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Orthocenters of triangles in the n-dimensional space

Ortocentros para triángulos en el espacio n-dimensional

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Abstract

We present a way to define a set of orthocenters for a triangle in the n-dimensional space $\mathbb{R}^n$, and we show some analogies between these orthocenters and the classical orthocenter of a triangle in the Euclidean plane. We also define a substitute of the orthocenter for tetrahedra which we call $\mathcal{G}-$orthocenter. We show that the $\mathcal{G}-$orthocenter of a tetrahedron has some properties similar to those of the classical orthocenter of a triangle.

Key words and phrases: orthocenter, triangle, tetrahedron, orthocentric system, Feuerbach sphere.

Resumen

Presentamos una manera de definir un conjunto de ortocentros de un triángulo en el espacio n-dimensional $\mathbb{R}^n$, y mostramos algunas analogías entre estos ortocentros y el ortocentro clásico de un triángulo en el plano euclidiano. También definimos un sustituto del ortocentro para tetraedros que llamamos $\mathcal{G}-$ortocentro. Se demuestra que el $\mathcal{G}-$ortocentro de un tetraedro tiene algunas propiedades similares a los del ortocentro clásico de un triángulo.

Palabras y frases clave: ortocentro, triángulo, tetraedro, sistema ortocéntrico, esfera de Feuerbach.

1 Introduction

In the Euclidean plane, the *orthocenter* H of a triangle $\triangle ABC$ is known as the unique point where the altitudes of the triangle intersect, i.e., the point at which the three lines perpendicular to

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the (prolonged) sides of the triangle and passing through the opposite vertex meet. Orthocenters of triangles are closely related to many theorems of elementary geometry (see below), and their higher dimensional analogues create the interesting class of *orthocentric simplices* (see [4, 5]). If O and G are the *circumcenter* and the *centroid* of the triangle, respectively, a classical theorem of Euler asserts that O , G, and H are collinear (creating the famous *Euler line* of the triangle) with $|OG| = 2|GH|$.

Another property of the orthocenter of a triangle is the following: the orthocenter is the point where concur the circles whose radii are equal to that of the circumcircle, suitably passing through two vertices of the triangle; i.e, if the circumcircle is reflected with respect to the midpoints of the sides of the triangle, then the three obtained circles concur in the orthocenter of the triangle. Since that definition of the orthocenter does not depend on the notion of orthogonality, we speak in this case about $\mathcal{C}$-*orthocenters* (where $\mathcal{C}$ comes from "circle"). It should be noted that the concept of $\mathcal{C}$-orthocenters can succesfully carried over to normed planes; see [2, 6, 7, 8, 9, 10]. Moreover, by the definition of $\mathcal{C}$-orthocenter of a triangle, this point is the circumcenter of the triangle whose vertices are the reflections of the circumcenter at the midpoints of the sides.

If the triangle $\triangle ABC$ is not a right triangle, then the triangles $\triangle HBC$, $\triangle AHC$, and $\triangle ABH$ have the points A, B, C as orthocenters, respectively. Thus, any triangle with vertices from the set $\{A, B, C, H\}$ has the remaining point as orthocenter, and so it makes sense to call a set of four points satisfying the above property an *orthocentric system*. Analogously, for $\mathcal{C}$-orthocenters we speak about a $\mathcal{C}$-*orthocentric system*. Basic references on $\mathcal{C}$-*orthocentric systems* in normed planes are [2, 6, 7, 8, 9, 10].

When we review the properties and notions related to the orthocenter (such as Euler line, Feuerbach circle, $\mathcal{C}$-orthocenter, orthocentric system, etc.), we realize that they essentially depend on the relationship between vertices and the circumcenter of the triangle, i.e., equidistance. In this paper we will use this idea to define an "orthocenter" associated with each point that is equidistant from the vertices of a triangle in n-dimensional space, and we will see some properties similar to those of the orthocenter in the Euclidean plane.

We will investigate notions like orthocentric systems, Euler-line properties and Feuerbach spheres of triangles and tetrahedra when embedding the starting figure into higher dimensional Euclidean space and creating important points of these systems by intersecting certain spheres. These are strongly related to the circumsphere of the starting figure. Clearly, some of our results could be obtained in a shorter way by using the ratio-invariance of a suitable parallel, or even orthogonal, projection in direction of the line connecting some point P in $\mathbb{R}^n$ (which is equidistant to A, B, C) and the circumcenter of the given triangle. This is particularly visible in the three-dimensional situation (see section 3.2 etc.), where the point H_P (see Theorem 3.1) is uniquely determined. It would finally yield only planar investigations. But our way via sphere intersections has some specific motivations, and we mention at least two of them. First, it opens the opportunity to speak about Monge points, orthocentric systems and Feuerbach spheres of tetrahedra also in (normed or) Minkowski spaces, a next research step in this direction. Namely, in $\mathbb{R}^n$ the concept of usual orthogonality is obviously needed for constructing the Monge point as analogue of the orthocenter in dimensions $n > 2$, but does no longer hold in Minkowski spaces (see [1] and also the remarks at the end of the paper); via sphere intersections, a reasonable analogue of the Monge point can be obtained. And second we believe that our approaches are a good "exercise" in spatial geometry, helping to develop the capacity of thinking in higher dimensions and thus to stimulate a better understanding of descriptive (advanced) Elementary Geometry, useful for readers like, e.g., math teachers.

2 Notation and preliminaries

Let $\mathbb{R}^n$ denote the classical n-dimensional Euclidean space, the elements of this vector space are identified with points and denoted by capital letters. If A and B are two points, then $\overrightarrow{AB} = B - A$ is written for their difference vector, whose norm is given by $\|B - A\|$, and AB denotes the standard *segment* with endpoints A and B, respectively. The *length* of AB is denoted by $|AB| = \|B - A\|$.

A *triangle* $\triangle A_0A_1A_2$ is determined by three non-collinear points A_0, A_1, and A_2 in the space $\mathbb{R}^n$. These points A_i are called *vertices* of the triangle, the segment denoted by a_i whose endpoints are the vertices other than A_i is called the *side opposite to* A_i. Denote by O, $\mathcal{C}$, r and G the *circumcenter*, the *circumcircle*, the *circumradius*, and the *centroid* of the triangle $\triangle A_0A_1A_2$, respectively, i.e., O is the only point in the plane determined by A_0, A_1, A_2 equidistant from them, with $r = |OA_0|$ and $G = \frac{1}{3}(A_0 + A_1 + A_2)$. By M_i we denote the *midpoint of the side* a_i. We also recall the *medial triangle* $\triangle M_0M_1M_2$ of $\triangle A_0A_1A_2$, and denote the circumcenter of $\triangle M_0M_1M_2$ by Q_O. Note that $Q_O = \frac{1}{2}(A_0 + A_1 + A_2 - O)$.

If P is a point of $\mathbb{R}^n$ and λ is a scalar, the *homothety* with center P and ratio λ is the mapping $\mathscr{H}_{P,\lambda} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\mathscr{H}_{P,\lambda}(X) = (1 - \lambda) P + \lambda X,$$

for all X in $\mathbb{R}^n$. The particular case $\mathscr{H}_{P,-1}$, also called *point reflection at* P, we will symbolize by $\mathscr{S}_P$.

For the triangle $\triangle A_0A_1A_2$, the orthocenter H is expressed as a function of the circumcenter O and the vertices of the triangle by the formula $H = A_0 + A_1 + A_2 - 2O$, and it is not difficult to see that H is the circumcenter of the triangle $\triangle B_0B_1B_2$, where B_i is the point symmetric to O with respect to M_i for $i = 0, 1, 2$; i.e, $B_i = A_j + A_k - O$. The points B_0, B_1, and B_2 are the circumcenters of the triangles $\triangle HA_1A_2$, $\triangle A_0HA_2$, and $\triangle A_0A_1H$, respectively; the circumcircles of these triangles are denoted by $\mathcal{C}_0$, $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. All of them have radius r. The triangles $\triangle A_0A_1A_2$ and $\triangle B_0B_1B_2$ are symmetric to each other, and in [8] the triangle $\triangle B_0B_1B_2$ is called the *antitriangle* of $\triangle A_0A_1A_2$ associated to O. The center of symmetry of the union of both triangles is the point Q_O.

The following list contains some of the properties satisfied by the orthocenter (see Figure 1).

1. The points O, G, and H are collinear, with G in between, and $2|OG| = |GH|$ (the *Euler-line property*).

2. If N_0, N_1, and N_2 are the midpoints of the sides of the triangle $\triangle B_0B_1B_2$, the circle with center Q_O and radius $\frac{r}{2}$ (called the *Feuerbach circle* of $\triangle A_1A_2A_3$) passes through the points M_0, M_1, M_2, N_0, N_1, and N_2. It also passes through the midpoints of the segments that join H with the points of the circumcircle of $\triangle A_0A_1A_2$, and through the midpoints of the segments that join O with the points of the circumcircle of $\triangle B_0B_1B_2$.

3. The points O, Q_O, G, and H form a *harmonic quadruple*, satisfying $\frac{|OG|}{|GQ_O|} = \frac{|OH|}{|HQ_O|} = 2$.

4. The following sets: $\{A_0, A_1, A_2, H\}$, $\{B_0, B_1, B_2, O\}$, $\{M_0, M_1, M_2, O\}$, $\{N_0, N_1, N_2, H\}$, and $\{G_0, G_1, G_2, G\}$, where G_0, G_1, and G_2 are the centroids of the triangles $\triangle HA_1A_2$, $\triangle A_0HA_2$, and $\triangle A_0A_1H$, respectively, are *orthocentric systems*.

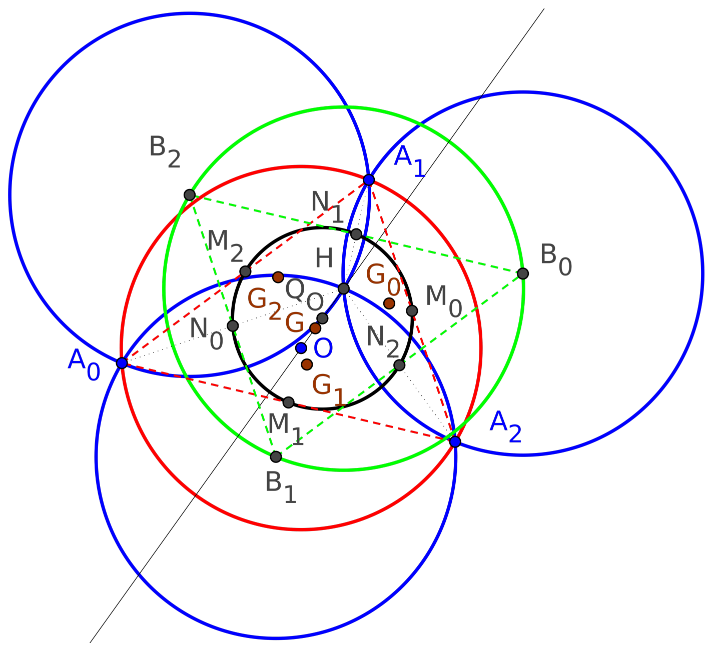

Figure 1: Properties of the orthocenter

5. If $\{A_0, A_1, A_2, A_3\}$ is an orthocentric system, then $\overrightarrow{A_iA_j} \perp \overrightarrow{A_kA_l}$, for $\{i, j, k, l\} = \{0, 1, 2, 3\}$, i.e., the vectors $\overrightarrow{A_iA_j}$ and $\overrightarrow{A_kA_l}$ are *orthogonal* to each other.

3 Results

We present now new results on orthocentric systems introduced for triangles in $\mathbb{R}^n$. More precisely, using points being equidistant to the vertices of the triangles, we will define orthocenters for triangles embedded into $\mathbb{R}^n$, and we compare them with the classical notion of orthocenter. We will see that this machinery allows to define a substitute of the orthocenter for tetrahedra, yielding further interesting analogies to the classical orthocenter of a triangle. All the figures that appear below, illustrate what happens in the three-dimensional case.

3.1 Orthocenters of triangles in $\mathbb{R}^n$

Given three non-collinear points A_0, A_1, A_2 in the Euclidean plane, there is only one point that is equidistant from them, being precisely the circumcenter of the triangle $\triangle A_0A_1A_2$. However, if the points A_0, A_1, A_2 are embedded in n-dimensional space $\mathbb{R}^n$, then the set of points equidistant from A_0, A_1, and A_2 forms an $(n-2)$-dimensional affine subspace, which we denote by $\mathscr{C}(\triangle A_0A_1A_2)$. Each point of this subspace is the center of an $(n-1)$-dimensional sphere passing through the points A_0, A_1, and A_2. The following theorem allows us to introduce the notion of an "orthocenter" associated with each point in $\mathscr{C}(\triangle A_0A_1A_2)$, and provides a generalization of the common notion of $\mathcal{C}$-orthocenter in the plane.

Theorem 3.1.1. *Let $\triangle A_0A_1A_2$ be a triangle in $\mathbb{R}^n$, G be its centroid and H its orthocenter. If $P \in \mathscr{C}(\triangle A_0A_1A_2)$ and $\mathcal{S}$ is the $(n-1)$-dimensional sphere with center P passing through the points A_0, A_1, and A_2 and having radius r, then the spheres $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$ that are symmetric to $\mathcal{S}$ with respect to the midpoints M_0, M_1, and M_2 of the sides of the triangle $\triangle A_0A_1A_2$ concur in a quadratic variety of dimension $n-3$ and, in particular, in the points H and $H_P = A_0 + A_1 + A_2 - 2P$. Furthermore, the following assertions hold:*

1. *If B_0, B_1, and B_2 are the centers of $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, then the triangles $\triangle A_0A_1A_2$ and $\triangle B_0B_1B_2$ are symmetric to each other with respect to the point $Q_P = \frac{1}{2}(A_0 + A_1 + A_2 - P)$.*
2. *The points P, G, and H_P are collinear having G between P and H_P, with $2|PG| = |GH_P|$ (the Euler-line property).*
3. *If N_0, N_1, and N_2 are the midpoints of the sides of the triangle $\triangle B_0B_1B_2$, the $(n-1)$-dimensional sphere $\mathcal{S}_M$ with center Q_P and radius $\frac{r}{2}$ passes through the points M_0, M_1, M_2, N_0, N_1, and N_2. It also passes through the midpoints of the segments that join H_P with the points of $\mathcal{S}$, and the midpoints of the segments that join O with the points of the $(n-1)$-dimensional sphere $\mathcal{S}_H$ with center H_P and radius r (the Feuerbach sphere associated to P).*
4. *The points P, Q_P, G, and H_P form a harmonic quadruple satisfying $\frac{|PG|}{|GQ_P|} = \frac{|PH_P|}{|H_PQ_P|} = 2$.*

Proof. Since the circumcircles $\mathcal{C}_0$, $\mathcal{C}_1$, and $\mathcal{C}_2$ are included in $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_2$, respectively, then the point H is in the spheres $\mathcal{S}_0$, $\mathcal{S}_1$, and $\mathcal{S}_2$. In order to see that $H_P = A_0 + A_1 + A_2 - 2P$ is in $\mathcal{S}_i$, it is enough to take a look at $|H_PB_i| = r$, for $i = 0, 1, 2$, where B_i is the center of $\mathcal{S}_i$. Note that $B_i = A_j + A_k - P$, for $\{i, j, k\} = \{0, 1, 2\}$, from which we get

$$|H_PB_i| = \|(A_j + A_k - P) - (A_0 + A_1 + A_2 - 2P)\| = \|P - A_i\| = r\,.$$

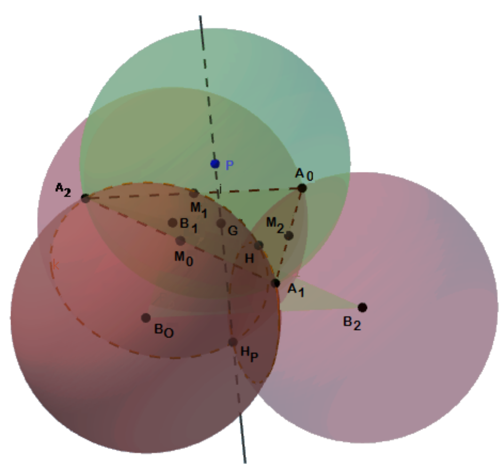

Figure 2: The orthocenter for a triangle in 3-dimesional space

1. Note that $A_i + B_i = A_i + A_j + A_k - P$, where $\{i, j, k\} = \{0, 1, 2\}$. Therefore, the midpoint of A_iB_i is $Q_P = \frac{1}{2}(A_0 + A_1 + A_2 - P)$, for $i = 0, 1, 2$.

2. Since $2(G - P) = \frac{2}{3}(A_0 + A_1 + A_2 - 3P) = \frac{2}{3}(H_P - P) = H_P - G$, it follows that P, G, and H_P are collinear and $2|PG| = |GH_P|$.

3. By item *1.* we know that $\mathscr{S}_{Q_P}(\triangle A_0A_1A_2) = \triangle B_0B_1B_2$, from which $\mathscr{S}_{Q_P}(M_i) = N_i$ for $i = 0, 1, 2$ is obtained. For the first part it only remains to show that $|M_iQ_P| = \frac{r}{2}$, for $i = 0, 1, 2$. Indeed,

$$\begin{aligned} |M_iQ_P| &= \left\| \frac{1}{2}(A_0 + A_1 + A_2 - P) - \frac{1}{2}(A_J + A_k) \right\| \\ &= \frac{1}{2}\|A_i - P\| = \frac{r}{2} \end{aligned}$$

for $i = 0, 1, 2$.

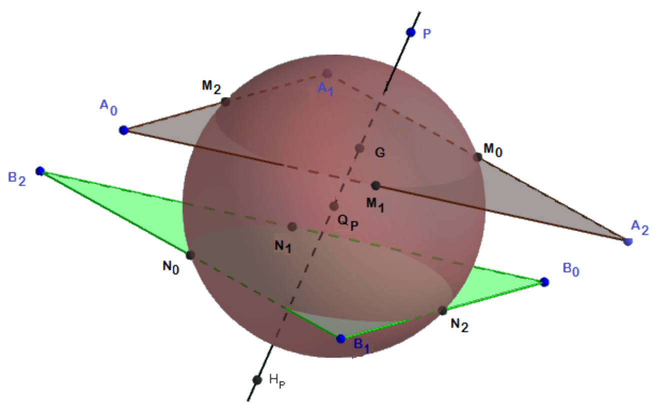

Figure 3: Triangle, antitriangle and Feuerbach sphere

For the second part note that $\mathscr{H}_{H_P,\frac{1}{2}}(\mathcal{S}) = \mathcal{S}_M$ and $\mathscr{H}_{P,\frac{1}{2}}(\mathcal{S}_H) = \mathcal{S}_M$, which implies the assertion.

4. Since 2. holds, $|PG| = \frac{1}{3}|PH_P|$. On the other hand,

$$\begin{aligned} |GQ_P| &= \left\| \left(\frac{1}{2}(A_0 + A_1 + A_2 - P) - \frac{1}{3}(A_0 + A_1 + A_2) \right) \right\| \\ &= \frac{1}{6}\|(A_0 + A_1 + A_2 - 3P)\| = \frac{1}{6}|PH_P| \end{aligned}$$

and

$$\begin{aligned} |Q_PH_P| &= \left\| \left((A_0 + A_1 + A_2 - 2P) - \frac{1}{2}(A_0 + A_1 + A_2 - P) \right) \right\| \\ &= \frac{1}{2}\|(A_0 + A_1 + A_2 - 3P)\| = \frac{1}{2}|PH_P|. \end{aligned}$$

Our statement follows from the above relations.

□

We call the point H_P the *orthocenter of the triangle* $\triangle A_0A_1A_2$ *associated to* P, and the set of all these orthocenters is denoted by $\mathcal{H}(\triangle A_0A_1A_2)$. The above theorem says that the Euler property is satisfied, i.e.,

$$\mathscr{H}_{G,\frac{1}{2}}\left(\mathcal{C}\left(\triangle A_0A_1A_2\right)\right) = \mathcal{H}\left(\triangle A_0A_1A_2\right).$$

Furthermore, the orthocenter of the triangle $\triangle H_PA_iA_j$ associated to B_k is the point A_k, where $\{i,j,k\} = \{0,1,2\}$. Thus, the notion of orthocentric system can be generalized to n-dimensional space, and we say that a set of four points $\{A_0, A_1, A_2, A_3\}$ is an *orthocentric system* if there is a point $P \in \mathcal{C}\left(\triangle A_0A_1A_2\right)$ such that $A_3 = A_0 + A_1 + A_2 - 2P$. We will see that the properties of orthocentric systems in the plane, previously listed, are also valid in this context. The following trivial, but useful, lemma is used for this purpose. In addition, this one is a generalization of Theorem 3.4 present in [10], which has a similar demonstration.

Lemma 3.1.2. *The homothetic image of a orthocentric system in $\mathbb{R}^n$ is a orthocentric system.*

Proof. Let $\{A_0, A_1, A_2, A_3\}$ be a orthocentric system, then there exists $P \in \mathcal{C}(\triangle A_0A_1A_2)$ such that $A_3 = A_0 + A_1 + A_2 - 2P$.

Let $B_i = \mathscr{H}_{C,\lambda}(A_i)$, for $i = 0,1,2,3$, and $R = \mathscr{H}_{C,\lambda}(P)$. Clearly, $R \in \mathcal{C}(\triangle B_0B_1B_2)$ and

$$\begin{aligned} B_0 + B_1 + B_2 - 2R &= ((1-\lambda)C + \lambda A_0) + ((1-\lambda)C + \lambda A_1) \\ &\quad + ((1-\lambda)C + \lambda A_2) - 2((1-\lambda)C + \lambda P) \\ &= (1-\lambda)C + \lambda\left(A_0 + A_1 + A_2 - 2P\right) \\ &= (1-\lambda)C + \lambda A_3 = B_3, \end{aligned}$$

which completes the proof. □

Theorem 3.1.3. *Let $\triangle A_0A_1A_2$ be a triangle in $\mathbb{R}^n$, G its centroid, $P \in \mathcal{C}\left(\triangle A_0A_1A_2\right)$, and H_P its orthocenter associated to P. Then the sets of points $\{A_0, A_1, A_2, H_P\}$, $\{B_0, B_1, B_2, P\}$, $\{M_0, M_1, M_2, O\}$, $\{N_0, N_1, N_2, H_P\}$ and $\{G_0, G_1, G_2, G\}$ are orthocentric systems, where G_0, G_1 and G_2 are the centroids of the triangles $\triangle H_PA_1A_2$, $\triangle A_0H_PA_2$, and $\triangle A_0A_1H_P$, respectively.*

Proof. We know that $M_i = \mathscr{H}_{G,-\frac{1}{2}}(A_i)$, for $i = 0,1,2,3$, and $\mathscr{H}_{G,-\frac{1}{2}}(H_P) = \frac{3}{2}G - \frac{1}{2}H_P = P$, from which $\{M_0, M_1, M_2, P\} = \mathscr{H}_{G,-\frac{1}{2}}\left(\{A_0, A_1, A_2, H_P\}\right)$ follows.

If $Q_P = \frac{1}{2}\left(A_0 + A_1 + A_2 - P\right)$, then $\mathscr{S}_{Q_P}\left(P\right) = H_P$. Thus,

$$\mathscr{S}_{Q_P}\left(\{A_0, A_1, A_2, H_P\}\right) = \{H_0, H_1, H_2, P\},$$

and

$$\mathscr{S}_{Q_P}\left(\{M_0, M_1, M_2, P\}\right) = \{N_0, N_1, N_2, H_P\}.$$

Finally, $G_i = \frac{1}{3}\left(A_i + 2A_j + 2A_k - 2P\right)$, from which we get

$$\mathscr{H}_{Q_P,-\frac{1}{3}}\left(A_i\right) = \frac{4}{3}Q_P - \frac{1}{3}A_i = \frac{2}{3}\left(A_0 + A_1 + A_2 - P\right) - \frac{1}{3}A_i = G_i,$$

for $i = 0,1,2,3$, and

$$\mathscr{H}_{Q_P,-\frac{1}{3}}\left(H_P\right) \;=\; \frac{4}{3}Q_P - \frac{1}{3}H_P = \frac{2}{3}\left(A_0 + A_1 + A_2 - P\right) - \frac{1}{3}\left(A_0 + A_1 + A_2 - 2P\right) = G.$$

Thus, $\mathscr{H}_{Q_P,-\frac{1}{3}}\left(\{A_0, A_1, A_2, H_P\}\right) = \{G_0, G_1, G_2, G\}$. □

Theorem 3.1.4. *If $\{A_0, A_1, A_2, A_3\}$ is an orthocentric system in $\mathbb{R}^n$, then the sets $\left\{\overrightarrow{A_iA_j}, \overrightarrow{A_kA_l}\right\}$ are orthogonal, for $\{i, j, k, l\} = \{0, 1, 2, 3\}$.*

Proof. Since usual orthogonality in $\mathbb{R}^n$ is equivalent to isosceles orthogonality (see [1]), we just need to see that $\left\|\overrightarrow{A_iA_j} - \overrightarrow{A_kA_l}\right\| = \left\|\overrightarrow{A_iA_j} + \overrightarrow{A_kA_l}\right\|$.

Indeed, consider the case $i = 0$, $j = 1$, $k = 2$, $l = 3$. Let $P \in \mathcal{C}\left(\triangle A_0A_1A_2\right)$ be such that $A_3 = A_0 + A_1 + A_2 - 2P$ and r is the radius of the sphere with center P passing through A_0, A_1, and A_2. Then

$$\left\|\overrightarrow{A_0A_1} - \overrightarrow{A_2A_3}\right\| = \|(A_1 - A_0) - (A_3 - A_2)\| = \|2\left(P - A_0\right)\| = 2r$$

and

$$\left\|\overrightarrow{A_0A_1} + \overrightarrow{A_2A_3}\right\| = \|(A_1 - A_0) + (A_3 - A_2)\| = \|2\left(A_1 - P\right)\| = 2r.$$

The other cases can be shown analogously. □

The above theorem tells us also that if $\{A_0, A_1, A_2, A_3\}$ is an orthocentric system and A_3 is not in the plane determined A_0, A_1, and A_2, then the tetrahedron $A_0A_1A_2A_3$ is an orthocentric tetrahedron, i.e, the altitudes of this tetrahedron concur. For properties of orthocentric tetrahedra and simplices we refer to [4, 5].

Conversely, if $A_0A_1A_2A_3$ is an orthocentric tetrahedron and consider the points B_i defined by $B_i = \frac{1}{2}\left(A_j + A_k + A_l - A_i\right)$, where $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Then, by using the orthogonality of opposite sides, it is not difficult to see that $B_i \in \mathcal{C}\left(\triangle A_jA_kA_l\right)$, and the orthocenter of the triangle $\triangle A_jA_kA_l$ associated to B_i is A_i.

Note that

$$\begin{aligned} Q_{B_i} &= \frac{1}{2}\left(A_j + A_k + A_l - B_i\right) \\ &= \frac{1}{2}\left(A_j + A_k + A_l - \frac{1}{2}\left(A_j + A_k + A_l - A_i\right)\right) \\ &= \frac{1}{4}\left(A_i + A_j + A_k + A_l\right), \end{aligned}$$

i.e., Q_{B_i} is the centroid G of the tetrahedron $A_0A_1A_2A_3$, and the Feuerbach sphere of the triangle associated to B_i is the *Feuerbach sphere of second kind* of the orthocentric tetrahedron $A_0A_1A_2A_3$, i.e., the sphere passing through the midpoints of the edges of $A_0A_1A_2A_3$.

On the other hand, the tetrahedra $A_0A_1A_2A_3$ and $B_0B_1B_2B_3$ are symmetric with respect to G. Thus G is also the centroid of the tetrahedron $B_0B_1B_2B_3$. As well, the Feuerbach spheres of second kind of the tetrahedra $A_0A_1A_2A_3$ and $B_0B_1B_2B_3$ coincide. In addition, the circumcenter of the tetrahedron $B_0B_1B_2B_3$ is the orthocenter of the tetrahedron $A_0A_1A_2A_3$.

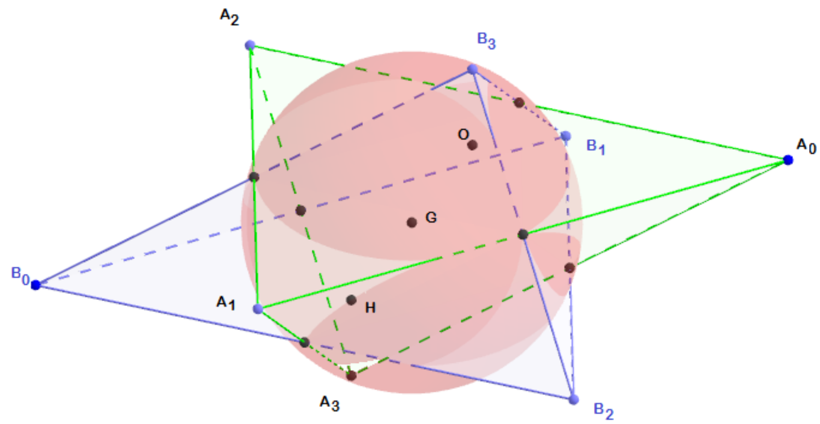

Figure 4: Orthocentrics tetrahedra $A_0A_1A_2A_3$ and $B_0B_1B_2B_3$ and Feuerbach spheres of second kind

This point gives rise to underline a nice analogy between the orthocenter of a triangle and the orthocenter of an orthocentric tetrahedron.The first one is the intersection of three circles, with radii equal to its circumradius, whose centers are equidistant to the vertices of the corresponding triangle side, in each case. And for the orthocentric tetrahedron, the orthocenter is the intersection of four spheres whose radii are equal to the circumradius of the tetrahedron, and whose centers are points equidistant to the three vertices of the respective tetrahedral face, in each case. In the latter case the spheres involved do not necessarilly pass through the corresponding vertices of the tetrahedron, but those which contain the vertices of their corresponding face have radius equal to twice the radius of the Feuerbach sphere of second kind of the orthocentric tetrahedron. We continue with results of this type in our next subsection.

3.2 Another orthocenter and Feuerbach spheres of tetrahedra

Recall that in a tetrahedron $A_0A_1A_2A_3$, the six planes perpendicular to the edges passing through their midpoints meet in a point called the *Monge point* M of $A_0A_1A_2A_3$. Properties of Monge points can be found in classical books on solid geometry, and in [3] and [5]. If O and G are the *circumcenter* and the *centroid* of the tetrahedron $A_0A_1A_2A_3$, respectively, then G is the midpoint of the segment with endpoints O and M, and therefore $M = \frac{1}{2}(A_0 + A_1 + A_2 + A_3 - 2O)$. The following results were inspired by results on cyclic quadrangles presented in [6, 5].

Theorem 3.2.1. *Let $A_0A_1A_2A_3$ be a tetrahedron, O its circumcenter, r its circumradius and M its Monge point. If $C_0C_1C_2C_3$ is the tetrahedron formed by the orthocenters of the faces of the tetrahedron $A_0A_1A_2A_3$ with respect to O (i.e., $C_i = A_j+A_k+A_l-2O$, for $\{i, j, k, l\} = \{0, 1, 2, 3\}$), then*

1. *$A_0A_1A_2A_3$ and $C_0C_1C_2C_3$ are symmetric with respect to M, i.e., for $i = 0,1,2,3$ we have $\mathscr{S}_M(A_i) = C_i$. In particular, $A_i - A_j = C_j - C_i$ for $\{i,j\} \subset \{0,1,2,3\}$,.*

2. *$\{C_i, C_j, C_k, A_l\}$ is an orthocentric system.*

3. *$\{C_i, C_j, A_k, A_l\}$ lie on a sphere with radius r.*

Proof. For the first assertion, notice that for each $i = 0,1,2,3$ we have

$$\frac{1}{2}(A_i + C_i) = \frac{1}{2}(A_i + A_j + A_k + A_l - 2O) = M\,,$$

where $\{i,j,k,l\} = \{0,1,2,3\}$. This proves that M is the midpoint of the segments with endpoints A_i and B_i, respectively.

For the second assertion we note that

$$\mathscr{S}_M(\{A_i, A_j, A_k, C_l\}) = \{C_i, C_j, C_k, A_l\}\,,$$

and $\{A_i, A_j, A_k, C_l\}$ is an orthocentric system. By the Lemma 2, also $\{C_i, C_j, C_k, A_l\}$ is an orthocentric system.

Finally, it is not difficult to see that the sphere with center $A_k + A_l - O$ and radius r contains the points $\{C_i, C_j, A_k, A_l\}$. □

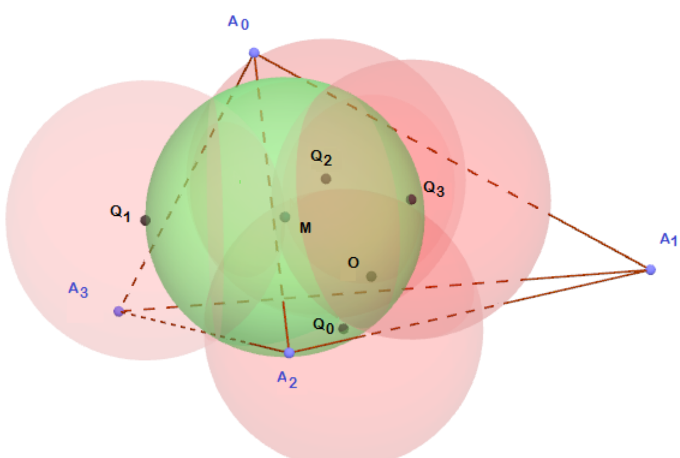

Figure 5: Feuerbach spheres of a tetrahedron

It is clear that the circumsphere of the tetrahedron $C_0C_1C_2C_3$ also has radius r, and that its center $H_{\mathcal{G}}$ (which we will call *$\mathcal{G}$-orthocenter* of the tetrahedron $A_0A_1A_2A_3$) is symmetric to O with respect to M. Note that $H_{\mathcal{G}}$ can be expressed by the vertices of the tetrahedron $A_0A_1A_2A_3$ and its circumcenter O via $H_{\mathcal{G}} = A_0 + A_1 + A_2 + A_3 - 3O$. The tetrahedron $C_0C_1C_2C_3$ we will call the *antitetrahedron* of $A_0A_1A_2A_3$. We will see that $H_{\mathcal{G}}$ has properties similar to those of the orthocenter of a triangle. For general orthocentric tetrahedra, $H_{\mathcal{G}}$ does not coincide with the orthocenter H. This holds only if the centroid G and the circumcenter O of the tetrahedron $A_0A_1A_2A_3$ coincide. The following theorem is analogous to Theorem 4.8 in [6], and will define a new Feuerbach sphere for tetrahedra analogous to the Feuerbach circle of cyclic quadrangles studied in [6].

Theorem 3.2.2. *Let $A_0A_1A_2A_3$ be a tetrahedron, O its circumcenter, r its circumradius, and M its Monge point. Then the four Feuerbach spheres of the faces of the tetrahedron $A_0A_1A_2A_3$ associated to O intersect in M. In addition, if Q_i is the center of the Feuerbach sphere of the face $A_jA_kA_l$ associated to O, for $\{i,j,k,l\} = \{0,1,2,3\}$, then $M - Q_i = \frac{1}{2}(A_i - O)$.*

Proof. Recall that

$$Q_i = \frac{1}{2}(A_j + A_k + A_l - O),$$

for $\{i,j,k,l\} = \{0,1,2,3\}$, and

$$M = \frac{1}{2}(A_0 + A_1 + A_2 + A_3 - 2O).$$

We get immediately $M - Q_i = \frac{1}{2}(A_i - O)$, and the proof is done. □

The relation between the Monge point and the Feuerbach spheres of the faces of the tetrahedron associated to the circumcenter of the tetrahedron, offers an alternative way to define the Monge point, without using orthogonality.

In the case of a cyclic quadrangle, the circle that passes through the centers of the Feuerbach circles of the triangles determined by the vertices of the quadrangle, is called the *Feuerbach circle of the quadrangle*. For that reason, we propose to call the sphere with center M and radius $r/2$ the *Feuerbach sphere of the tetrahedron* $A_0A_1A_2A_3$. In the planar case, the Feuerbach circle of a triangle coincides with the Feuerbach circle of its antitriangle. The existing symmetry between the tetrahedron $A_0A_1A_2A_3$ and its antitetrahedron $C_0C_1C_2C_3$ ensures the following corollary of the previous theorem, which is the analogue of Theorem 4.14 in [6].

Corollary 3.2.3. *The Feuerbach spheres of a tetrahedron and its antitetrahedron coincide.*

Furthermore, we easily get

Corollary 3.2.4. *The Feuerbach sphere of the tetrahedron $A_0A_1A_2A_3$ contains the midpoints of the segments that join the $\mathcal{G}-$orthocenter $H_{\mathcal{G}}$ with the points of the circumsphere S of the tetrahedron, and the midpoints of the segments that join the circumcenter O of the tetrahedron $A_0A_1A_2A_3$ with the points of the circumsphere S_1 of the antitetrahedron $C_0C_1C_2C_3$.*

Proof. The proof is straightforward since $\mathscr{H}_{O,\frac{1}{2}}(H_{\mathcal{G}}) = M$ and $\mathscr{H}_{H_{\mathcal{G}},\frac{1}{2}}(O) = M$. □

The following theorem shows further analogues between the notions of $\mathcal{G}-$orthocenter of a tetrahedron and orthocenter of a triangle. Recall that the *Feuerbach spheres of first kind* of a tetrahedron $A_0A_1A_2A_3$ is the circumspheres of the tetrahedron with vertices in the centroids of the faces of the tetrahedron $A_0A_1A_2A_3$.

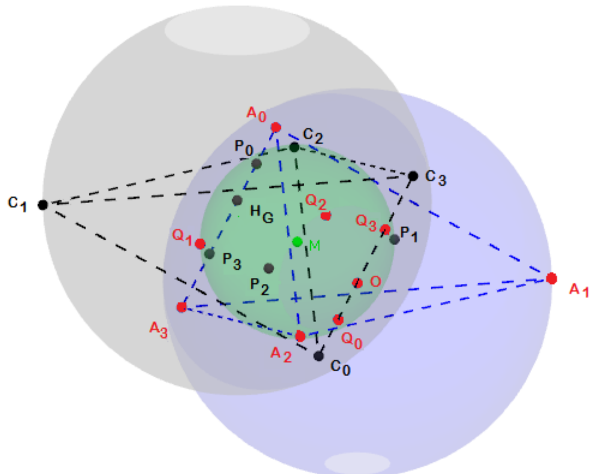

Figure 6: Circumspheres and Feuerbach spheres of a tetrahedron and its antitetrahedron

Theorem 3.2.5. *Let $A_0A_1A_2A_3$ be a tetrahedron, O its circumcenter, G its centroid, $C_0C_1C_2C_3$ the antitetrahedron associated to O, $H_{\mathcal{G}}$ its $\mathcal{G}-$orthocenter, $\mathcal{S}_1$ the circumsphere of $A_0A_1A_2A_3$, $\mathcal{S}_2$ the circumsphere of $C_0C_1C_2C_3$, S_3 and S_4 the Feuerbach spheres of first kind of the tetrahedra $A_0A_1A_2A_3$ and $C_0C_1C_2C_3$, respectively. Then*

1. *O, G and $H_{\mathcal{G}}$ are collinear, with G between O and $H_{\mathcal{G}}$ and $3|OG| = |GH_{\mathcal{G}}|$ (Euler-line property).*

2. *$S_4 = \mathscr{H}_{H_{\mathcal{G}},\frac{1}{3}}(S_1)$ and $S_3 = \mathscr{H}_{O,\frac{1}{3}}(S_2)$.*

Proof. The first assertion is immediate by the definitions of G and $H_{\mathcal{G}}$.

For the second assertion, recall that the centers of the spheres S_3 and S_4 are the points

$$Q = \frac{1}{3}\left(A_0 + A_1 + A_2 + A_3 - O\right)$$

and

$$Q_1 = \frac{1}{3}\left(C_0 + C_1 + C_2 + C_3 - H_{\mathcal{G}}\right) = \frac{2}{3}(A_0 + A_1 + A_2 + A_3) - \frac{5}{3}O,$$

respectively. It is enough to see that $\mathscr{H}_{H_{\mathcal{G}},\frac{1}{3}}(O) = Q_1$ and $\mathscr{H}_{O,\frac{1}{3}}(H_{\mathcal{G}}) = Q$. Indeed,

$$\mathscr{H}_{H_{\mathcal{G}},\frac{1}{3}}(O) = \frac{2}{3}\left(A_0 + A_1 + A_2 + A_3 - 3O\right) + \frac{1}{3}O = Q_1$$

and

$$\mathscr{H}_{O,\frac{1}{3}}(H_{\mathcal{G}}) = \frac{2}{3}O + \frac{1}{3}H_{\mathcal{G}} = \frac{1}{3}(A_0 + A_1 + A_2 + A_3 - O) = Q.$$

$\square$

The previous theorem shows that the ratio in which G divides the segment OH is equal to the ratio in which G divides every median of the tetrahedron. The points O, Q, G, and $H_{\mathcal{G}}$ do not form a harmonic quadruple, but they satisfy the following relations:

$$3\,(G-Q) = O - \frac{1}{4}(A_0 + A_1 + A_2 + A_3) = O - G$$

$$-4\,(O-G) = 4O + (A_0 + A_1 + A_2 + A_3) = H_G - O.$$

In case of a triangle $\triangle A_0A_1A_2$ we know that the homotheties $\mathscr{H}_{O,\frac{1}{2}}$ and $\mathscr{H}_{H,\frac{1}{2}}$ transform the circumcircles of the antitriangle $\triangle B_0B_1B_2$ associated to O and of the triangle $\triangle A_0A_1A_2$ into the Feuerbach circles of the triangle $\triangle A_0A_1A_2$ and the antitriangle $\triangle B_0B_1B_2$, respectively. In this case, additionally both circumcircles coincide. The previous theorem shows that the analogue for the tetrahedron is valid with the notion of Feuerbach sphere of first kind. In particular, we have

Corollary 3.2.6. *Let $A_0A_1A_2A_3$ be a tetrahedron, O its circumcenter, $H_{\mathcal{G}}$ its $\mathcal{G}-$orthocenter, $C_0C_1C_2C_3$ the antitetrahedron associated to O, $\mathcal{S}_1$ the circumsphere of $A_0A_1A_2A_3$, $\mathcal{S}_2$ the circumsphere of $C_0C_1C_2C_3$, S_3 and S_4 the Feuerbach spheres of first kind of the tetrahedra $A_0A_1A_2A_3$ and $C_0C_1C_2C_3$, respectively. Then the following statements hold.*

1. *The points Y of the segments that join O with the points X of the circumsphere $\mathcal{S}_2$ of the antitetrahedron such that $|YO| = \frac{1}{3}|XO|$ belong to S_3.*
2. *The points Z of the segments that join $H_{\mathcal{G}}$ with the points W of the circumsphere $\mathcal{S}_1$ of the tetrahedron $A_0A_1A_2A_4$ such that $|ZH_{\mathcal{G}}| = \frac{1}{3}|WH_{\mathcal{G}}|$ lie in S_4.*

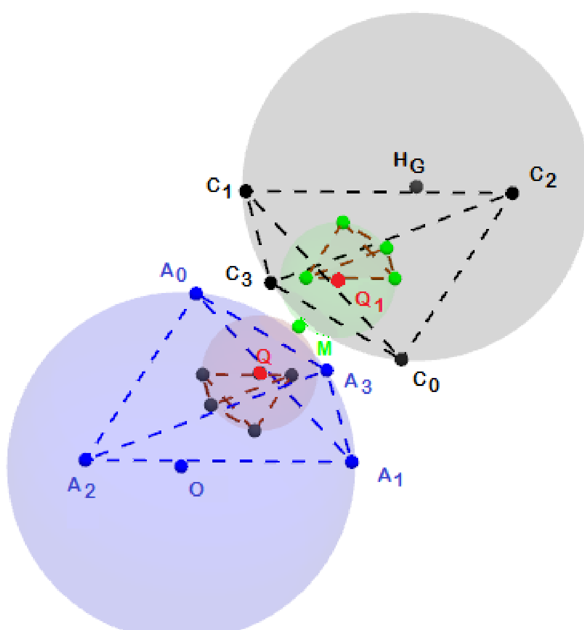

Figure 7: Circumspheres and Feuerbach spheres of first kind of a tetrahedron and its antitetrahedron

On the other hand, the circumcenter O of the tetrahedron $A_0A_1A_2A_3$ is the $\mathcal{G}-$orthocenter of the following tetrahedra: the antitetrahedron $C_0C_1C_2C_3$; the medial tetrahedron $M_1M_2M_3M_4$ consisting of the centroids of the faces of $A_0A_1A_2A_3$ (i.e., $M_i = \frac{1}{3}(A_j + A_k + A_l)$ for $\{i,j,k,l\} = \{0,1,2,3\}$); and the tetrahedron $Q_0Q_1Q_2Q_3$ formed by the centers of the Feuerbach spheres of the faces of the tetrahedron associated to O. Moreover, it is not difficult to see that for the homothety $\mathscr{H}_{P,\lambda}$ the points $P_i = \mathscr{H}_{P,\lambda}(A_i)$ for $\{i,j,k,l\} = \{0,1,2,3\}$, where $A_0A_1A_2A_3$ is a tetrahedron and $H_{\mathcal{G}}$ is its $\mathcal{G}-$orthocenter. Then $\mathscr{H}_{P,\lambda}(H_{\mathcal{G}})$ is the $\mathcal{G}-$orthocenter of the tetrahedron $P_0P_1P_2P_3$. As a consequence of all this we get that if G_i are the centroids of the tetrahedra $A_jA_kA_lH_G$ for $\{i,j,k,l\} = \{0,1,2,3\}$, then the $\mathcal{G}-$orthocenter of the tetrahedron $G_0G_1G_2G_3$ is the centroid G of the tetrahedron $A_0A_1A_2A_3$.

One should note that in our proofs we do not use orthogonality properties of $\mathbb{R}^n$. This means that if one replaces the orthogonal projection method mentioned in the introduction by suitable parallel projections in direction of the line connecting P and the circumcenter of the given triangle, then most of our statements can successfully be extended to normed spaces (using the concept of isosceles orthogonality, (see [1]). This is the subject of forthcoming investigations.

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