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Generalized q-Mittag-Leffler function and its properties

Función q-Mittag-Leffler generalizada y sus propiedades

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Abstract

Motivated essentially by the success of the applications of the Mittag-Leffler functions in Science and Engineering, we propose here a unification of certain q-extensions of generalizations of Mittag-Leffler function together with Saxena-Nishimoto's function, Bessel-Maitland function, Dotsenko function, Elliptic Function, etc. We obtain Mellin-Barnes contour integral representation, a q-difference equation, Eigen function property. As a specialization, a generalization of q-Konhauser polynomial is considered for which the series inequality relations and inverse series relations are obtained.

Key words and phrases: q-Mittag-Leffler function, q-Bessel function, q-difference equation, q-inverse series, eigen function, generalized q-Konhauser polynomial, series inequality relations.

Resumen

Motivados esencialmente por el éxito de las aplicaciones de las funciones de Mittag-Leffler en Ciencia e Ingeniería, proponemos aquí una unificación de ciertas q-extensiones de generalizaciones de la función de Mittag-Leffler incluyendo la función de Saxena-Nishimoto, la función de Bessel-Maitland, función de Dotsenko, función elíptica, etc. Obtenemos la representación integral de contorno de Mellin-Barnes, una ecuación de q-diferencia, propiedad de función Eigen. Como especialización, se considera un polinomio generalizado de q-Konhauser para el cual se obtienen las relaciones de desigualdad en serie y relaciones en serie inversa.

Palabras y frases clave: Función q-Mittag-Leffler, función q-Bessel, ecuación de q-diferencia, series q-inversas, función Eigen, polinomios q-Konhauser generalizados, relaciones de desigualdad en serie.

1 Introduction

Since the time of Wiman [18], many researchers have proposed and studied various generalizations of the Mittag-Leffler function (ML-function) [11] (also [5], [7], [12], [14], [15], [17]).

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We propose here a generalized structure of the Mittag-Leffler function which provides a q-extension to the function:

$$E_{\alpha,\beta}^{\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)n!}, \tag{1}$$

due to Shukla and Prajapati [17], where $\Re(\alpha,\beta,\gamma)>0, q\in(0,1)\cup\mathbb{N}$.
Interestingly, the proposed function ((12) and (13) below) also enables us to define and include the q-analogues of

(i) Bessel-Maitland function [6, Eq.(1.7.8), p.19] :

$$J_\nu^\mu(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\nu+n\mu+1)\ n!},$$

(ii) Dotsenko function [6, Eq.(1.8.9), p.24] :

$${}_2R_1(a,b;c,\omega;\mu;z)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n\frac{\omega}{\mu})}{\Gamma(c+n\frac{\omega}{\mu})}\frac{z^n}{n!},$$

(iii) A particular form $(m=2)$ of extension of Mittag-Leffler function:

$$E_{\gamma,K}[(\alpha_j,\beta_j)_{1,2};z]=\sum_{n=0}^{\infty}\frac{(\gamma)_{Kn}}{\Gamma(\alpha_1 n+\beta_1)\Gamma(\alpha_2 n+\beta_2)n!}z^n,$$

due to Saxena and Nishimoto [16], where

$$z,\gamma,\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{C},\Re(\alpha_1+\alpha_2)>\Re(K)-1,\Re(K)>0,$$

(iv) The Elliptic function [9, Eq.(1), p.211] :

$$K(k)=\frac{\pi}{2}\ {}_2F_1\left(\begin{matrix}\frac{1}{2}, & \frac{1}{2}; & k^2\\ 1; & & \end{matrix}\right).$$

The following definitions and formulas will be used in this work. For $a\in\mathbb{C}$, and $0<|q|<1$, the q-shifted factorial is defined by [4, Eq.(1.2.15), p.3 and Eq.(1.2.30), p.6]

$$(a;q)_n=\begin{cases}1 & if\quad n=0\\ (1-a)(1-aq)...(1-aq^{n-1}) & if\quad n\in\mathbb{N}.\end{cases} \tag{2}$$

For any n,

$$(a;q)_n=\frac{(q;q)_\infty}{(aq^n;q)_\infty},$$

where

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k)\ ,\quad |q|<1.$$

A q-binomial coefficient is (cf. [4, Ex.(1.2), p.20] with r=1):

$$\begin{bmatrix} n \\ m \end{bmatrix}_r = \frac{(q^r;q^r)_n}{(q^r;q^r)_{n-m}\ (q^r;q^r)_m}, r \neq 0. \tag{3}$$

A q-Gamma function is defined as [4, Eq.(1.10.1), p.16]:

$$\Gamma_q(\alpha) \ = \ \frac{(q;q)_\infty\ (1-q)^{1-\alpha}}{(q^\alpha;q)_\infty}, \tag{4}$$

where $\alpha \neq 0, -1, -2, ...$ and $0 < q < 1$.

A q-Stirling's asymptotic formula [10, Eq.(2.25), p.482] is given by

$$\Gamma_q(x) \sim (1+q)^{\frac{1}{2}}\ \Gamma_{q^2}\left(\frac{1}{2}\right)\ (1-q)^{\frac{1}{2}-x}\ e^{\mu_q(x)}, \tag{5}$$

where $\mu_q(x) = \dfrac{\theta\ q^x}{1-q-q^x}$, $0 < \theta < 1$.

Theorem 1.1. *If* $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ *is an entire function then the order* $\varrho(f)$ *of* f *is given by [2, Eq.(1.2)]*

$$\varrho(f) = \lim_{n\to\infty} \sup \frac{n\ \log n}{\log(1/|a_n|)}. \tag{6}$$

and the type of the function σ *is given by [8]*

$$e\varrho\sigma = \lim_{n\to\infty} \sup \left(n\,|a_n|^{\varrho/n}\right). \tag{7}$$

For every positive ϵ, the asymptotic estimate [8, Eq.(16)]

$$|f(z)| < exp\left((\sigma+\epsilon)\ |z|^\varrho\right),\ |z| \geq r_0 > 0 \tag{8}$$

holds with ϱ, σ as in (6), (7) for $|z| \geq r_0(\epsilon),\ r_0(\epsilon)$ sufficiently large.

The two q-exponential functions are defined as [4, Eq.(II.1), p.236]

$$e_q(x) \ = \ \sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty},\ |x| < 1 \tag{9}$$

and [4, Eq.(II.2), p.236]

$$E_q(x) \ = \ \sum_{n=0}^{\infty} q^{n(n-1)/2} \frac{x^n}{(q;q)_n} = (-x;q)_\infty,\ |x| < \infty. \tag{10}$$

The q-derivative of a function $f(x)$ is defined by [4, Ex.1.12, p.22]

$$D_q f(x) = \frac{f(x)-f(xq)}{x(1-q)}. \tag{11}$$

In view of two q-analogues of exponential function, we define q-generalized Mittag-Leffler functions in the forms:

Definition 1.1. If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with $\Re(\alpha, \beta, \gamma, \lambda) > 0,\ r \in \{-1, 0\} \cup \mathbb{N},\ \delta, \mu > 0,\ s \in \mathbb{N} \cup \{0\}$ then

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma + \delta n)]^s}{\Gamma_q(\beta + \alpha n)\ [\Gamma_q(\lambda + \mu n)]^r\ (q;q)_n} z^n, \tag{12}$$

where $p = \alpha^2 + r\mu^2 - s\delta^2 + 1$ with $\Re(p) > 0$.

Definition 1.2. If $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with $\Re(\alpha, \beta, \gamma, \lambda) > 0,\ r \in \{-1, 0\} \cup \mathbb{N}, \delta, \mu > 0,\ s \in \mathbb{N} \cup \{0\}$ and $\alpha^2 + r\mu^2 + 1 = s\delta^2$ then

$$e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q) \quad = \quad \sum_{n=0}^{\infty} \frac{[\Gamma_q(\gamma + \delta n)]^s}{\Gamma_q(\beta + \alpha n)\ [\Gamma_q(\lambda + \mu n)]^r\ (q;q)_n} z^n. \tag{13}$$

Alternatively, in view of (4) these q-forms can also be put in the form:

$$\begin{aligned} E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q) \quad &= \quad \sum_{n=0}^{\infty} (-1)^{pn}\ q^{pn(n-1)/2}\ \frac{(q^{\alpha n+\beta}; q)_\infty\ [(q^{\lambda+\mu n}; q)_\infty]^r}{[(q^{\gamma+\delta n}; q)_\infty]^s} \\ &\quad \times \frac{z^n}{(q;q)_n}, \end{aligned} \tag{14}$$

and

$$e^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q) \quad = \quad \sum_{n=0}^{\infty} \frac{(q^{\alpha n+\beta}; q)_\infty\ [(q^{\lambda+\mu n}; q)_\infty]^r}{[(q^{\gamma+\delta n}; q)_\infty]^s\ (q;q)_n} z^n. \tag{15}$$

We shall refer to these functions as q-*gml*.

The objective of constructing this function is to:

(i) Include certain existing generalizations of Mittag-Leffler function.

(ii) Include Bessel-Maitland function, Dotsenko function, Saxena-Nishimoto function, Elliptic function.

(iii) Obtain inverse inequality relations and some other inequalities by means of the parameter "s".

The q-analogues of the above stated Shukla and Prajapati's function (1) and those functions listed above from (i) through (iv) are all yielded by the q-gml (12) or (13). They are tabulated below (see Table 1) together with the indicated substitutions.

The explicit forms of the functions mentioned in thi Table 1 are as stated below.

- q-Mittag-Leffler function:

$$E_\alpha(z|q) = \sum_{n=0}^{\infty} \left[(-1)^n\ q^{n(n-1)/2}\right]^{\alpha^2}\ (q^{\alpha n+1}; q)_\infty\ z^n.$$

- q-Analogue of Wiman's function:

$$E_{\alpha,\beta}(z|q) = \sum_{n=0}^{\infty} \left[(-1)^n\ q^{n(n-1)/2}\right]^{\alpha^2}\ (q^{\alpha n+\beta}; q)_\infty\ z^n.$$

q-Function of	**r**	**s**	$\boldsymbol{\alpha}$	$\boldsymbol{\beta}$	$\boldsymbol{\gamma}$	$\boldsymbol{\delta}$	$\boldsymbol{\lambda}$	$\boldsymbol{\mu}$	**Particular case of**
Mittag-Leffler	0	1	α	1	1	1	-	-	(12)
Wiman	0	1	α	β	1	1	-	-	(12)
Prabhakar	0	1	α	β	γ	1	-	-	(12)
Shukla and Prajapati	0	1	α	β	γ	q	-	-	(12)
Bessel-Maitland	0	0	μ	$\nu+1$	-	-	-	-	(12)
Dotsenko	-1	1	ω/ν	c	a	1	b	ω/ν	(13)
Saxena-Nishimoto	1	1	α_1	β_1	γ	K	β_2	α_2	(12)
Elliptic	-1	1	1	1	$\frac{1}{2}$	1	$\frac{1}{2}$	1	(13)

Table 1: q-Functions

- q-Analogue of Prabhakar's generalized ML-function:

$$E^{\gamma}_{\alpha,\beta}(z|q) = \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{\alpha^2} \; (q^{\alpha n+\beta};q)_\infty}{(q^{\gamma+n};q)_\infty \; (q;q)_n} z^n.$$

- q-ML-function of Shukla and Prajapati (q is replaced by δ):

$$E^{\gamma,\delta}_{\alpha,\beta}(z|q) = \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{(\alpha^2-\delta^2+1)} \; (q^{\alpha n+\beta};q)_\infty}{(q^{\gamma+\delta n};q)_\infty \; (q;q)_n} z^n.$$

- q-Bessel-Maitland function:

$$J^{\mu}_{\nu}(-z;q) = \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{(\mu^2+1)} \; (q^{\mu n+\nu+1};q)_\infty}{(q;q)_n} z^n.$$

(Later on, this will be referred to this as q-BMF)

- q-Dotsenko function:

$${}_2R_1(a,b;c,\omega;\nu;z;q) = \sum_{n=0}^{\infty} \frac{(q^{c+\frac{\omega}{\nu}n};q)_\infty}{(q^{b+\frac{\omega}{\nu}n};q)_\infty \; (q^{n+a};q)_\infty \; (q;q)_n} z^n.$$

- q-Form (of the particular case $m=2$) of the function due to Saxena and Nishimoto:

$$\begin{aligned} E_{\gamma,K}[(\alpha_j,\beta_j)_{1,2};z|q] &= \sum_{n=0}^{\infty} \frac{\left[(-1)^n \; q^{n(n-1)/2}\right]^{(\alpha_1^2+\alpha_2^2-K^2+1)}}{(q^{\gamma+Kn};q)_\infty \; (q;q)_n} \\ &\quad \times (q^{\alpha_1 n+\beta_1};q)_\infty \; (q^{\alpha_2 n+\beta_2};q)_\infty \; z^n. \end{aligned}$$

(Later on, this will be referred to as q-SNF)

- q-Elliptic function:

$$K(\sqrt{z}|q) = \frac{\pi}{2}\ {}_2\phi_1\left(\begin{array}{ccc} \frac{1}{2}, & \frac{1}{2}; & z \\ 1; & & \end{array}\right).$$

We first show the convergence of series in (12) and (13); this is followed by Mellin-Barnes integral representation, difference equation and eigen function property. As a special case of (12), a q-extension of the Konhauser polynomial is illustrated and, associated inequalities are established.

2 Main Results

In this section, we prove the following results.

2.1 Convergence

Theorem 2.1.1. *Let $0 < q < 1$, $\Re(\alpha,\beta,\gamma,\lambda) > 0$, $\Re(\alpha^2) + r\mu^2 - s\delta^2 + 1 > 0$, $\delta,\mu > 0$, $r \in \{-1,0\} \cup \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$ and $\alpha,\beta,\gamma,\lambda \in \mathbb{C}$. Then $E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q)$ is an entire function of order zero.*

Proof. Put

$$V_n = \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)} \tag{16}$$

to get

$$E^{\gamma,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q) = \sum_{n=0}^{\infty} V_n\ z^n.$$

Then in view of (5), we get after some simplification,

$$\begin{aligned} V_n \quad \sim \quad & \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ (1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{s-r-2}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{-s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}} \\ & \times e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}. \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt[n]{|V_n|} \quad \sim \quad & \left|\frac{(1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{(s-r-2)}\ (1-q)^{s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}}\right|^{\frac{1}{n}} \\ & \times \left|e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}\right|^{\frac{1}{n}} \\ & \times \left|(-1)^p\ q^{p(n-1)/2}\right|. \end{aligned}$$

Making limit $n \to \infty$, this gives

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|V_n|} \sim \left|(1-q)^{\alpha+r\mu-s\delta+1}\right| \lim_{n\to\infty} \left|q^{p(n-1)/2}\right| = 0$$

when $\Re(\alpha^2)+r\mu^2-s\delta^2+1>0$. Thus, the function (12) is an *entire* function. Its order may be determined by using Theorem 1.1. In fact, by choosing $f(z)=E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)$ and $u_n=V_n$, Theorem 1.1 gets particularized to

$$\varrho(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q))=\lim_{n\to\infty}\sup\frac{n\ \log n}{\log(1/|V_n|)},$$

where

$$\begin{aligned}\log\left(\frac{1}{|V_n|}\right) &= \log\left(\left|\frac{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)}{q^{n(n-1)(\alpha^2+r\mu^2-s\delta^2+1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}\right|\right)\\ &= \log|\Gamma_q(\alpha n+\beta)+r\log|\Gamma_q(\lambda+\mu n)|\\ &\quad+\log|\Gamma_q(n+1)|-\frac{1}{2}n(n-1)[\Re(\alpha^2+r\mu^2-s\delta^2+1)]\log q\\ &\quad-s\log|\Gamma_q(\gamma+\delta n)|.\end{aligned}\tag{17}$$

From the definition (4) of q-Gamma function, one finds

$$\begin{aligned}\log|\Gamma_q(\alpha n+\beta)| &= \log\left|\frac{(q;q)_\infty}{(q^{\alpha n+\beta};q)_\infty}(1-q)^{1-\alpha n-\beta}\right|\\ &= \log\left|\frac{(q;q)_\infty}{(q^{\alpha n+\beta};q)_\infty}\right|(1-q)^{1-n\Re(\alpha)-\Re(\beta)}\\ &= \log|(q;q)_\infty|+(1-n\Re(\alpha)-\Re(\beta))\log(1-q)\\ &\quad-\log\left|(q^{\alpha n+\beta};q)_\infty\right|;\end{aligned}\tag{18}$$

in which

$$\begin{aligned}\log\left|(q^{\alpha n+\beta};q)_\infty\right| &= \log\left(\prod_{k=0}^{\infty}\left|1-q^{\alpha n+\beta+k}\right|\right)\\ &= \log\left(\lim_{m\to\infty}\prod_{k=0}^{m}\left|1-q^{\alpha n+\beta+k}\right|\right)\\ &= \lim_{m\to\infty}\sum_{k=0}^{m}\log\left|1-q^{\alpha n+\beta+k}\right|\\ &= \sum_{k=0}^{\infty}\log\left|1-q^{\alpha n+\beta+k}\right|.\end{aligned}$$

Here it may be noted that [2, p.207]

$$\log\left|1-q^{\alpha n+\beta+k}\right|\le\log(1+\left|q^{\alpha n+\beta+k}\right|)\le\left|q^{\alpha n+\beta+k}\right|=q^{n\Re(\alpha+\beta)+k}$$

which leads us to

$$\sum_{k=0}^{\infty}\log\left|1-q^{\alpha n+\beta+k}\right|\le\sum_{k=0}^{\infty}q^{n\Re(\alpha+\beta)+k}=\frac{q^{n\Re(\alpha+\beta)}}{1-q}.$$

This implies that

$$\lim_{n\to\infty} \frac{\log\left|(q^{\alpha n+\beta};q)_\infty\right|}{n\ \log n} = 0.$$

Consequently from (18), it follows that

$$\lim_{n\to\infty} \frac{\log\left|\Gamma_q(\alpha n+\beta)\right|}{n\ \log n} = 0.$$

This last limit and the trivial limit

$$\lim_{n\to\infty} \frac{n-1}{\log n} = \infty$$

when used in (17), yields

$$\lim_{n\to\infty} \frac{\log(1/\left|V_n\right|)}{n \log n} = \infty.$$

Thus,

$$\varrho(E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q)) = 0.$$

□

Theorem 2.1.2. *The function* $e^{\gamma,\delta}_{\alpha,\ \beta,\ \lambda,\ \mu}(z;s,r|q)$ *represents absolutely convergent series for* $|z| < \left|(1-q)^{(s\delta-\alpha-r\mu-1)}\right|$ *and* $|q|<1$.

Proof. Take

$$U_n = \frac{[\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)} \tag{19}$$

then

$$e^{\gamma,\delta}_{\alpha,\ \beta,\ \lambda,\ \mu}(z;s,r|q) = \sum_{n=0}^{\infty} U_n\ z^n.$$

Now in view of the q- analogue of Stirling's asymptotic formula (5), we get

$$\begin{aligned} U_n \quad \sim \quad & \frac{(1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{s-r-2}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{-s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}} \\ & \times e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}. \end{aligned}$$

This gives

$$\begin{aligned} \sqrt[n]{|U_n|} \quad \sim \quad & \left|\frac{(1+q)^{\frac{1}{2}(s-r-2)}\ (\Gamma_{q^2}(\frac{1}{2}))^{(s-r-2)}\ (1-q)^{s(\frac{1}{2}-\gamma-\delta n)}\ (1-q)^{n+\frac{1}{2}}}{(1-q)^{\frac{1}{2}-\beta-\alpha n}\ (1-q)^{r(\frac{1}{2}-\lambda-\mu n)}}\right|^{\frac{1}{n}} \\ & \times \left| e^{\frac{\theta q^{\gamma+\delta n}}{1-q-q^{\gamma+\delta n}}}\ e^{-\frac{\theta q^{\beta+\alpha n}}{1-q-q^{\beta+\alpha n}}}\ e^{-\frac{\theta q^{\lambda+\mu n}}{1-q-q^{\lambda+\mu n}}}\ e^{-\frac{\theta q^{1+n}}{1-q-q^{1+n}}}\right|^{\frac{1}{n}} \end{aligned}$$

whence

$$\frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|U_n|} \sim \left|(1-q)^{\alpha+r\mu-s\delta+1}\right|.$$

Thus, the series in (13) converges absolutely if $|z| < R = (1-q)^{s\delta-\Re(\alpha)-r\mu-1}$. □

2.2 Contour integral

Theorem 2.2.1. *Let $\alpha > 0; \beta, \gamma, \lambda \in \mathbb{C}$ with $\Re(\beta,\gamma,\lambda) > 0$ and $\delta,\ \mu > 0$. Then the function $E^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q)$ is expressible as the Mellin - Barnes q-integral given by*

$$E^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z;s,r|q) = \frac{1}{2\pi i}\int_L \frac{(-1)^{-pS}\ q^{-pS(-S-1)/2}\ \Gamma_q(S)\ [\Gamma_q(\gamma-\delta S)]^s}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r} \times (-z)^{-S}\ \mathrm{d}_q S, \quad (20)$$

where $|argz| < \pi$. The contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles of integrand at $S = -n$ to the left; and the poles at $S = (\gamma+n)/\delta$ to the right of the path for all $n \in \mathbb{N}\cup\{0\}$.

Proof. The integral on the right hand side of (20) may be evaluated as the sum of the residues at the poles $S = 0, -1, -2, \ldots$. In fact, in view of the definition of residue,

$$\begin{aligned}
I &= \frac{1}{2\pi i}\int_L \frac{(-1)^{-pS}\ q^{-pS(-S-1)/2}\ \Gamma_q(S)\ [\Gamma_q(\gamma-\delta S)]^s\ (-z)^{-S}}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r}\ \mathrm{d}_q S \\
&= \sum_{n=0}^{\infty} \underset{S=-n}{Res} \left[\frac{(-1)^{-pS}\ q^{-pS(-S-1)/2}\ \Gamma_q(S)\ (-z)^{-S}}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r\ [\Gamma_q(\gamma-\delta S)]^{-s}}\right] \\
&= \sum_{n=0}^{\infty} \lim_{S\to -n} \frac{\pi(S+n)}{sin\pi S} \frac{(-1)^{-pS}\ q^{-pS(-S-1)/2}\ [\Gamma_q(\gamma-\delta S)]^s\ (-z)^{-S}}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r\ \Gamma_q(1-S)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [\Gamma_q(\gamma+\delta n)]^s}{\Gamma_q(\beta+\alpha n)\ [\Gamma_q(\lambda+\mu n)]^r\ \Gamma_q(n+1)} z^n \\
&= E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q).
\end{aligned}$$

□

By dropping the factor $q^{N(N-1)/2}$ in this proof, we get

Theorem 2.2.2. *Let $\alpha \in \mathbb{R}_+; \beta,\ \gamma,\ \lambda \in \mathbb{C}$, with $\Re(\beta,\gamma,\lambda) > 0$ and $\delta,\ \mu > 0$. Then the function $e^{\gamma,\,\delta}_{\alpha,\,\beta,\,\lambda,\,\mu}(z; s, r|q)$ is expressible as the Mellin - Barnes q-integral given by*

$$e^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z;s,r|q) = \frac{1}{2\pi i}\int_L \frac{\Gamma_q(S)\ [\Gamma_q(\gamma-\delta S)]^s\ (-z)^{-S}}{\Gamma_q(\beta-\alpha S)\ [\Gamma_q(\lambda-\mu S)]^r}\ \mathrm{d}_q S, \quad (21)$$

where $|argz| < \pi$; the contour L of integration begins from $-i\infty$ and proceeds towards $+i\infty$, and is indented to keep the poles of integrand at $S = -n$ to the left; and the poles at $S = (\gamma+n)/\delta$ to the right of the path, for all $n \in \mathbb{N}\cup\{0\}$.

2.3 Difference equation

With the aid of the following operators, the difference equations of both the q-analogues will be derived. Put

$$\Lambda_q f(x) = f(x) - f(xq^{-1}), \quad \Theta f(x) = f(x) - f(xq), \tag{22}$$

$$\mathcal{D}_q\ f(x) = (1-q)\ D_q f(x) := (1-q)\ \frac{f(x)-f(xq)}{x-xq} = \frac{f(x)-f(xq)}{x}, \tag{23}$$

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{1-(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}} = \Phi_{u,v}^{(a,b,c;m)} \tag{24}$$

and

$$\frac{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[\Theta + c^{-u}q^{(b+v)/a} - 1]^m\right\}}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{-(b+v)/a}]^m\right\}} = \Psi_{u,v}^{(a,b,c;m)}. \tag{25}$$

In these notations, the q-difference equation satisfied by (12) is derived in the following theorem.

Theorem 2.3.1. *Let $\alpha, \mu, \delta \in \mathbb{N}$, then $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$ satisfies the equation*

$$\begin{aligned}&\left[\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta\right] E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)\\ &-\left[(-1)^p\ z\ \Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\right] E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(zq^p; s, r|q) = 0,\end{aligned} \tag{26}$$

in which ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

Proof. In the first place, the coefficient of z^n in the series representation of $E^{\gamma,\delta}_{\alpha,\beta,\lambda,\mu}(z; s, r|q)$ will be expressed in q-factorial notation with the help of the set of formulas [4, Appendix I]:

$$(a;q)_{kn} = (a, aq, \ldots, aq^{k-1}; q^k)_n,$$

$$(a^k;q^k)_n = (a, a\omega_k, \ldots, a\omega_k^{k-1}; q^k)_n \text{ in which } \omega_k = e^{(2\pi i)/k},$$

$$(A;q^n)_{\nu k} = (A^{1/n};q)_{\nu k}\ (A^{1/n}\omega;q)_{\nu k} \ldots (A^{1/n}\omega^{n-1};q)_{\nu k}, \text{ where } \omega^n = 1,$$

and

$$(q^\gamma;q^\delta)_n = (q^{\gamma/\delta};q)_n\ (\varpi q^{\gamma/\delta};q)_n \ldots (\varpi^{\delta-1}q^{\gamma/\delta};q)_n = \prod_{i=0}^{\delta-1}(\varpi^i q^{\gamma/\delta};q)_n,$$

where $\varpi^\delta = 1$. Then following the notation used in (16) for the coefficient of z^n, we get

$$V_n \quad = \quad \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^\gamma;q)_{\delta n}]^s}{[(q^\lambda;q)_{\mu n}]^r\ (q^\beta;q)_{\alpha n}\ (q;q)_n}$$

$$\begin{aligned}
&= \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\gamma};q)_{\delta n}]^s}{[(q^{\lambda};q)_{\mu n}]^r\ (q^{\beta};q)_{\alpha n}\ (q;q)_n} \\
&= \frac{(-1)^{pn}\ q^{pn(n-1)/2}\ [(q^{\gamma};q^{\delta})_n]^s\ [(q^{\gamma+1};q^{\delta})_n]^s\cdots[(q^{\gamma+\delta-1};q^{\delta})_n]^s}{[(q^{\lambda};q^{\mu})_n]^r\ [(q^{\lambda+1};q^{\mu})_n]^r\cdots[(q^{\lambda+\mu-1};q^{\mu})_n]^r} \\
&\quad\times\frac{1}{(q^{\beta};q^{\alpha})_n\ (q^{\beta+1};q^{\alpha})_n\cdots(q^{\beta+\alpha-1};q^{\alpha})_n(q;q)_n} \\
&= \frac{(-1)^{pn}\ q^{pn(n-1)/2}}{(q;q)_n}\ \left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}^{-1} \\
&\quad\times\left\{\prod_{j=0}^{\delta-1}\ \prod_{i=0}^{\delta-1}[(\zeta^{j}\ q^{(\gamma+i)/\delta};q)_n]^s\right\}\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{h}\ q^{(\beta+m)/\alpha};q)_n\right\}^{-1} \qquad (27)
\end{aligned}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity. Now take

$$\prod_{j=0}^{\delta-1}\ \prod_{i=0}^{\delta-1}[(\zeta^{j}\ q^{(\gamma+i)/\delta};q)_n]^s=\mathcal{A}_n,\ \prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r=\mathcal{B}_n, \qquad (28)$$

and

$$\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{h}\ q^{(\beta+m)/\alpha};q)_n=\mathcal{C}_n,\ (-1)^{pn}\ q^{pn(n-1)/2}=D_n \qquad (29)$$

then

$$\sum_{n=0}^{\infty}V_n\ z^n=\sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^n=W,\quad \text{say.}$$

Since the series in (12) converges, we have

$$\Theta W=\sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ D_n\ \Theta z^n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}=\sum_{n=0}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_n}\frac{1-q^n}{(q;q)_n}z^n=\sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_n}\frac{z^n}{(q;q)_{n-1}}.$$

Next operating by $\Phi_{h,m}^{(\alpha,\beta,\sigma;1)}$, we get

$$\begin{aligned}
\Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\Theta\ W &= \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ (q;q)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\Theta+\sigma^{-h}q^{1-(\beta+m)/\alpha}-1)\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{-h}q^{1-(\beta+m)/\alpha})\right\}} \\
&\quad\times\frac{z^n}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{h}q^{(\beta+m)/\alpha};q)_n\right\}} \\
&= \sum_{n=1}^{\infty}\frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ (q;q)_{n-1}}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(1-\sigma^{h}q^{n-1+(\beta+m)/\alpha})\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{h}q^{(\beta+m)/\alpha};q)_n\right\}}z^n
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{\mathcal{A}_n\ D_n}{\mathcal{B}_n\ \mathcal{C}_{n-1}\ (q;q)_{n-1}} z^n.$$

Finally,

$$\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta W$$
$$= \sum_{n=1}^{\infty} \frac{\mathcal{A}_n\ D_n}{\mathcal{C}_{n-1}\ (q;q)_{n-1}} \frac{1}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}$$
$$\times \frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\Theta+\eta^{-\ell}q^{1-(\lambda+k)/\mu}-1)]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}}\ z^n$$
$$= \sum_{n=1}^{\infty} \frac{\mathcal{A}_n\ D_n}{\mathcal{C}_{n-1}\ (q;q)_{n-1}} \frac{1}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}$$
$$\times \frac{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(-q^n+\eta^{-\ell}q^{1-(\lambda+k)/\mu})]^r\right\}}{\left\{\prod_{\ell=0}^{\mu-1}\prod_{k=0}^{\mu-1}[(\eta^{\ell}\ q^{(\lambda+k)/\mu};q)_n]^r\right\}}\ z^n$$
$$= \sum_{n=1}^{\infty} \frac{\mathcal{A}_n\ D_n}{\mathcal{B}_{n-1}\ \mathcal{C}_{n-1}\ (q;q)_{n-1}} z^n.$$

Thus,

$$\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta W = \sum_{n=0}^{\infty} \frac{\mathcal{A}_{n+1}\ D_{n+1}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n} z^{n+1}. \tag{30}$$

On the other hand,

$$\Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\ E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zq^p;s,r|q)$$
$$= \sum_{n=0}^{\infty} \frac{\mathcal{A}_n\ D_n\ q^{pn}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n} \frac{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\Theta+\zeta^{-j}q^{-(\gamma+i)/\delta}-1)]^s\right\}}{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^{-j}q^{-(\gamma+i)/\delta})]^s\right\}} z^n$$
$$= \sum_{n=0}^{\infty} \frac{D_n\ q^{pn}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n} \left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^{j}\ q^{(\gamma+i)/\delta};q)_n]^s\right\}$$
$$\times \left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(1-\zeta^{j}q^{n+(\gamma+i)/\delta})]^s\right\} z^n,$$

that is,

$$z\ (-1)^p\ \Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\quad E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(zq^p;s,r|q)=\sum_{n=0}^{\infty}\frac{\mathcal{A}_{n+1}\ D_{n+1}}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^{n+1}. \tag{31}$$

On comparing (30) and (31), the equation (26) is obtained. □

The q-difference equation satisfied by the function (13) is given in following theorem whose proof follows line-to-line just dropping the factor $q^{n(n-1)/2}$ that is, dropping D_n in (29).

Theorem 2.3.2. *Let $\alpha,\mu,\delta\in\mathbb{N}$, then $Y=e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ satisfies the equation*

$$\left[\Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ \Theta - z\ \Psi_{j,i}^{(\delta,\gamma,\zeta;s)}\right]Y=0, \tag{32}$$

where ζ is δ^{th} root of unity, η is μ^{th} root of unity, σ is α^{th} root of unity.

2.4 Eigen function property

Take

$$\frac{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[(\Lambda_q+c^{-u}q^{1-(b+v)/a}-1)]^m}{\left\{\prod_{u=0}^{a-1}\prod_{v=0}^{a-1}[c^{-u}q^{1-(b+v)/a}]^m\right\}}=\Omega_{u,v}^{(a,b,c;m)}, \tag{33}$$

and

$$\Delta_q=\mathcal{D}_q\ \Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}\ \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}. \tag{34}$$

Here the operators $\Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}, \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}, \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}$ in (34) are not commutative with the operator $\mathcal{D}_q$. This property does not hold for the function $E_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$, but it is established for the function $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ in

Theorem 2.4.1. *Let $\alpha,\mu,\delta\in\mathbb{N}$ and the q-difference operator Θ be defined by (22), then $e_{\alpha,\beta,\lambda,\mu}^{\gamma,\delta}(z;s,r|q)$ is an eigen function with respect to the operator Δ_q defined by (34). That is, for any non zero c,*

$$\Delta_q\ e_{\alpha,\ \beta,\ \lambda,\ \mu}^{\gamma,\delta}(cz;s,r|q)=c\ e_{\alpha,\ \beta,\ \lambda,\ \mu}^{\gamma,\delta}(cz;s,r|q). \tag{35}$$

Proof. With $\mathcal{A}_n$, $\mathcal{B}_n$ and $\mathcal{C}_n$ as in (28) and in (29),

$$e_{\alpha,\ \beta,\ \lambda,\ \mu}^{\gamma,\delta}(cz;s,r|q)\quad=\quad\sum_{n=0}^{\infty}\frac{c^n\ \mathcal{A}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^n.$$

Now if $e_{\alpha,\ \beta,\ \lambda,\ \mu}^{\gamma,\delta}(cz;s,r|q)=Y_c$ then in the notation (24),

$$\Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ Y_c\quad=\quad\sum_{n=0}^{\infty}\frac{c^n\ \mathcal{A}_n}{\mathcal{B}_n\ (q;q)_n}\frac{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\Theta+\sigma^{-h}q^{1-(\beta+m)/\alpha}-1)\right\}}{\left\{\prod_{h=0}^{\alpha-1}\prod_{m=0}^{\alpha-1}(\sigma^{-h}q^{1-(\beta+m)/\alpha})\right\}}$$

$$\times \frac{z^n}{\left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\sigma^h q^{(\beta+m)/\alpha}; q)_n \right\}}$$

$$= \sum_{n=0}^{\infty} \frac{c^n \mathcal{A}_n}{\mathcal{B}_n \ (q;q)_n} \frac{\left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (1-\sigma^h q^{n-1+(\beta+m)/\alpha}) \right\}}{\left\{ \prod_{h=0}^{\alpha-1} \prod_{m=0}^{\alpha-1} (\sigma^h q^{(\beta+m)/\alpha}; q)_n \right\}} z^n$$

$$= \sum_{n=0}^{\infty} \frac{c^n \mathcal{A}_n}{\mathcal{B}_n \ \mathcal{C}_{n-1} \ (q;q)_n} z^n.$$

Next

$$\begin{aligned} \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)} \ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)} \ Y_c &= \sum_{n=0}^{\infty} \frac{c^n \mathcal{A}_n}{\mathcal{C}_{n-1} \ (q;q)_n} \frac{1}{\left\{ \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^\ell \ q^{(\lambda+k)/\mu}; q)_n]^r \right\}} \\ &\quad \times \frac{\left\{ \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\Theta + \eta^{-\ell} q^{1-(\lambda+k)/\mu} - 1)]^r \right\}}{\left\{ \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^{-\ell} q^{1-(\lambda+k)/\mu})]^r \right\}} \ z^n \\ &= \sum_{n=0}^{\infty} \frac{c^n \mathcal{A}_n}{\mathcal{C}_{n-1} \ (q;q)_n} \frac{1}{\left\{ \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^{-\ell} q^{1-(\lambda+k)/\mu})]^r \right\}} \\ &\quad \times \frac{\left\{ \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(-q^n + \eta^{-\ell} q^{1-(\lambda+k)/\mu})]^r \right\}}{\left\{ \prod_{\ell=0}^{\mu-1} \prod_{k=0}^{\mu-1} [(\eta^\ell \ q^{(\lambda+k)/\mu}; q)_n]^r \right\}} \ z^n \\ &= \sum_{n=0}^{\infty} \frac{c^n \mathcal{A}_n}{\mathcal{B}_{n-1} \ \mathcal{C}_{n-1} \ (q;q)_n} z^n. \end{aligned}$$

Further using (33),

$$\begin{aligned} \Omega_{j,i}^{(\delta,\gamma,\zeta;-s)} \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)} \Phi_{h,m}^{(\alpha,\beta,\sigma;1)} Y_c &= \sum_{n=0}^{\infty} \frac{c^n}{\mathcal{B}_{n-1} \ \mathcal{C}_{n-1} \ (q;q)_n} \\ &\quad \times \frac{\left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\zeta^{-j} q^{1-(\gamma+i)/\delta})]^s \right\}}{\left\{ \prod_{j=0}^{\delta-1} \prod_{i=0}^{\delta-1} [(\Delta_q + \zeta^{-j} q^{1-(\gamma+i)/\delta} - 1)]^s \right\}} \end{aligned}$$

$$
\begin{aligned}
&\times\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\ q^{(\gamma+i)/\delta};q)_n]^s\right\}\\
&= \sum_{n=0}^{\infty}\frac{c^n}{\mathcal{B}_{n-1}\ \mathcal{C}_{n-1}\ (q;q)_n}\\
&\times\frac{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^{-j}q^{1-(\gamma+i)/\delta})]^s\right\}}{\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(-q^n+\zeta^{-j}q^{1-(\gamma+i)/\delta})]^s\right\}}\\
&\times\left\{\prod_{j=0}^{\delta-1}\prod_{i=0}^{\delta-1}[(\zeta^j\ q^{(\gamma+i)/\delta};q)_n]^s\right\} z^n\\
&= \sum_{n=0}^{\infty}\frac{c^n\ \mathcal{A}_{n-1}}{\mathcal{B}_{n-1}\ \mathcal{C}_{n-1}\ (q;q)_n}z^n.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\Delta_q\ Y_c &= \mathcal{D}_q\ \Omega_{j,i}^{(\delta,\gamma,\zeta;-s)}\ \Phi_{\ell,k}^{(\mu,\lambda,\eta;r)}\ \Phi_{h,m}^{(\alpha,\beta,\sigma;1)}\ Y_c\\
&= \sum_{n=1}^{\infty}\frac{c^n\ \mathcal{A}_{n-1}\ z^{n-1}}{\mathcal{B}_{n-1}\ \mathcal{C}_{n-1}(q;q)_{n-1}}\\
&= \sum_{n=0}^{\infty}\frac{c^{n+1}\ \mathcal{A}_n}{\mathcal{B}_n\ \mathcal{C}_n\ (q;q)_n}z^n\\
&= c\ e_{\alpha,\ \beta,\ \lambda,\ \mu}^{\gamma,\delta}(cz;s,r|q).
\end{aligned}
$$

□

2.5 Generalized q-Konhauser polynomial

The well known q-Konhauser polynomial [1]

$$
Z_m^{\beta}(x;k|q)=\frac{[q^{\beta+1}]_{km}}{(q^k;q^k)_m}\sum_{n=0}^{m}\frac{q^{kn(kn-1)/2+kn(m+\beta+1)}(q^{-mk};q^k)_n}{[q^{\beta+1}]_{kn}\ (q^k;q^k)_n}\ x^{kn}, \tag{36}
$$

with $\Re(\mu)>-1$, admits a generalization by means of the q-*gml* (12) by taking $\alpha,\delta,\mu,r,s\in\mathbb{N}$, $\gamma=$ a negative integer: $-m$, replacing β by $\beta+1$, and z by a real variable x^k, $k\in\mathbb{N}$, and denoting the polynomial thus obtained by $B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r)$ as follows.

Definition 2.5.1. For $\alpha,\beta,\lambda>0$, $m,\delta,\mu,k,s\in\mathbb{N}$, $r\in\mathbb{N}\cup\{0\}$, $m^*=\lfloor\frac{m}{\delta}\rfloor$, the greatest integer part of $\frac{m}{\delta}$, define

$$
\begin{aligned}
B_{m^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q) &= \frac{(q^{\beta+1};q)_{\alpha m}}{[(q^k;q^k)_m]^s}\sum_{n=0}^{m^*}\frac{q^{sk\delta n(m+(\delta nk-1)/2)+\delta n(\alpha\beta+\alpha+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}\\
&\times\frac{[(q^{-mk};q^k)_{\delta n}]^s}{(q^k;q^k)_n}\ x^{kn}.
\end{aligned}
\tag{37}
$$

Note 1. Here (36) is a particular case $s = 1, r = 0, \delta = 1$, and $\alpha = k$ of (37).

The presence of parameter "s" yields the *unusual* inverse series relations involving the inequalities. In fact, for $s = 1$ the usual inverse series relations occur whereas for other values of s the inverse inequality relations occur. This is shown in the following theorems.

If the real functions $F(x, n; s|q)$, $G(x, n; s|q)$, are such that

$$F(x, n; s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q),\ G(x, n; s|q) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q),$$

then there hold the following inequality relations.

Theorem 2.5.1. *Let $F(x, n; s|q)$ and $G(x, n; s|q)$ be real valued functions, $\alpha, \beta, \lambda > 0$, and $\mu, k, s \in \mathbb{N}$, $r, n \in \mathbb{N} \cup \{0\}$ and $n^* = \lfloor \frac{n}{m} \rfloor$, then*

$$F(x, n; s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q) \tag{38}$$

implies

$$\begin{aligned} x^{kn} &> q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(kmn-1)/2}\ \frac{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s} \\ &\times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}} F(x, j; s|q); \end{aligned} \tag{39}$$

and

$$\begin{aligned} x^{kn} &< q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(kmn-1)/2}\ \frac{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s} \\ &\times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}} G(x, j; s|q) \end{aligned} \tag{40}$$

implies

$$G(x, n; s|q) \quad > \quad B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k; s, r|q). \tag{41}$$

Proof. Assume that the inequality (38) holds. Putting

$$\begin{aligned} \omega_n &= q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(kmn-1)/2}\ \frac{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s} \\ &\times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}} F(x, j; s|q), \end{aligned}$$

and substituting the series inequality (38) for $F(x, j; s|q)$, one gets

$$\begin{aligned} \omega_n &< q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(kmn-1)/2}\ \frac{(q^{\beta+1}; q)_{\alpha n}\ [(q^\lambda; q)_{\mu n}]^r\ (q^k; q^k)_n}{[(q^k; q^k)_{mn}]^s} \\ &\times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn}; q^k)_j]^s}{(q^{\beta+1}; q)_{\alpha j}} \frac{(q^{\beta+1}; q)_{\alpha j}}{[(q^k; q^k)_j]^s} \end{aligned}$$

$$
\begin{aligned}
&\times\sum_{i=0}^{\lfloor\frac{j}{m}\rfloor}\frac{q^{s(kmi(kmi-1)/2+kmij)+mi(\alpha\beta+\alpha+r\mu\lambda)}[(q^{-kj};q^k)_{mi}]^s\ x^{ki}}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i}\\
=\ & q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(mn-1)/2}\ \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s}\\
&\times\sum_{j=0}^{mn}\frac{q^{skj}\ (-1)^{sj}q^{skj(j-1)/2-skmnj}\left[(q^k;q^k)_{mn}\right]^s}{\left[(q^k;q^k)_{mn-j}\right]^s(q^{\beta+1};q)_{\alpha j}}\\
&\times\frac{(q^{\beta+1};q)_{\alpha j}}{((q^k;q^k)_j)^s}\sum_{i=0}^{\lfloor\frac{j}{m}\rfloor}\frac{(-1)^{smi}\ q^{s(kmi(kmi-1)/2+kmij)+mi(\alpha\beta+\alpha+r\mu\lambda)}}{\left[(q^k;q^k)_{j-mi}\right]^s(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r}\\
&\times\frac{q^{skmi(kmi-1)/2-skjmi}\left((q^k;q^k)_j\right)^s\ x^{ki}}{(q^k;q^k)_i}\\
=\ & \sum_{j=0}^{mn}\sum_{i=0}^{\lfloor\frac{j}{m}\rfloor}\frac{(-1)^{sj+smi}\ q^{s(kmi(kmi-1)/2+kmij)+mi(\alpha\beta+\alpha+r\mu\lambda)}}{\left[(q^k;q^k)_{j-mi}\right]^s\left[(q^k;q^k)_{mn-j}\right]^s}\\
&\times\frac{q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(kmn-1)/2+skmi(mi-1)/2-skjmi}}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i}\\
&\times q^{skj+skj(j-1)/2-skmnj}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n\ x^{ki}.
\end{aligned}
$$

Now in view of the double series relation

$$
\sum_{i=0}^{mn}\sum_{j=0}^{\lfloor\frac{i}{m}\rfloor}f(i,j)=\sum_{j=0}^{n}\sum_{i=0}^{mn-mj}f(i+mj,j),
$$

we get

$$
\begin{aligned}
\omega_n\ <\ & \sum_{i=0}^{n}\sum_{j=0}^{mn-mi}\frac{(-1)^{sj}\ q^{skmi(kmi-1)/2}\ q^{mi(\alpha\beta+\alpha+r\mu\lambda)}}{\left((q^k;q^k)_j\right)^s\left[(q^k;q^k)_{mn-mi-j}\right]^s}\\
&\times q^{skj(mi-mn+1)+skmi(mi-mn)-mn(\alpha\beta+\alpha+r\mu\lambda)+skj(j-1)/2}\\
&\times q^{-skmn(kmn-1)/2}\ \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i}\ x^{ki}\\
=\ & x^{kn}+\sum_{i=0}^{n-1}\frac{q^{s(kmi(kmi-1)/2+(mi-mn)(skmi+\alpha\beta+\alpha)+r\mu\lambda)}}{\left[(q^k;q^k)_{mn-mi}\right]^s\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i}\\
&\times(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r(q^k;q^k)_n\ x^{ki}\\
&\times\sum_{j=0}^{mn-mi}(-1)^{sj}\ q^{skj(j-1)/2}\ q^{skj(mi-mn+1)}\begin{bmatrix}mn-mi\\ j\end{bmatrix}_{q^k}^s\\
\leq\ & x^{kn}+\sum_{i=0}^{n-1}\frac{q^{skmi(kmi-1)/2+s(kmi(kmi-1)/2+(mi-mn)(skmi+\alpha\beta+\alpha)+r\mu\lambda)}}{\left[(q^k;q^k)_{mn-mi}\right]^s\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r\ (q^k;q^k)_i}\\
&\times(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r(q^k;q^k)_n\ x^{ki}
\end{aligned}
$$

$$\times\left(\sum_{j=0}^{mn-mi}(-1)^j\ q^{kj(j-1)/2}\ q^{skj(mi-mn+1)}\begin{bmatrix}mn-mi\\ j\end{bmatrix}_{q^k}\right)^s$$

$$= x^{kn}+\sum_{i=0}^{n-1}\frac{q^{s(kmi(kmi-1)/2+skmi(mi-mn)}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{[(q^k;q^k)_{mn-mi}]^s\ (q^{\beta+1};q)_{\alpha i}\ [(q^\lambda;q)_{\mu i}]^r}$$

$$\times\frac{(q^k;q^k)_n}{(q^k;q^k)_i}\ x^{ki}\ \left\{\prod_{j=1}^{mn-mi}(1-q^{k(mi-mn+j)})\right\}^s.$$

Here the product on the right hand side vanishes, hence $\omega_n < x^{kn}$. Next, the proof of another inequality relations stated above runs as follows. Here assume that (40) holds true. Now if

$$\nu_n = \frac{(q^{\beta+1};q)_{\alpha n}}{[(q^k;q^k)_n]^s}\sum_{j=0}^{\lfloor\frac{n}{m}\rfloor}\frac{q^{s(kmj(kmj-1)/2+skmjn)+mj(\alpha\beta+\alpha)+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^\lambda;q)_{\mu j}]^r}$$

$$\times\frac{[(q^{-nk};q^k)_{mj}]^s}{(q^k;q^k)_j}\ x^{kj}$$

then substituting the series inequality (40) for x^{kn} , we get

$$\nu_n < \frac{(q^{\beta+1};q)_{\alpha n}}{[(q^k;q^k)_n]^s}\sum_{j=0}^{\lfloor\frac{n}{m}\rfloor}\frac{q^{s(kmj(kmj-1)/2+kmjn)+mj(\alpha\beta+\alpha+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^\lambda;q)_{\mu j}]^r\ (q^k;q^k)_j}$$

$$\times q^{-mj(\alpha\beta+\alpha+r\mu\lambda)-skmj(kmj-1)/2}\ \frac{[(q^{-nk};q^k)_{mj}]^s\ (q^{\beta+1};q)_{\alpha j}}{[(q^k;q^k)_{mj}]^s}$$

$$\times[[q^\lambda]_{\mu j}]^r\ (q^k;q^k)_j\ \sum_{i=0}^{mj}\frac{q^{ski}\ [(q^{-kmj};q^k)_i]^s}{(q^{\beta+1};q)_{\alpha i}}G(x,i;s|q)$$

$$= \frac{(q^{\beta+1};q)_{\alpha n}}{((q^k;q^k)_n)^s}\sum_{j=0}^{\lfloor\frac{n}{m}\rfloor}\frac{(-1)^{smj}\ q^{skmjn-snmj+skmj(mj-1)/2}\ ((q^k;q^k)_n)^s}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ [(q^k;q^k)_{mj}]^s}$$

$$\times\sum_{i=0}^{mj}\frac{(-1)^{is}\ q^{ski}\ q^{ski(i-1)/2-skimj}\ \left[(q^k;q^k)_{mj}\right]^s}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ (q^{\beta+1};q)_{\alpha i}}G(x,i;s|q)$$

$$= \sum_{mj=0}^{n}\sum_{i=0}^{mj}\frac{(-1)^{smj+is}\ q^{ski(i+1)/2+skmj(mj-1)/2-skimj}\ (q^{\beta+1};q)_{\alpha n}}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ \left[(q^k;q^k)_{(mj-i)}\right]^s\ (q^{\beta+1};q)_{\alpha i}}$$

$$\times G(x,i;s|q).$$

In view of double series relation

$$\sum_{k=0}^{n}\sum_{j=0}^{k}f(k,j)=\sum_{j=0}^{n}\sum_{k=j}^{n}f(k,j).$$

this takes the form:

$$\nu_n < \sum_{i=0}^{n}\sum_{mj=i}^{n}\frac{(-1)^{smj+is}\ q^{ski(i+1)/2+skmj(mj-1)/2-skimj}\ (q^{\beta+1};q)_{\alpha n}}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ \left[(q^k;q^k)_{(mj-i)}\right]^s\ (q^{\beta+1};q)_{\alpha i}}$$

$$
\begin{aligned}
& \times G(x,i;s|q) \\
= \quad & G(x,n;s|q) + \sum_{i=0}^{n-1} \frac{(-1)^{is}\ q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}}\ G(x,i;s|q) \\
& \times \sum_{mj=i}^{n} \frac{(-1)^{smj}\ q^{skmj(mj-1)/2-skimj}}{\left[(q^k;q^k)_{(n-mj)}\right]^s\ \left[(q^k;q^k)_{(mj-i)}\right]^s} \\
= \quad & G(x,n;s|q) + \sum_{i=0}^{n-1} \frac{(q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}}\ G(x,i;s|q) \\
& \times \sum_{mj=0}^{n-i} \frac{(-1)^{smj}\ q^{skmj(mj-1)/2}}{\left[(q^k;q^k)_{(n-i-mj)}\right]^s\ \left[(q^k;q^k)_{mj}\right]^s} \\
= \quad & G(x,n;s|q) + \sum_{i=0}^{n-1} \frac{q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}\left[(q^k;q^k)_{(n-i)}\right]^s}\ G(x,i;s|q) \\
& \times \sum_{mj=0}^{n-i} (-1)^{smj}\ q^{skmj(mj-1)/2} \begin{bmatrix} n-i \\ mj \end{bmatrix}_k^s \\
\leq \quad & G(x,n;s|q) + \sum_{i=0}^{n-1} \frac{q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}\left[(q^k;q^k)_{(n-i)}\right]^s}\ B_{i^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q) \\
& \times \left(\sum_{mj=0}^{n-i} (-1)^{mj}\ \ q^{kmj(mj-1)/2} \begin{bmatrix} n-i \\ mj \end{bmatrix}_k \right)^s \qquad (42) \\
= \quad & G(x,n;s|q) + \sum_{i=0}^{n-1} \frac{q^{ski(i+1)/2}\ (q^{\beta+1};q)_{\alpha n}}{(q^{\beta+1};q)_{\alpha i}\left[(q^k;q^k)_{(n-i)}\right]^s}\ G(x,i;s|q) \\
& \times \left\{ \prod_{mj=1}^{n-i} \left(1-q^{kmj-k}\right) \right\}^s .
\end{aligned}
$$

This gives $\nu_n < G(x,n;s|q)$. □

Towards the converse of these inequality relations, we obtain

Theorem 2.5.2. *With the same restrictions as stated in Theorem 9, to the parameters involved,*

$$
\begin{aligned}
x^{kn} \quad > \quad & q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(kmn-1)/2}\ \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r\ (q^k;q^k)_n}{\left[(q^k;q^k)_{mn}\right]^s} \\
& \times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}}\ F(x,j;s|q) \qquad (43)
\end{aligned}
$$

implies

$$F(x,n;s|q) < B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q); \qquad (44)$$

and

$$G(x,n;s|q) > B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;s,r|q)m \qquad (45)$$

implies

$$\begin{aligned} x^{kn} &< q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-skmn(kmn-1)/2}\ \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{[(q^k;q^k)_{mn}]^s} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{skj}\ [(q^{-kmn};q^k)_j]^s}{(q^{\beta+1};q)_{\alpha j}}\ G(x,j;s|q)m. \end{aligned} \tag{46}$$

The proof runs parallel to that of Theorem 2.5.1, hence is omitted. For $s=1$, the polynomial (37) possesses the following inverse series relation.

Theorem 2.5.3. *For $\alpha,\beta,\lambda>0,\ m,\mu,k\in\mathbb{N},\ r\in\mathbb{N}\cup\{0\}$,*

$$\begin{aligned} B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) &= \frac{(q^{\beta+1};q)_{\alpha n}}{(q^k;q^k)_n} \sum_{j=0}^{\lfloor \frac{n}{m}\rfloor} \frac{q^{k(mj(mj-1)/2+mjn)+mj(\alpha\beta+\alpha+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^{\lambda};q)_{\mu j}]^r} \\ &\quad \times \frac{(q^{-nk};q^k)_{mj}\ x^{kj}}{(q^k;q^k)_j} \end{aligned} \tag{47}$$

if and only if

$$\begin{aligned} \frac{x^{kn}}{(q^k;q^k)_n} &= q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-kmn(kmn-1)/2}\ \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}{(q^k;q^k)_{mn}} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q), \end{aligned} \tag{48}$$

and for $n\neq ml,\ l\in\mathbb{N}$,

$$\sum_{j=0}^{n} \frac{q^{kj}\ (q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) = 0. \tag{49}$$

Proof. The proof of (47) implies (48) runs as follows. Here the equality (47) holds. Putting

$$\begin{aligned} J_n &= q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-kmn(kmn-1)/2}\ \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^k;q^k)_{mn}} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \end{aligned}$$

and substituting the series equality (47) for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q)$, we get

$$\begin{aligned} J_n &= q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-kmn(kmn-1)/2}\ \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^k;q^k)_{mn}} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} \frac{(q^{\beta+1};q)_{\alpha j}}{(q^k;q^k)_j} \\ &\quad \times \sum_{i=0}^{\lfloor \frac{j}{m}\rfloor} \frac{q^{kmi(kmi-1)/2+kmij+mi(\alpha\beta+\alpha+r\mu\lambda)}\ (q^{-kj};q^k)_{mi}\ x^{ki}}{(q^{\beta+1};q)_{\alpha i}\ [(q^{\lambda};q)_{\mu i}]^r\ (q^k;q^k)_i} \end{aligned}$$

$$
\begin{aligned}
&= q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-kmn(mn-1)/2} \frac{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r\ (q^k;q^k)_n}{(q^k;q^k)_{mn}} \\
&\quad\times\sum_{j=0}^{mn} \frac{q^{kj}\ (-1)^j q^{kj(j-1)/2-kmnj}(q^k;q^k)_{mn}}{(q^k;q^k)_{mn-j}(q^{\beta+1};q)_{\alpha j}} \\
&\quad\times\frac{(q^{\beta+1};q)_{\alpha j}}{(q^k;q^k)_j}\sum_{i=0}^{\lfloor\frac{j}{m}\rfloor}\frac{(-1)^{mi}\ q^{kmi(kmi-1)/2+kmij+mi(\alpha\beta+\alpha+r\mu\lambda)}}{(q^k;q^k)_{j-mi}(q^{\beta+1};q)_{\alpha i}\ [(q^{\lambda};q)_{\mu i}]^r} \\
&\quad\times\frac{q^{kmi(kmi-1)/2-kjmi}(q^k;q^k)_j\ x^{ki}}{(q^k;q^k)_i} \\
&= \sum_{j=0}^{mn}\sum_{i=0}^{\lfloor\frac{j}{m}\rfloor}(-1)^{j+mi}\ q^{kmi(kmi-1)/2+kmi(mi-1)/2+m(i-n)(\alpha\beta+\alpha+r\mu\lambda)} \\
&\quad\times\frac{q^{-kmn(kmn-1)/2+kj(j+1)/2-kmnj}\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}{(q^k;q^k)_{j-mi}(q^k;q^k)_{mn-j}(q^{\beta+1};q)_{\alpha i}\ [(q^{\lambda};q)_{\mu i}]^r\ (q^k;q^k)_i} \\
&\quad\times(q^k;q^k)_n\ x^{ki}.
\end{aligned}
$$

Now in view of the double series relation

$$
\sum_{i=0}^{mn}\sum_{j=0}^{\lfloor\frac{i}{m}\rfloor} f(i,j) = \sum_{j=0}^{n}\sum_{i=0}^{mn-mj} f(i+mj,j),
$$

we get

$$
\begin{aligned}
J_n &= \sum_{i=0}^{n}\sum_{j=0}^{mn-mi}(-1)^j\ q^{kmi(kmi-1)/2+m(i-n)(\alpha\beta+\alpha+r\mu\lambda)+kj(mi-mn+1)} \\
&\quad\times\frac{q^{kj(j-1)/2-kmn(kmn-1)/2+kmi(mi-mn)}\ (q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r}{(q^k;q^k)_j\ (q^k;q^k)_{mn-mi-j}(q^{\beta+1};q)_{\alpha i}\ [(q^{\lambda};q)_{\mu i}]^r} \\
&\quad\times\frac{(q^k;q^k)_n}{(q^k;q^k)_i}\ x^{ki} \\
&= \frac{x^{kn}}{(q^k;q^k)_n} + \sum_{i=0}^{n-1} q^{kmi(kmi-1)/2+kmi(mi-mn)+(mi-mn)(\alpha\beta+\alpha+r\mu\lambda)} \\
&\quad\times\frac{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r(q^k;q^k)_n}{(q^k;q^k)_{mn-mi}\ (q^{\beta+1};q)_{\alpha i}\ [(q^{\lambda};q)_{\mu i}]^r(q^k;q^k)_i}\ x^{ki} \\
&\quad\times\sum_{j=0}^{mn-mi}(-1)^j\ q^{kj(j-1)/2}\ q^{kj(mi-mn+1)}\begin{bmatrix} mn-mi \\ j\end{bmatrix}_{q^k} \\
&= \frac{x^{kn}}{(q^k;q^k)_n} + \sum_{i=0}^{n-1} q^{kmi(kmi-1)/2+(mi-mn)(kmi+\alpha\beta+\alpha+r\mu\lambda)} \\
&\quad\times\frac{(q^{\beta+1};q)_{\alpha n}\ [(q^{\lambda};q)_{\mu n}]^r(q^k;q^k)_n}{(q^k;q^k)_{mn-mi}\ (q^{\beta+1};q)_{\alpha i}\ [(q^{\lambda};q)_{\mu i}]^r(q^k;q^k)_i}\ x^{ki}
\end{aligned}
$$

$$
\begin{aligned}
& \times \sum_{j=0}^{mn-mi} (-1)^j \; q^{kj(j-1)/2} \; q^{kj(mi-mn+1)} \begin{bmatrix} mn-mi \\ j \end{bmatrix}_{q^k} \\
= \;& \frac{x^{kn}}{(q^k;q^k)_n} + \sum_{i=0}^{n-1} \frac{q^{kmi(kmi-1)/2+kmi(mi-mn)} \; (q^{\beta+1};q)_{\alpha n} \; [(q^\lambda;q)_{\mu n}]^r}{(q^k;q^k)_{mn-mi} \; (q^{\beta+1};q)_{\alpha i} \; [(q^\lambda;q)_{\mu i}]^r \; (q^k;q^k)_i} \\
& \times (q^k;q^k)_n \; x^{ki} \prod_{j=1}^{mn-mi} (1-q^{k(mi-mn+j)}) \\
= \;& \frac{x^{kn}}{(q^k;q^k)_n}
\end{aligned}
$$

as the product on the right hand side vanishes. To show further that (47) also implies (49), we may substitute for $B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q)$ from (47) to the left hand side of (49), to get

$$
\begin{aligned}
& \sum_{j=0}^{n} \frac{q^{kj} \; (q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \\
= \;& \sum_{j=0}^{n} \frac{q^{kj} \; (-1)^j \; q^{kj(j-1)/2-knj} (q^k;q^k)_n \; (q^k;q^k)_j}{(q^k;q^k)_{n-j} (q^{\beta+1};q)_{\alpha j}} \\
& \times \sum_{i=0}^{\lfloor \frac{j}{m} \rfloor} \frac{(-1)^{mi} \; q^{kmi(kmi-1)/2+mi(\alpha\beta+\alpha+r\mu\lambda)+kmi(kmi-1)/2}}{(q^{\beta+1};q)_{\alpha i} \; [(q^\lambda;q)_{\mu i}]^r \; (q^k;q^k)_{j-mi} \; (q^k;q^k)_i} \; x^{ki} \\
= \;& \sum_{i=0}^{\lfloor \frac{n}{m} \rfloor} \frac{q^{3kmi(kmi-1)/2+kmi-knmi+mi(\alpha\beta+\alpha+r\mu\lambda)} \; (q^k;q^k)_n}{(q^{\beta+1};q)_{\alpha i} \; [(q^\lambda;q)_{\mu i}]^r \; (q^k;q^k)_{n-mi} \; (q^k;q^k)_i} \; x^{ki} \\
& \times \sum_{j=0}^{n-mi} (-1)^j \; q^{kj(j-1)/2} \; q^{kj(mi-n+1)} \; \begin{bmatrix} n-mi \\ j \end{bmatrix}_{q^k}.
\end{aligned}
$$

Here the inner sum on the r.h.s. is actually the product

$$
\prod_{j=1}^{n-mi} \left(1-q^{k(mi-n+j)}\right)
$$

which vanishes for $j = n-mi$ and n not an integer multiple of m. Thus completing the first part. The proof of converse part runs as follows [3]. In order to show that both the series (48) and the condition (49) together imply the series (47), a simplest inverse series relations [13, Eq.(1), p.43]:

$$
\Delta_n = \sum_{j=0}^{n} \frac{q^{knj} \; (q^{-kn};q^k)_j}{(q^k;q^k)_j} \; \Psi_j \; \Leftrightarrow \; \Psi_n = \sum_{j=0}^{n} \frac{q^{kj} \; (q^{-kn};q^k)_j}{(q^k;q^k)_j} \; \Delta_j
$$

may be used. Here putting

$$
\Psi_j = \frac{q^{kj} \; (q^k;q^k)_j}{(q^{\beta+1};q)_{\alpha j}} \; B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q),
$$

and considering one sided relation that is, the series on the left hand side implies the series on the right side, we get

$$\Delta_n = \sum_{j=0}^{n} \frac{q^{kj}\ (q^{-kn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \tag{50}$$

$$\Rightarrow\ B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) = \frac{(q^{\beta+1};q)_{\alpha j}}{(q^k;q^k)_n}\ \sum_{j=0}^{n} \frac{(q^{-kn};q^k)_j}{(q^k;q^k)_j}\ \Delta_j. \tag{51}$$

Since the condition (49) holds, $\omega_n = 0$ for $n \neq ml,\ l \in \mathbb{N}$, whereas

$$\Delta_{mn} = \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}}\ B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q).$$

But since the series (48) holds true.

$$\Delta_{mn} = \frac{q^{mn(\alpha\beta+\alpha+r\mu\lambda)}\ q^{kmn(kmn-1)/2}\ (q^k;q^k)_{mn}\ x^{kn}}{(q^k;q^k)_n\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}.$$

Consequently, the inverse pair (50) and (51) assume the form:

$$\begin{aligned} B_{n^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q) \quad &= \quad \frac{(q^{\beta+1};q)_{\alpha n}}{(q^k;q^k)_n} \sum_{j=0}^{\lfloor \frac{n}{m} \rfloor} \frac{q^{k(mj(mj-1)/2+mjn)+mj(\alpha\beta+\alpha+r\mu\lambda)}}{(q^{\beta+1};q)_{\alpha j}\ [(q^\lambda;q)_{\mu j}]^r} \\ &\quad \times \frac{(q^{-nk};q^k)_{mj}}{(q^k;q^k)_j}\ x^{kj}, \end{aligned}$$

$$\begin{aligned} \Rightarrow\ \frac{x^{kn}}{(q^k;q^k)_n} \quad &= \quad \frac{q^{-mn(\alpha\beta+\alpha+r\mu\lambda)-kmn(kmn-1)/2}\ (q^{\beta+1};q)_{\alpha n}\ [(q^\lambda;q)_{\mu n}]^r}{(q^k;q^k)_{mn}} \\ &\quad \times \sum_{j=0}^{mn} \frac{q^{kj}\ (q^{-kmn};q^k)_j}{(q^{\beta+1};q)_{\alpha j}} B_{j^*}^{(\alpha,\beta,\lambda,\mu)}(x^k;1,r|q), \end{aligned}$$

subject to the condition (49). □

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