dot{\mathbf{r}} \times \mathbf{B} = \frac{q\mu}{l^3}\dot{\mathbf{r}} \times \mathbf{r} \quad [6]$$

which act in the charged particle. Explicitly the Lorentz's force are:

$$\frac{q\mu}{l^3}\dot{\mathbf{r}} \times \mathbf{r} = \frac{q\mu}{l^3}\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}\dot{\theta} - \frac{q\mu}{l^3}\begin{pmatrix}-\cos\theta\sin\theta\cos\phi\\ \cos\theta\sin\theta\sin\phi\\ \sin^2\theta\end{pmatrix}\dot{\phi} \quad [7]$$

Adding this force in the left side of equation [2] and tacking inner product in this modificated equation [2] with

$$\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}$$

we obtain the modified form of [3] as

$$\frac{q\mu}{ml^4}\dot{\theta} = 2\dot{\theta}\dot{\phi}\cos\theta + \ddot{\phi}\sin\theta \quad [8]$$

and taking inner product with

$$\begin{pmatrix}\cos\theta\sin\phi\\ \cos\theta\cos\phi\\ \sin\theta\end{pmatrix}s$$

we obtain the modified form of [4] as

$$-\frac{g}{l}\sin\theta-\frac{q\mu}{ml^4}\sin\theta\dot{\phi}=\ddot{\theta}-\phi^2\sin\theta\cos\theta. \quad [9]$$

We observe that [8] can be multiplied with $\dot{\theta}$ and written also in the form

$$\frac{d}{dt}\left[ml^2\dot{\phi}^2\sin^2\theta+\frac{q\mu}{l^3}\cos\theta\right]=0. \quad [10]$$

Clearly the quantity within the brackets is a conserved quantity.

However, it may no longer be identified with the z-component of the angular momentum about the origin. A new contribution of the conserved quantities associated with the magnetig monopole apear how are expecting. In this case, is posible that the energy associated to magnetic pendulum change to.

The motion equation are direct starightforward obtain in terms of the cartesian quantities, and are:

$$\ddot{x}+\omega_0^2x-\frac{\omega_0^2}{2l^2}(x^2+y^2)x+\frac{q\mu\dot{x}}{ml^4}+$$
$$\frac{(\dot{x}^2+\dot{y}^2)x}{l^2}+\frac{q\mu(x\dot{x}+y\dot{y})(l^2-x^2)}{2ml^6y}-$$
$$\frac{q\mu\dot{x}}{2ml^6}(x^2+y^2)=0, \quad [11]$$

and for y an similar equation.

For infinitesimal motions the cubic terms can be negleted, in this case the equation [11], become into expression

$$\frac{d^2}{dt^2}x(t)+\frac{q\mu}{ml^4}\frac{d}{dt}x(t)+\omega_0^2x(t)=0 \quad [12]$$

wich solution is directly shown

$$x(t)=C_1e^{(\alpha-\beta)t}+C_2e^{(\alpha+\beta)t} \quad [13]$$

with

$$\alpha:=-\frac{q\mu}{2ml^4},\ \beta:=\frac{\sqrt{q^2\mu^2-4m^2l^8\omega_0^2}}{2ml^4}. \quad [14]$$

Note that for $\mu = 0$ we found the trivial solution for the spherical pendulum making use that the pendulum is started by a displacement in the x coordinate and an impulse in the y direction. This is a infinitesimal solution determined by

$$x(t)=a\cos\omega t,\ y(t)=b\sin\omega t, \quad [15]$$

describing an ellipse whose axes lie along the x and y coordinate.

In our case, the infinitesimal solution for $q\mu<2ml^4\omega_0$, provies the solution

$$x:=Ae^{\alpha t}\cos\beta t,\ y:=Ae^{\alpha}\sin\beta t \quad [16]$$

wich describe a region dominated for the equation,

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}=e^{2\alpha t}. \quad [17]$$

3. Finite displacements

The effect of the cubic terms in equation [11] will cause to change slightly in frecuency and introduces small higher harmonic term. To account for this we introduce a coordinate system x' and y' that rotate at angular velocity Ω relative to the x, y system

$$x=x'\cos\Omega t-y'\sin\Omega t \quad [18]$$

$$y=x'\sin\Omega t-y'\cos\Omega t \quad [19]$$

where in the rotating system the orbit is stationary and is described by

$$x'=a\cos\omega t+\textit{small higher harmonics terms} \quad [20]$$

$$y'=b\sin\omega t+\textit{small higher harmonics terms} \quad [21]$$

The new frecuency ω differ slightly from natural frecuency ω_0.

Ours strategy will be to require that equations [19] and [20], satisfy the equation of motion [11] for each harmonic and to look in detail at the lowest harmonic. Combining equation [19] and [21], our trial solution for the lowest harmonic, is

$$x' = a\cos\omega t\cos\Omega t - b\sin\omega t\sin\Omega t \qquad [22]$$

$$y' = a\cos\omega t\sin\Omega t + b\sin\omega t\cos\Omega t. \qquad [23]$$

By direct substitution of equation [22] into the equation of motion [11], we verify that our trial solution works, but only for special values of Ω and ω. This substitution is perfectly straightforward, but takes a few steps. Some intermediate quantities are (see eq. [24-28])

$$T2{:=}-\frac{\omega_0^2}{2l^2}(x^2+y^2)x = \\ = \frac{\omega_0^2}{2l^2}(-a\cos\omega t\cos\Omega t + b\sin\omega t\sin\Omega t) \\ (b^2\sin^2\omega t + a^2\cos^2\omega t) \qquad [25]$$

$$T3{:=}\frac{(\dot{x}^2+\dot{y}^2)}{l^2}x = \\ = -\frac{1}{l^2}(-a\cos\omega t\cos\Omega t + b\sin\omega t\sin\Omega t) \\ \left(b^2(a\Omega+b\omega)^2 + a^2(b\Omega+a\omega)^2\right) \qquad [26]$$

$$T4{:=}\frac{q\mu(x\dot{x}+y\dot{y})(l^2-x^2)}{2ml^6y} = \\ = \frac{q\mu\cos\omega t\sin\omega t(b^2-a^2)}{ml^6(a\cos\omega t\sin\Omega t + b\sin\omega t\cos\Omega t)} \\ \left(l^2 - (a\cos\omega t\cos\Omega t + b\sin\omega t\sin\Omega t)^2\right) \qquad [27]$$

$$T5{:=}-\frac{q\mu\dot{x}(x^2+y^2)}{2ml^6} = -((b\omega+a\Omega)\times \\ \cos\omega t\sin\Omega t + (a\omega+b\Omega)\sin\omega t\cos\Omega t) \\ \frac{q\mu(b^2\sin^2\omega t + a^2\cos^2\omega t)}{2ml^6} \qquad [28]$$

Therefore, combining all the terms and taking the lower-order harmonics (which in this case will be terms of $\cos 2\omega t$, $\cos 2\Omega t$, $\sin 2\omega t$ and $\sin 2\omega t$, respectively) shall have the following important equations, which are the equations of motion for the spherical magnetic pendulum:

$$q\mu(a^2-b^2)\left(\left(b^2+\frac{8l^2}{3}+a^2\right)\Omega + \frac{2\omega ab}{3}\right) = 0 \qquad [29]$$

$$\frac{3l^4}{4}\left(b^2+\frac{8l^2}{3}+a^2\right)m\omega_0^2 + 2ml^6(\Omega^2+\omega^2) - \\ \frac{3l^4m}{2}\left(\left(\frac{a^2}{3}+\frac{b^2}{3}\right)\omega^2 + \frac{8ab\Omega\omega}{3} + \Omega^2(b^2+a^2)\right) \\ +bq\mu\omega a = 0 \qquad [30]$$

$$q\mu\omega(b^4+a^4-4a^2l^2+4b^2l^2) - \\ 4l^6m\omega\Omega(b^2-a^2) = 0 \qquad [31]$$

Here, making use of the equation [29], can be obtain the expression for the relative velocity Ω in terms of ω, in this case

$$\Omega = -\delta\omega, \quad \delta{:=}\frac{2ab}{3a^2+3b^2+8l^2} \qquad [32]$$

and making use of (30) elimining ω for this equation can be found that

$$T1{:=}\ \ddot{x} + \frac{q\mu}{ml^4}\dot{x} + \omega_0^2 x = \frac{-q\mu(a\Omega+b\omega) + ml^4\sin\omega t(2a\omega\Omega + b\omega^2 + b\Omega^2 - \omega_0^2 b)}{ml^4}\sin\Omega t \\ -\frac{q\mu\sin\omega t(a\Omega+b\omega) + ml^4\cos\omega t(a\omega^2 + a\Omega^2 + b\Omega\omega - a\omega_0^2)}{ml^4}\cos\Omega t \qquad [24]$$

$$\omega = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma\omega_0^2}}{2\alpha} \quad [33]$$

with de choice

$$\alpha := l^2 - \frac{a^2+b^2}{4} + \delta^2\left(l^2 - \frac{3(a^2+b^2)}{4}\right) + 2ab\delta \quad [34]$$

$$\beta := \frac{q\mu ab}{2ml^4} \quad [35]$$

$$\gamma := \frac{3}{8}(a^2+b^2) - l^2 \quad [36]$$

and therefore

$$\Omega = -\delta\frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma\omega_0^2}}{2\alpha}. \quad [37]$$

Therefore, the frequency of the rotated system and the relative frequency, depending on the natural frequency ω_0, the integration constant a and b, the length of the rope l, the mass of the charge m, and the coupling parameters of magnetic monopole q and μ. In addition, the coupling parameter is given by equation [31], such that:

$$\mu = -4\frac{ml^6\Omega}{q(a^2+b^2-4l^4)}. \quad [38]$$

4. Angle precession for spherical magnetic pendulum

Since β is a function of μ, this value can be inserted in the equation [37] and therefore is achieved immediately:

$$\Omega = \sqrt{\frac{\gamma}{D\delta - \alpha}}\delta\omega_0, \text{ con } D := -\frac{2abl^2}{(a^2+b^2-4l^2)} \quad [39]$$

Now the angle of precession for the spherical pendulum is:

$$\theta_n = 2\pi n\frac{\Omega}{\omega_0} = 2\pi n\sqrt{\frac{\gamma}{D\delta - \alpha}}\delta \quad [40]$$

this concludes our consideration.

5. Conclusions

We get a solution of the equations of motion for the spherical pendulum with a magnetic monopolo localed in the origin of coordinate system, such that the solution is the analytical, ie without resorting to numerical methods or computer, and the solution is cuasi-exact, under some minimum conditions harmonics in the general equation of this dynamic.

In conclusion it can be seen that the frequency of spherical magnetic pendulum is given in terms of the natural frecuency through the equation [40].

The coupling constant of a monopole to the system is given by

$$\mu = -4\frac{ml^6\Omega}{q(a^2+b^2-4l^4)}$$

in terms of parameters of system a, b, ω, and length of the pendulum, as well as particle charge.

Recall that the frequency of the spherical pendulum in rotated system: $\omega^2 = \omega_0^2\left(1 - \frac{a^2+b^2}{8l^2}\right)$, so if $a = b = 0$, the system does not rotated and the above equation are: $\omega = \omega_0$, the frequency is equal to the natural frequency of the system (as expected). Now see if the frequency of the system of the spherical magnetic pendulum keeps this condition as well. Therefore, it must meet the condition that if the pendulum is not under the influence of monopolo magnetic and whether $a = b = 0$, the frequency should be equal to the natural frequency of the system, in summary is due to meet if $\beta = 0$, (without monopole) then

$\omega = \omega_0$. We replaced these conditions, and to make this possible must be met:

$$\sqrt{-\frac{\gamma}{\alpha}} = 1 \Rightarrow \sqrt{\frac{l^2}{l^2}} = 1$$

Now in the spherical magnetic pendulum is not conserved the momentun angle L_Z, but a linear combination of this with specific parameters of monopole, see:

$$\dot{\phi}\sin^2 q + \frac{q\mu}{ml^2}\cos\theta = constant,$$

$$\longrightarrow L_Z + q\mu\cos\theta = constant \qquad [41]$$

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