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A theoretical improvement of the Riemann-Liouville and Caputo fractional derivatives

Una mejora teórica de las derivadas fraccionarias de Riemann-Liouville y Caputo

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Abstract

In this paper we propose new fractional derivatives which, from the theoretical viewpoint, improve the Riemann-Liouville and Caputo fractional derivatives. Furthermore, some useful properties of the new fractional derivatives are presented.

Key words and phrases: Riemann-Liouville fractional derivative; Caputo fractional derivative; Laplace transform.

Resumen

En este trabajo se proponen nuevas derivadas fraccionarias que, desde el punto de vista teórico, mejoran las derivadas fraccionarias de Riemann-Liouville y Caputo. Por otro lado, se introducen algunas propiedades importantes de estas nuevas derivadas fraccionarias.

Palabras y frases clave: Derivada fraccionaria de Riemann-Liouville; Derivada fraccionaria de Caputo; Transformada de Laplace.

1 Introduction

Fractional calculus (FC) is an extension of ordinary calculus with more than 300 years of history. The history of the Fractional Calculus goes back to seventeenth century, when in 1695 the derivative of order $\alpha = \frac{1}{2}$ was described by Leibnitz in his letter to L'Hospital (cf. [2]). That date is regarded as the exact birthday of the fractional calculus. Since then this branch has been treated by eminent mathematicians, such as Euler, Laplace, Fourier, Liouville, Riemann, Laurent, Weyl and Abel. Therefore many definitions of fractional derivative have been proposed (cf. [1, 3, 7, 10, 11], [13]-[20] and [4, 5, 6, 8, 9, 12]), as follows (just to name a few):

Definition 1.1. The Grunwald and Letnikov fractional derivative is given by the formula

$$(D^{\alpha} f)(x) = \lim_{n\to 0} \frac{(\nabla_h^{\alpha} f)(x)}{h^{\alpha}}$$

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where

$$(\nabla_h^\alpha f)(x) = \sum_{j=0}^{n} (-1)^j \binom{\alpha}{j} f(x - jh)$$

with $n = [\alpha]$, where $[\alpha]$ denotes the integer part of a real number α.

Definition 1.2. Suppose $\alpha, a, t \in \mathbb{R}$, with $\alpha > 0$ and $t > a$. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I_{at}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds$$

where Γ is the gamma function, given by $\Gamma(\alpha) = \displaystyle\int_0^\infty e^{-s} s^{\alpha-1} ds$.

Definition 1.3. Suppose $\alpha, a, t \in \mathbb{R}$, with $\alpha > 0$ and $t > a$. The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$^{RL}D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-(1+\alpha)} f(s) ds \tag{1}$$

where $n = [\alpha] + 1$.

Definition 1.4. Suppose $\alpha, a, t \in \mathbb{R}$, with $\alpha > 0$ and $t > a$. The Caputo fractional derivative of order $\alpha > 0$ is defined by

$$^{C}D_{at}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-(1+\alpha)} f^{(n)}(s) ds \tag{2}$$

where $n = [\alpha] + 1$.

In the present work, we shall introduce new fractional derivatives to improve theoretically the Riemann-Liouville and Caputo fractional derivatives. The outline of the paper is as follows: in section 2, new definitions of fractional derivative are introduced. Section 3 presents properties of new fractional derivatives. Finally, conclusions are summarized in section 4.

2 New fractional derivatives

Our new definitions are motivated by the following reasoning. Integrating (1) and (2) by parts for $n = 1$, we have

$$\begin{aligned} ^{RL}D_{at}^{\alpha} f(t) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds \\ &= \frac{1}{\Gamma(1-\alpha)} \left[0^{-\alpha} f(t) - \alpha \int_a^t (t-s)^{-(\alpha+1)} f(s) ds \right] \end{aligned} \tag{3}$$

and

$$ {}^{C}D^{\alpha}_{at}f(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha} f^{(1)}(s)ds $$
$$ = \frac{1}{\Gamma(1-\alpha)}\Big[0^{-\alpha}f(t) - (t-a)^{-\alpha}f(a) - \alpha\int_a^t (t-s)^{-(\alpha+1)}f(s)ds\Big] \qquad (4) $$

respectively. Considering the first term in (3) and (4), we can see that both definitions loose sense. To avoid this issue, we propose the following definitions:

Definition 2.1. Suppose $\alpha, a, t \in \mathbb{R}$, with $\alpha > 0$ and $t > a$, then the new fractional derivative (Asumu fractional derivative in Riemann-Liouville sense) is given as:

$$ {}^{ARL}D^{\alpha}_{at}f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{d^n}{dt^n}\displaystyle\int_a^t (t-s)^{n-(1-\alpha)}f(s)ds, & n-1<\alpha<n \\ \dfrac{d^n}{dt^n}f(t), & \alpha = n \end{cases} $$

Definition 2.2. Suppose $\alpha, a, t \in \mathbb{R}$, with $\alpha > 0$ and $t > a$, then the new fractional derivative (Asumu fractional derivative in Caputo sense) is given as:

$$ {}^{AC}D^{\alpha}_{at}f(t) := \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-s)^{n-(1-\alpha)}f^{(n)}(s)ds, & n-1<\alpha<n \\ \dfrac{d^n}{dt^n}f(t), & \alpha = n \end{cases} $$

3 Basic properties of new fractional order derivatives

Before we establish the main properties of the new fractional derivatives, we present their Laplace transform formulas in the following two theorems:

Theorem 3.1. *Suppose that $n-1 < \alpha < n$ and f such that $f(0), f'(0), \cdots, f^{(n-1)}(0)$ exist. Then*

$$ \mathfrak{L}\{{}^{AC}D^{\alpha}_{0t}f(t)\} = \frac{\Gamma(n+\alpha)}{s^{n+\alpha}\cdot\Gamma(n-\alpha)}\Big\{s^n\mathfrak{L}\{f(t)\} - \sum_{i=0}^{n-1}s^{n-i-1}f^{(i)}(0)\Big\} \qquad (5) $$

Proof. From Definition 2.2 we have

$$ \mathfrak{L}\{{}^{AC}D^{\alpha}_{0t}f(t)\} = \frac{1}{\Gamma(n-\alpha)}\cdot\mathfrak{L}\Big\{\int_a^t (t-s)^{n-(1-\alpha)}f^{(n)}(s)ds\Big\} $$
$$ = \frac{1}{\Gamma(n-\alpha)}\cdot\mathfrak{L}\Big\{t^{n-(1-\alpha)} * f^{(n)}(t)\Big\} $$
$$ = \frac{1}{\Gamma(n-\alpha)}\cdot\mathfrak{L}\Big\{t^{n-(1-\alpha)}\Big\}\cdot\mathfrak{L}\Big\{f^{(n)}(t)\Big\} $$
$$ = \frac{\Gamma(n+\alpha)}{s^{n+\alpha}\cdot\Gamma(n-\alpha)}\cdot\Big\{s^n\cdot\mathfrak{L}\{f\} - \sum_{i=0}^{n-1}s^{n-1-i}\cdot f^{(i)}(0)\Big\} $$

as required. □

Theorem 3.2. *Suppose that $n-1<\alpha<n$. Then*

$$\mathfrak{L}\{^{ARL}D_{0t}^{\alpha}f(t)\} = \frac{1}{\Gamma(n-\alpha)} \cdot \left\{ \frac{\Gamma(n+\alpha)}{s^{\alpha}} \cdot \mathfrak{L}\{f\} - \sum_{i=0}^{n-1} s^{n-1-i} \cdot \frac{d^i}{dt^i}\left\{ \int_0^t (t-s)^{n-(1-\alpha)} f(s)ds \right\}(0) \right\} \tag{6}$$

Proof. By Definition 2.1 we have that

$$\begin{aligned}
\mathfrak{L}\{^{ARL}D_{0t}^{\alpha}f(t)\} &= \frac{1}{\Gamma(n-\alpha)} \cdot \mathfrak{L}\left\{ \frac{d^n}{dt^n} \int_0^t (t-s)^{n-(1-\alpha)} f(s)ds \right\} \\
&= \frac{1}{\Gamma(n-\alpha)} \cdot \left\{ s^n \cdot \mathfrak{L}\left\{ \int_0^t (t-s)^{n-(1-\alpha)} f(s)ds \right\} \right. \\
&\quad \left. - \sum_{i=0}^{n-1} s^{n-1-i} \cdot \frac{d^i}{dt^i}\left\{ \int_0^t (t-s)^{n-(1-\alpha)} f(s)ds \right\}(0) \right\} \\
&= \frac{1}{\Gamma(n-\alpha)} \cdot \left\{ \frac{s^n\Gamma(n+\alpha)}{s^{n+\alpha}} \cdot \mathfrak{L}\{f\} - \sum_{i=0}^{n-1} s^{n-1-i} \right. \\
&\quad \left. \cdot \frac{d^i}{dt^i}\left\{ \int_0^t (t-s)^{n-(1-\alpha)} f(s)ds \right\}(0) \right\}
\end{aligned}$$

This completes the proof □

The following two theorems permit us to write the new fractional derivatives in other way:

Theorem 3.3. *Let $n \in \mathbb{N}$, $a, \alpha \in \mathbb{R}$ such that $n-1<\alpha<n$ and $f(a), f'(a), \cdots, f^{(n-1)}(a)$ exist. Then*

$$\begin{aligned}
{}^{AC}D_{at}^{\alpha}f(t) = &-\frac{1}{\Gamma(n-\alpha)} \sum_{i=0}^{n-1} \frac{(\alpha+n-1)!}{(\alpha+n-1-i)!}(t-a)^{n-1+\alpha-i} f^{(n-1-i)}(a) \\
&+ \frac{1}{\Gamma(n-\alpha)} \cdot \left[\prod_{i=0}^{n-1} (i+\alpha) \right] \cdot \int_a^t (t-s)^{\alpha-1} f(s)ds
\end{aligned} \tag{7}$$

Proof. We will use the principle of mathematical induction. Let $P(n)$ be

$$\begin{aligned}
P(n) \equiv^{AC} D_{at}^{\alpha}f(t) \\
= &-\frac{1}{\Gamma(n-\alpha)} \sum_{i=0}^{n-1} \frac{(\alpha+n-1)!}{(\alpha+n-1-i)!}(t-a)^{n-1+\alpha-i} f^{(n-1-i)}(a) \\
&+ \frac{1}{\Gamma(n-\alpha)} \cdot \left[\prod_{i=0}^{n-1} (i+\alpha) \right] \cdot \int_a^t (t-s)^{\alpha-1} f(s)ds
\end{aligned}$$

For our base case, we need to show $P(1)$ is true, meaning that

$$
\begin{aligned}
{}^{AC}D^{\alpha}_{at}f(t) = \frac{1}{\Gamma(1-\alpha)}\Bigg[&- \sum_{i=0}^{1-1} \frac{(\alpha+0)!}{(\alpha+0-i)!}(t-a)^{0+\alpha-i} f^{(0-i)}(a) \\
&+ \left[\prod_{i=0}^{1-1}(i+\alpha)\right] \int_a^t (t-s)^{\alpha-1} f(s)ds \Bigg],
\end{aligned}
$$

This is trivial, since

$$
\begin{aligned}
{}^{AC}D^{\alpha}_{at}f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{\alpha} f^{(1)}(s)ds \\
&= \frac{1}{\Gamma(1-\alpha)}\left[-(t-a)^{\alpha} f(a) + \alpha \int_a^t (t-s)^{\alpha-1} f(s)ds \right] \\
&= \frac{1}{\Gamma(1-\alpha)}\Bigg[- \sum_{i=0}^{1-1} \frac{(\alpha+0)!}{(\alpha+0-i)!}(t-a)^{0+\alpha-i} f^{(0-i)}(a) \\
&\quad + \left[\prod_{i=0}^{1-1}(i+\alpha)\right] \int_a^t (t-s)^{\alpha-1} f(s)ds \Bigg],
\end{aligned}
$$

For the inductive step, assume that for some n, $P(n)$ holds, so

$$
\begin{aligned}
{}^{AC}D^{\alpha}_{at}f(t) &= \frac{1}{\Gamma(k-\alpha)} \int_a^t (t-s)^{k-(1-\alpha)} f^{(k)}(s)ds \\
&= \frac{1}{\Gamma(k-\alpha)}\Bigg[- \sum_{i=0}^{k-1} \frac{(\alpha+k-1)!}{(\alpha+k-1-i)!}(t-a)^{k-1+\alpha-i} f^{(k-1-i)}(a) \\
&\quad + \left[\prod_{i=0}^{k-1}(i+\alpha)\right] \cdot \int_a^t (t-s)^{\alpha-1} f(s)ds \Bigg]
\end{aligned} \tag{8}
$$

We need to show that $P(n+1)$ holds, meaning that

$$
\begin{aligned}
{}^{AC}D^{\alpha}_{at}f(t) = &-\frac{1}{\Gamma(n+1-\alpha)} \sum_{i=0}^{n} \frac{(\alpha+n)!}{(\alpha+n-i)!}(t-a)^{n+\alpha-i} f^{(n-i)}(a) \\
&+ \frac{1}{\Gamma(n+1-\alpha)} \cdot \left[\prod_{i=0}^{n}(i+\alpha)\right] \cdot \int_a^t (t-s)^{\alpha-1} f(s)ds
\end{aligned}
$$

To see this, note that

$$
\begin{aligned}
&{}^{AC}D_{at}^{\alpha}f(t)\\
&= \frac{1}{\Gamma(k+1-\alpha)}\int_a^t (t-s)^{k+1-(1-\alpha)} f^{(k+1)}(s)ds\\
&= \frac{1}{\Gamma(k+1-\alpha)}\Big[-(t-a)^{k+\alpha}f^{(k)}(a)+(k+\alpha)\int_a^t (t-s)^{k+\alpha-1}f^{(k)}(s)ds\Big]\\
&= \frac{1}{\Gamma(k+1-\alpha)}\Big\{-(t-a)^{k+\alpha}f^{(k)}(a)+(k+\alpha)\Big[-\sum_{i=0}^{k-1}\frac{(\alpha+k-1)!}{(\alpha+k-1-i)!}(t\\
&\quad -a)^{k-1+\alpha-i}f^{(k-1-i)}(a)+\Big[\prod_{i=0}^{k-1}(i+\alpha)\Big]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds\Big]\Big\}\\
&= \frac{1}{\Gamma(k+1-\alpha)}\Big[-(t-a)^{k+\alpha}f^{(k)}(a)-\sum_{i=1}^{k}\frac{(\alpha+k)!}{(\alpha+k-i)!}(t\\
&\quad -a)^{k+\alpha-i}f^{(k-i)}(a)+\Big[\prod_{i=0}^{k}(i+\alpha)\Big]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds\Big]\\
&= \frac{1}{\Gamma(k+1-\alpha)}\Big[-\sum_{i=0}^{k}\frac{(\alpha+k)!}{(\alpha+k-i)!}(t-a)^{k+\alpha-i}f^{(k-i)}(a)\\
&\quad +\Big[\prod_{i=0}^{k}(i+\alpha)\Big]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds\Big] \qquad (9)
\end{aligned}
$$

Thus $P(n+1)$ holds when $P(n)$ is true, so $P(n)$ is true for all natural numbers n. □

Theorem 3.4. *Let $n \in \mathbb{N}$, $a, \alpha \in \mathbb{R}$ such that $n-1 < \alpha < n$. Then*

$$
{}^{ARL}D_{at}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\cdot\Big[\prod_{i=0}^{n-1}(i+\alpha)\Big]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds \qquad (10)
$$

Proof. Let $P(n)$ be

$$
P(n) \equiv^{ARL} D_{at}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\cdot\Big[\prod_{i=0}^{n-1}(i+\alpha)\Big]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds.
$$

We will show, by induction, that $P(n)$ holds for all $n \in \mathbb{N}$. We note that

$$
\begin{aligned}
{}^{ARL}D_{at}^{\alpha}f(t) &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_a^t (t-s)^{\alpha}f(s)ds\\
&= \frac{1}{\Gamma(1-\alpha)}\cdot\alpha\int_a^t (t-s)^{\alpha-1}f(s)ds\\
&= \frac{1}{\Gamma(1-\alpha)}\Big[\prod_{i=0}^{1-1}(i+\alpha)\Big]\int_a^t (t-s)^{\alpha-1}f(s)ds.
\end{aligned}
$$

Thus, $P(1)$ is true. Asume that $P(n)$ is true for some natural number n, i.e.,

$$\begin{aligned} {}^{ARL}D^{\alpha}_{at}f(t) &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-(1-\alpha)}f(s)ds \\ &= \frac{1}{\Gamma(n-\alpha)}\cdot\left[\prod_{i=0}^{n-1}(i+\alpha)\right]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds \end{aligned}$$

We need to proof that $P(n+1)$ is true whenever $P(n)$ is true. We have

$$\begin{aligned} {}^{ARL}D^{\alpha}_{at}f(t) &= \frac{1}{\Gamma(n+1-\alpha)}\frac{d^{n+1}}{dt^{n+1}}\int_a^t (t-s)^{n+1-(1-\alpha)}f(s)ds \\ &= \frac{1}{\Gamma(n+1-\alpha)}\frac{d^n}{dt^n}\left[\frac{d}{dt}\int_a^t (t-s)^{n+\alpha}f(s)ds\right] \\ &= \frac{n+\alpha}{\Gamma(n+1-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-s)^{n-(1-\alpha)}f(s)ds \\ &= \frac{n+\alpha}{\Gamma(n+1-\alpha)}\cdot\left[\prod_{i=0}^{n-1}(i+\alpha)\right]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds \\ &= \frac{1}{\Gamma(n+1-\alpha)}\cdot\left[\prod_{i=0}^{n}(i+\alpha)\right]\cdot\int_a^t (t-s)^{\alpha-1}f(s)ds. \end{aligned}$$

Thus, we get what we want. Hence, by the principle of mathematical induction, $P(n)$ is true for all natural numbers n. □

The following theorem can therefore be established:

Theorem 3.5. *Let $n \in \mathbb{N}$, $a, \alpha \in \mathbb{R}$ such that $n-1 < \alpha < n$ and $f(a), f'(a), \cdots, f^{(n-1)}(a)$ exist. Then, the following relation is obtained*

$$ {}^{ARL}D^{\alpha}_{at}f(t) = {}^{AC}D^{\alpha}_{at}f(t) + \frac{1}{\Gamma(n-\alpha)}\sum_{i=0}^{n-1}\frac{(\alpha+n-1)!}{(\alpha+n-1-i)!}(t-a)^{n-1+\alpha-i}f^{(n-1-i)}(a). \quad (11)$$

Proof. In terms of (7) and (10), then it follows (11). □

The following two theorems characterize the well-posed of the new fractional derivatives:

Theorem 3.6. *Let f such that $I^{\alpha}_{at}f(t)$ exists, then the operator ${}^{ARL}D^{\alpha}_{at}f(t)$ is well-defined.*

Proof. The proof follows from Theorem 3.4. □

Theorem 3.7. *Let f such that $f(a), f'(a), \cdots, f^{(n-1)}(a)$ and $I^{\alpha}_{at}f(t)$ exist, then the operator ${}^{AC}D^{\alpha}_{at}f(t)$ is well-defined.*

Proof. The proof follows from Theorem 3.3. □

4 Conclusions

The aim of this paper was to suggest new fractional derivatives to improve theoretically the Riemann-Liouville and Caputo fractional derivatives. In this sense, one of the derivative is based upon the Riemann-Liouville viewpoint and the other one on the Caputo approach. Also we have given some properties of the proposed new fractional derivatives.

Conflict of Interest
The author declare that he has no conflict of interest.

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