Divulgaciones Matemáticas Vol. 22, No. 1(2021), pp. 22–30

A note on the Banach contraction principle in b-metric spaces

Una nota sobre el principio de contracción de Banach en b-espacios métricos

Mohamed Akkouchi (akkm555@yahoo.fr)

Department of Mathematics
Faculty of Sciences-Semlalia, University Cadi Ayyad
Av. Prince My. Abdellah, BP: 2390, Marrakesh;
(40.000-Marrakech), Morocco (Maroc).

Abstract

Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$. Let T a contractive map on X, that is a selfmap T of X satisfying

$$d(Tx, Ty) \leq \lambda d(x, y),\ \forall x, y \in X, \qquad (B_\lambda)$$

with some $\lambda \in [0, 1)$. In 1989, Bakhtin established an alogous to the Banach contraction principle in the context of complete b-metric spaces. Precisely, he proved that if $\lambda \in [0, \frac{1}{s})$. Then T has a unique fixed point. The aim of this note is to give a simple proof of the Banach contraction principle in X for all $\lambda \in [0, 1)$. So, in particular, we provide some complements to Bakhtin's result. We establish a fundamental contraction inequality for T and use it to prove convergence of Picard sequences. For such sequences, we give an evaluation of the order of convergence and a posteriori error estimate. We estimate the diameter of the T-orbits. As applications, we deduce two stopping rules indicating the sufficient number of iterations of the Picard process which allows a satisfactory approximation for the fixed point of T.

Key words and phrases: Banach fixed point principle; b-metric space; stopping rules.

Resumen

Sea $(X, d; s)$ un espacio b-métrico completo con el parámetro $s \geq 1$. Sea T un mapa contractivo en X, que es un automapa T de X satisfactorio

$$d(Tx, Ty) \leq \lambda d(x, y),\ \forall x, y \in X, \qquad (B_\lambda)$$

con algo de $\lambda \in [0, 1)$. En 1989, Bakhtin estableció un principio análogo al principio de contracción de Banach en el contexto de espacios b-métricos completos. Precisamente, demostró que si $\lambda \in [0, \frac{1}{s})$, entonces T tiene un punto fijo único. El objetivo de esta nota es dar una prueba simple del principio de contracción de Banach en X para todos los $\lambda \in [0, 1)$. Entonces, en particular, brindamos algunos complementos al resultado de Bakhtin. Establecemos una desigualdad de contracción fundamental para T y la usamos para probar la convergencia de las secuencias de Picard. Para tales secuencias, damos una evaluación del orden de convergencia y una estimación del error a posteriori. Estimamos el diámetro de las órbitas T.

Received 31/01/2021. Revised 23/02/2021. Accepted 25/05/2021.
MSC (2010): Primary 47H10; Secondary 54H25.
Corresponding author: Mohamed Akkouchi

Como aplicaciones, deducimos dos reglas de parada que indican el número suficiente de iteraciones del proceso Picard que permite una aproximación satisfactoria para el punto fijo de T.

Palabras y frases clave: Principio de punto fijo de Banach; b-espacio métrico; detener las reglas.

1 Introduction

An important generalization of metric spaces is given by the concept of b-metric spaces. We recall (see [2, 6, 7]) the following definition.

Definition 1.1. *Let X be a non-empty set and let $d : X \times X \to [0, +\infty)$ be a function. Then d is said to be a b-metric on the set X, if the following conditions are satisfied:*

(i) $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$.

(ii) $d(x, y) = d(y, x)$.

(iii) there exists a real number $s \geq 1$ such that:

$$d(x, y) \leq s\,[d(x, u) + d(u, y)] \quad \text{for all } \ x, y, u \in X.$$

The triplet $(X, d; s)$ is said to be a b-metric space with parameter s. The inequality (iii) is also called the s-triangle inequality.

Throughout this paper, $(X, d; s)$ will designate a b-metric space with parameter $s \geq 1$. As in the metric case, we have a topological setting. For the sequel, we denote the set of nonnegative integers by $\mathbb{N}_0$. As usual, $\mathbb{N}$ designates the set of positive integers. A sequence $\{x_n\}$ $(n \in \mathbb{N}_0)$ of elements of a set X will be also denoted by $(x_n)_{n\in\mathbb{N}_0}$ and its range set will be denoted by $\{x_n : n \in \mathbb{N}_0\}$.

Definition 1.2. *Let $(X, d; s)$ be a b-metric space, $\{x_n\}$ be a sequence in X and $x \in X$. Then, the following are defined as follows:*

(i) The sequence $\{x_n\}$ is said to be a Cauchy sequence, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n \geq N$ and all $p \in \mathbb{N}$, we have $d(x_n, x_{n+p}) < \epsilon$.

(ii) The sequence $\{x_n\}$ is said to be convergent to x, if for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that, for all $n \geq N$, we have $d(x_n, x) < \epsilon$. In this case, we write:

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad \text{as} \quad n \to \infty.$$

(iii) $(X, d; s)$ is said to be complete b-metric space if every Cauchy sequence in X converges to some $x \in X$.

We observe that every converging sequence in a b-metric space is Cauchy sequence but, in general, the converse is not true. It is easy to see that the limit of a converging sequence (in a b-metric space) is unique.

Definition 1.3. *Let $(X, d; s)$ be a b-metric space and $\{x_n\}$ be a sequence in X. Let A be a non empty subset of X and $\delta(A) := \sup\{d(x, y) : (x, y) \in A \times A\}$. Then A is said to be bounded if $\delta(A)$ is finite.*

We observe that every Cauchy sequence in a b-metric space is bounded but, in general, the converse is not true.

Definition 1.4. *Let X be a non empty set and let T be a selfmapping of X. Then, for every $x \in X$, the set $O_T(x) := \{x, Tx, T^2x, T^3x, \dots\}$ is called the orbit of T at x.*

Now, we define the concepts of T-orbitally completeness.

Definition 1.5. *Let $(X, d; s)$ be a b-metric space and T be a selfmapping on X. Then X is said to be T-orbitally complete if, for any $x \in X$, every Cauchy sequence of the orbit $O_T(x) := \{x, Tx, T^2x, \dots\}$ is convergent in X.*

In the case of metric spaces, the concept of orbitally completness was first introduced in 1974 by Ćirić in [4].

Fixed point theory in b-metric spaces started with the extension of the Banach contraction principle. Let T be a selfmapping of X, we say that T is contraction on X, if T satisfies the following inequality:

$$d(Tx, Ty) \leq \lambda d(x, y), \forall x, y \in X, \qquad (B_\lambda)$$

with some $\lambda \in [0, 1)$.

In 1989, Bakhtin established in [2] the following result, which may be considered as the analogous of Banach contraction principle for b-metric spaces.

Theorem 1.1 ([2])**.** *Let $(X, d; s)$ be a complete b-metric space with parameter s and $f : X \to X$ a mapping such that, for some $\lambda > 0$,*

$$d(f(x), f(y)) \leq \lambda\, d(x, y), \quad \text{for all } x, y \in X. \qquad (1.1)$$

If $0 < \lambda < 1/s$, then f has a unique fixed point z and, for every $x \in X$, the sequence $\left(f^n(x)\right)_{n\in\mathbb{N}}$ converges to z as $n \to \infty$. Furthermore, the following evaluation of the order of convergence holds

$$d(x_n, z) \leq \frac{sd(x_0, x_1)}{1 - \lambda s} \cdot \lambda^n\,, \quad \text{for all} \quad n \in \mathbb{N}. \qquad (1.2)$$

The articles [6] and [7] published in 1993 and 1998 by S. Czerwik, respectively, and the article [3] published by V. Berinde in 1993 have initiated investigation of fixed points in b-metric spaces. During the last two decades, a very intensive research work was conducted in b-metric spaces and in their generalizations. The survey [5] of S. Cobzaş contains a large view on the evolution and recent developpements of the theory of b-metric spaces and fixed point theory therein. The survey [8] of E. Karapinar contains a short survey on some recent fixed point results obtained in the context of b-metric spaces. The reader is invited to consult the articles listed in the references of this work and the references therein.

In 2007, R. S. Palais (see [9]) provided a simple proof of the Banach contraction principle in complete metric spaces and established a stopping rule. The aim of this paper is to give a simple proof of the Banach contraction principle in X for all $\lambda \in [0, 1)$. The result obtained here is a variant of the Banach contraction in a complete b-metric space (See Theorem 2.1). This note is motivated by the article [9] of R. S. Palais concerning contractions in complete metric spaces.

This note extends the results of [9] to the case of b-metric spaces and may be considered as a continuation to [9].

To prove convergence of Picard sequences, we start by proving a fundamental contraction inequality for any given contraction T on any complete b-metric space (X, d, s). Thus we show that for all values of $\lambda \in [0,1)$, and for every point $x \in X$, the Picard sequence $(T^n(x))_n$ converges to a unique fixed point z of T. So, by this we provide complements to Bakhtin's result (see Theorem (1.1)) by investigating the remaining case, where the parameter $\lambda \in [\frac{1}{s}, 1)$ $(s > 1)$.

For each Picard sequence, we give an evaluation of the order of its convergence and an a posteriori error estimate. Also, we estimate the diameter of the T-orbits. As applications, we deduce two stopping rules precizing the sufficient number of iterations of the Picard process which allows a satisfactory approximation for the fixed point of T.

This paper is organized as follows: In section two, we establish a variant of the Banach contraction principle, where we have stated all our remarks and observations (see Theorem 2.1). Section three contains two stopping rules for contractive selfmappings in complete b-metric spaces. In particular, our results extend those obtained by R. S. Palais ([9]) for the Banach contractions in complete metric spaces.

2 A version of the Banach contraction principle

Let $(X, d; s)$ be a b-metric space with parameter $s \geq 1$. We need to introduce the following notation. Let

$$\mathbb{B}_2 := \{(\lambda, s, p) \in [0,1) \times [1, +\infty) \times (0, +\infty) : s^2 \lambda^p < 1\}.$$

We observe that if $(\lambda, s, p) \in \mathbb{B}_2$, then $s\lambda^p < 1$. For all $(\lambda, s, p) \in \mathbb{B}_2$, we set

$$C(\lambda, s, p) := \frac{s(1+s)}{2(1 - s^2\, \lambda^p)}.$$

Before stating our version of the Banach contraction principle in the setting of b-metric spaces, we need the following lemma.

Lemma 2.1. *Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$ and $T : X \to X$ a mapping such that, for some $\lambda \in [0,1)$,*

$$d(T(x), T(y)) \leq \lambda\, d(x, y), \quad \textit{for all } x, y \in X. \tag{2.1}$$

Let p_2 be the smallest positive integer satisfying: $s^2\, \lambda^{p_2} < 1$. Then for all $x, y \in X$, we have the following inequality:

$$d(x, y) \leq C(\lambda, s, p_2) \left[d(x, T^{p_2}(x)) + d(y, T^{p_2}(y)) \right]. \tag{2.2}$$

Proof. By using the s-triangle inequality, we have

$$d(x, y) \leq sd(x, T^{p_2}(x)) + s^2 d(T^{p_2}(x), T^{p_2}(y)) + s^2 d(T^{p_2}(y), y).$$

Therefore, we get

$$d(x, y) \leq sd(x, T^{p_2}(x)) + s^2 d(T^{p_2}(y), y) + s^2 \lambda^{p_2} d(x, y).$$

By a similar manner, we get

$$d(y,x) \le sd(y,T^{p_2}(y)) + s^2 d(T^{p_2}(x),x) + s^2\lambda^{p_2} d(y,x).$$

By adding left members and right members of the inequalities above, we obtain

$$2d(x,y) \le s(1+s)[d(x,T^{p_2}(x)) + d(y,T^{p_2}(y))] + 2s^2\lambda^{p_2} d(x,y),$$

from which, we deduce that

$$d(x,y) \le \frac{s(1+s)}{2(1-s^2\lambda^{p_2})}[d(x,T^{p_2}(x)) + d(y,T^{p_2}(y))],$$

which is the desired inequality. □

The following result provides some complements to Theorem 1.1.

Theorem 2.1. *Let $(X,d;s)$ be a b-metric space with parameter $s \ge 1$ and $T : X \to X$ a mapping such that*

$$d(T(x),T(y)) \le \lambda\, d(x,y), \quad \text{for all } x,y \in X, \tag{2.3}$$

for some $\lambda \in [0,1)$. Suppose that X is T-orbitally complete. Then

(P1) *T has a unique fixed point z in X.*

(P2) *For every $x \in X$, the Picard sequence $(x_n)_{n\ge 0}$ defined by*

$$x_0 := x \quad \text{and} \quad x_n := T^n(x), \quad \text{for all integer} \quad n \ge 1,$$

converges to z as $n \to \infty$.

(P3) *If $0 \le \lambda < \frac{1}{s}$ then the following evaluation of the order of convergence holds*

$$d(T^n(x),z) \le \frac{s\,d(x,Tx)}{1-\lambda s} \cdot \lambda^n\,, \quad \text{for all} \quad n \ge 0. \tag{2.4}$$

(P4) *If $s > 1$ and $\frac{1}{s} \le \lambda < 1$, let p_2 is the smallest positive integer satisfying: $s^2\,\lambda^{p_2} < 1$. Then:*

(i) *The following evaluation of the order of convergence holds*

$$d(T^n(x),z) \le s\,C(\lambda,s,p_2)\,d(x,T^{p_2}(x))\,\lambda^n, \quad \text{for all} \quad n \ge 0, \tag{2.5}$$

where the constant $C(\lambda,s,p_2)$ is given by

$$C(\lambda,s,p_2) := \frac{s(1+s)}{2(1-s^2\,\lambda^{p_2})}.$$

(ii) *The following a posteriori error estimate holds:*

$$d(x_n,z) \le \frac{s\lambda}{1-s\,\lambda^{p_2}} \cdot d(x_{n-1},T^{p_2-1}x_n)\,, \quad \text{for all} \quad n \ge 1. \tag{2.6}$$

(iii) *The orbit $O_T(x)$ is bounded and we have the following estimate for its diameter:*

$$\delta(O_T(x)) \leq 2\,C(\lambda, s, p_2)\, d(x, T^{p_2}(x)). \tag{2.7}$$

(P5) *The rate of convergence of Picard iteration is given by*

$$d(x_n, z) \leq \lambda\, d(x_{n-1}, z), \quad \textit{for all integer} \quad n \geq 1. \tag{2.8}$$

Proof. (1) The case where $0 \leq \lambda < \frac{1}{s}$ was studied in Theorem 1.1 of Bakhtin. In this case, the properties (P1), (P2) and (P3) are given by Bakhtin's theorem. So, we are led to consider only the case where $s > 1$ and $\frac{1}{s} \leq \lambda < 1$.

(2) For every $x_0 \in X$, we consider the sequence $x_n := T^n(x_0)$ (for all $n \in \mathbb{N}$). Let n and m be arbitrary nonnegative integers. By setting $x := x_n$ and $y := x_{n+m}$ in the inequality (2.2), we get

$$\begin{aligned} d(x_n, x_{n+m}) &\leq C(\lambda, s, p_2)\,[d(x_n, T^{p_2}(x_n)) + d(x_{n+m}, T^{p_2}(x_{n+m}))] \\ &= C(\lambda, s, p_2)\left[d(T^n x, T^n(T^{p_2} x)) + d(T^{n+m} x, T^{n+m}(T^{p_2} x))\right] \\ &\leq C(\lambda, s, p_2)\left[\lambda^n d(x, T^{p_2} x) + \lambda^{n+m} d(x, T^{p_2} x)\right] \\ &= \lambda^n (1 + \lambda^m)\, C(\lambda, s, p_2)\, d(x, T^{p_2} x), \end{aligned}$$

from which, we infer that

$$d(x_n, x_{n+m}) \leq 2\lambda^n\, C(\lambda, s, p_2)\, d(x, T^{p_2} x), \tag{2.9}$$

for all integers n and m. From inequality (2.9), we deduce that the sequence $(x_n)_n$ is a cauchy sequence in X. Since the b-metric space X is supposed to be T-orbitally complete, this sequence converges to a point (say) $z \in X$.

For evey nonnegative integer n, we have

$$d(T(x_n), T(z)) \leq \lambda\, d(x_n, z).$$

The above inequality, implies that the subsequence $(x_{n+1})_n$ converges to Tz. By uniqueness of the limit, we infer that $z = Tz$. Thus, z is a fixed point of T.

It is clear, by the condition (2.1), that T has a unique fixed point. We observe the following facts concerning the integer p_2:

(a) $s\,\lambda^{p_2} \leq s^2 \lambda^{p_2} < 1$.

(b) Since $s \geq 1$ and $1 \leq s\lambda$, then we have $p_2 \geq 2$.

(3) Let $x \in X$. Next we show the estimate (2.5). Indeed, for all positive integer n, we have

$$\begin{aligned} d(T^n x, z) &\leq s\left[d(T^n x, T^{n+p_2} x) + d(T^{n+p_2} x, T^{p_2} z)\right] \\ &\leq s\,\lambda^n d(x, T^{p_2} x) + s\lambda^{p_2} d(T^n x, z), \end{aligned}$$

from which we get the inequality (2.5).

(4) Next, we prove the estimate (2.6). Let $x \in X$ and let $(x_n)_{n\geq 0}$ be the Picard sequence associated to x. Then for every positive integer n, we have

$$\begin{aligned} d(x_n, z) &= d(Tx_{n-1}, Tz) \leq \lambda d(x_{n-1}, z) \\ &\leq s\lambda \left[d(x_{n-1}, T^{p_2-1}x_n) + d(T^{p_2-1}x_n, T^{p_2-1}z)\right] \\ &\leq s\,\lambda d(x_{n-1}, T^{p_2-1}x_n) + s\lambda^{p_2} d(x_n, z), \end{aligned}$$

from which we get the inequality (2.6).

(5) Next, we prove the estimate (2.7). Let $x \in X$ and let $(x_n)_{n\geq 0}$ be the Picard sequence associated to x. Then, by virtue of the inequality (2.9), we deduce that

$$\delta(O_T(x)) = \sup\{d(x_n, x_{n+m}) : n, m \in \mathbb{N}_0\} \leq 2\,C(\lambda, s, p_2)\,d(x, T^{p_2}(x)).$$

which is the desired inequality.

(6) The property (P5) is clear.

□

We point out that a different proof of the parts (P1) and (P2) of Theorem 2.1 was given in [1] by T. V. An and N. V. Dung. The proof of [1] used a metrization method.

3 Application: Stopping Rules

We point out that a stopping rule was established by R.S. Palais in [9] for contractions in complete metric spaces. We give here two stopping rules in the setting of complete b-metric spaces depending on two cases:

(i) where the parameter $\lambda \in [0, \frac{1}{s})$.

(ii) where $\lambda \in [\frac{1}{s}, 1)$ and $s > 1$.

Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$ and $T : X \to X$ a mapping such that

$$d(T(x), T(y)) \leq \lambda\, d(x, y), \quad \text{for all } x, y \in X, \tag{3.1}$$

for some $\lambda \in [0, 1)$. According to Theorem 2.1, we know that T has a unique fixed point z in X and that for all $x \in X$ the Picard sequence $(x_n)_n$, with $x_n := T^n x$, converges to z.

3.1 A first stopping rule.

Suppose that $\lambda \in [0, \frac{1}{s})$. By virtue of Theorem 2.1, we have the following inequality

$$d(T^n(x), z) \leq K(s, \lambda)\, d(x, Tx)\, \lambda^n\,, \quad \text{for all} \quad n \geq 0. \tag{3.2}$$

where $K(s, \lambda) := \frac{s}{1-\lambda s}$.

An application of the last inequality is as follows. Suppose that we accept an error of order ε, i.e., instead of the actual fixed point z of T we are satisfied with a point w of X satisfying $d(z, w) < \varepsilon$. Suppose also that we are starting our iteration at some point x in X. Then from the inequality (3.2) it is easy to find an integer N so that $w = T^N(x)$ will give the satisfactory answer. Indeed, since we desire that $d(T^N(x), p) < \varepsilon$, it is sufficient to require that

$$K(s,\lambda)\, d(x,Tx)\, \lambda^N < \varepsilon.$$

This is possible because $\lim_{n\to+\infty} K(s,\lambda)\, d(x,Tx)\, \lambda^n = 0$. Now the first displacement $\tau(x) = d(x, T(x))$ is a quantity that we can compute after the first iteration and we can then compute how large N has to be by taking the log of the above inequality and solving for N (remembering that $\ln(\lambda)$ is negative).

Under the assumptions above we have the first stopping rule.

Theorem 3.1 (First Stopping Rule)**.** *If* $\tau(x) = d(x,Tx)$ *and*

$$N > \frac{\ln(\varepsilon) - \ln(s) + \ln(1 - s\lambda) - \ln(\tau(x))}{\ln(\lambda)}, \tag{3.3}$$

then $d(T^N(x), z) < \varepsilon$.

Another interpretation of (3.3) is the following: Suppose we take $\varepsilon = 10^{-m}$ in our first stopping rule inequality. We see that the growth of N with m is a constant plus $\frac{m}{|\ln(\lambda)|}$, or in other terms, to get one more decimal digit of precision we have to do (roughly) $\frac{1}{|\ln(\lambda)|}$ more iteration steps. In other words, if we need N iterative steps to get m decimal digits of precision, then we need another N in order to arrive to $2m$ decimal digits of precision.

3.2 A second stopping rule.

Suppose that $s > 1$ and $\lambda \in [\frac{1}{s}, 1)$. Let $x \in X$ and let $(x_n := T^n x)_n$ be the corresponding Picard sequence. Let p_2 be the smallest positive integer satisfying: $s^2\, \lambda^{p_2} < 1$. By virtue of Theorem 2.1, we have the following inequality:

$$d(T^n(x), z) \le s\, C(\lambda, s, p_2)\, d(x, T^{p_2}(x))\, \lambda^n, \quad \text{for all} \quad n \ge 0, \tag{3.4}$$

where the constant $C(\lambda, s, p_2)$ is given by

$$C(\lambda, s, p_2) := \frac{s(1+s)}{2(1 - s^2\, \lambda^{p_2})}.$$

Under the assumptions above, by using the inequality (3.4) and similar arguments to those exposed in the first subsection, we are led to state our second stopping rule.

Theorem 3.2 (Second Stopping Rule)**.** *If* $\sigma(x) = d(x, T^{p_2}x)$ *and*

$$N > \frac{\ln(\varepsilon) - \ln(s) - \ln(C(\lambda, s, p_2)) - \ln(\sigma(x))}{\ln(\lambda)}, \tag{3.5}$$

then $d(T^N(x), z) < \varepsilon$.

References

[1] T. V. An and N. V. Dung. *Remarks on Frink's metrization technique and applications*, arXiv preprint, arXiv:1507.01724 (2015), 15 p.

[2] I. A. Bakhtin. *The contraction mapping principle in quasimetric spaces*, Funktionalnyi Analyz, Ulianovskii Gos. Ped. Inst. **30** (1989), 26–37 (Russian).

[3] V. Berinde. *Generalized contractions in quasimetric spaces*, Seminar on Fixed Point Theory, 1993, 3-9.

[4] Lj. B. Ćirić. *On some maps with non-unique fixed points*. Publ. Inst. Math. (Beograd), **13**(31) (1974), 52–58.

[5] S. Cobzaş. *B-metric spaces, fixed points and Lipschitz functions*, arXiv:1802.02722v2 [math.FA], (2018), 35 pp.

[6] S. Czerwik. *Contraction mapping in b-metric spaces.* Acta Math. Inform. Univ. Ostrav., **1**(1) (1993), 5–11.

[7] S. Czerwik. *Nonlinear set-valued contraction mappings in b-metric spaces.* Atti Semin. Mat. Fis. Univ. Modena, **46** (1998), 263–276.

[8] E. Karapinar. *A Short survey on the recent fixed point results on b-metric spaces*, Constructive Mathematical Analysis. **1**(1), (2018), 15–44.

[9] R. S. Palais. *A simple proof of the Banach contraction principle*, J. Fixed Point Theory Appl., **2**, (2007), 221–223.