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Lungs and blood oxygenation; Mathematical modeling

Oxigenación de pulmones y sangre; Modelo matemático

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Abstract

In this article, a general study is made about the lungs, their characteristics, their main function, detailing some aspects of the respiratory process; the main pulmonary diseases are indicated and how to prevent them. A model is made by means of a system of differential equations that simulates the blood oxygenation process, a qualitative study is made, and conclusions are given regarding the functioning in a healthy person; in the critical case of a zero and a negative eigenvalue, the system is reduced to the quasi-normal form to facilitate qualitative study.

Key words and phrases: Lung, qualitative study, mathematical model, breathing.

Abstract

En este artículo, se realiza un estudio general sobre los pulmones, sus características, su función principal, que detalla algunos aspectos del proceso respiratorio. Se indican las principales enfermedades pulmonares y cómo prevenirlas. Se elabora un modelo mediante un sistema de ecuaciones diferenciales que simula el proceso de oxigenación de la sangre, se realiza un estudio cualitativo y se dan conclusiones sobre el funcionamiento en una persona sana; en el caso crítico de un valor propio cero y uno negativo, el sistema se reduce a la forma casi normal para facilitar el estudio cualitativo.

Palabras y frases clave: Pulmón, estudio cualitativo, modelo matemático, respiración.

1 Introduction

The lungs are spongy and elastic organs formed by millions of air-filled alveoli. It is approximately 25 cm long and 700 g in weight. The right lung is larger in width than the left, as it has three lobes, one more than the left, but it is shorter in height, because on the right side the liver is present, causing the diaphragm to be higher. In the left lung there is a cardiac notch.

The lungs are attached to the pericardium through pulmonary ligaments and to the trachea and heart by structures called the hilum, comprising pulmonary vessels, lymphatic vessels, bronchial vessels, main bronchi and nerves that arrive and leave the lungs. The lungs are covered by a thin layer, the pleura that consists of a transparent and thin membrane. The inner pleura is attached to the pulmonary surface, and the outer pleura is attached to the wall of the rib cage. In the intermediate space of the pleura there is a small space, occupied by a lubricating liquid secreted by the pleura, this liquid is what holds the two pleuras together, due to surface tension, causing them to slide during breathing movements (cf. [6, 7]).

The base of each lung rests on the diaphragm, an organ that separates the chest from the abdomen, present only in mammals, promoting, together with intercostal muscles, respiratory movements. In the lungs, the bronchi ramify intensely, giving rise to increasingly thin tubes, the bronchioles. The highly branched set of bronchioles is the bronchial tree or respiratory tree.

In pulmonary breathing, air enters and leaves the lungs due to contraction and relaxation of the diaphragm. When the diaphragm contracts, it decreases the pressure in the lungs and the air outside the body enters rich in oxygen; process called inspiration. When the diaphragm relaxes, the pressure inside the lungs increases and the air that was inside now comes out with carbon dioxide; process called expiration.

People can stop breathing but no one can stop breathing for more than a few minutes, because the concentration of carbon dioxide in the blood gets so high that the body can no longer supply energy to the cells and the bulb, part of the nervous system that it forms the brain, sends nerve impulses to the diaphragm and intercostal muscles, so that they contract and breathing is resumed normally.

The person may suffer from different lung diseases such as, Bronchitis, Tuberculosis, Pulmonary emphysema, Pneumonia, Asthma, Lung cancer, etc. When inflammation occurs in an individual's lungs, more specifically in the alveoli, we call it pneumonia, due to infection caused by bacteria, viruses, fungi and other infectious agents. Pneumonia can cause death if left untreated; these diseases can damage the pulmonary alveoli, decreasing the lung's ability to perform its function (cf. [1, 2, 14, 16]).

The main purpose of the lungs is to supply our blood with oxygen, which is transported to the cells of the body. The other respiratory organs have the function of directing the air to the lungs, it is in them that the conversion of venous blood, blood low in oxygen and rich in carbon dioxide, into arterial blood, blood rich in oxygen occurs. When we breathe, we start a complex path, the air enters through the nose, or through the mouth, goes to the trachea following small tubes, the bronchi. From the bronchi, air is taken to other pulmonary regions; an involuntary movement that is controlled by the brain controls the entry and exit of air from the lungs.

Respiratory movement is controlled by a nerve center located in the spinal cord; under normal conditions, this center produces an impulse every 5 seconds, stimulating the contraction of the thoracic muscles and the diaphragm, where we inhale. However, when the blood becomes more acid due to the increase in carbon dioxide, the medullary respiratory center induces the acceleration of respiratory movements.

In the event of a decrease in the concentration of oxygen gas in the blood, the respiratory rate is also increased; this reduction is detected by chemical receptors located on the walls of the aorta and the carotid artery; however, when the air enters or leaves the organism through the mouth, however, the moistening and heating of the air is incomplete without the filtration of particles of dust, smoke, and even microscopic living beings, such as viruses and bacteria, capable of causing damage to our health. Some impurities are *filtered* in different organs of the respiratory system, but others are able to pass to the lungs, causing diseases.

Human beings have neurons in the bulb region that guarantee the regulation of breathing. The bulb perceives changes in the pH of the surrounding tissue liquid and triggers responses that guarantee changes in the respiratory rhythm. When carbon dioxide levels rise in the blood and cerebrospinal fluid, a drop in pH occurs. This happens due to the fact that the carbon dioxide present in these places can react with water and trigger the formation of carbonic acid.

The bulb then notices these changes, signals are sent to the intercostal muscles and diaphragm to increase the intensity and rate of breathing. When the pH returns to normal, there is a reduction in respiratory rate and intensity. It is worth noting that changes in the level of oxygen in the blood trigger few effects on the bulb. However, when the levels are very low, the breathing rate increases (cf. [3]).

Several works, books and articles related to processes in human life are known in which real problems are simulated by means of differential equations and systems of equations with the aim of being able to give conclusions about the processes, among others [5, 10, 12]. In cite 8 the authors simulate the process of polymer formation in the blood using autonomous systems of third and fourth order differential equations, giving conclusions about the formation of polymers and domains.

In [10] different problems of real life are treated by means of equations and systems of differential equations, all of them only in the autonomous case; where examples are further developed, and problems and exercises are placed so that they can be developed by the reader. The authors of [12] indicate a set of articles forming a collection of several problems that are modeled in different ways, but in general the qualitative and analytical theory of differential equations is used in both autonomous and non-autonomous cases.

A compartment system essentially consists of a finite number of interconnected subsystems, called compartments, which exchange between and with the environment, amount of concentration of materials or substances, each compartment is defined by its physical properties; in particular, the dynamics of a drug in the human body were treated; not all drugs have the same route, but in the ingestible case in [8], sufficient conditions are given for their elimination; the case of an inhalable drug is treated in [9] and injectable in [17], in all cases after the qualitative study of the system used in the modeling the future situation of the process is predicted.

Insulin is a hormone produced by the pancreas; its function is to act in the reduction of glycemia (blood glucose rate). It is responsible for the absorption of glucose by cells; when insulin-glucose dynamics are not natural in the human body, diabetes can be produced, this dynamics in both a normal and diabetic person is modeled in [4] and [13]. The case of tissue replacement is simulated in [11], the case of diabetic foot is treated.

Studies carried out in [13] have allowed the development of a mathematical model for the transmission of infectious diseases; this dynamic of contagion is modeled by means of sexual activity and, here, the concept of individuals susceptible to contagion with the disease is used.

The authors in [15], developed a mathematical model on the transmission of blennorrhagia, where they study exhaustively the behavior of the trajectories of the system that simulates the process in a neighborhood of the equilibrium points, offering conclusions regarding the future of the disease; giving a method for the identification of the coefficients, where there is a series of data corresponding to a given population.

2 Model formulation

In order to simulate the process of blood oxygenation through the lungs, it is necessary to take into account some basic principles regarding this process, firstly that the lungs do not provide more oxygen than our body needs, so in their variation will increase proportionally to the concentration of carbon dioxide and decrease proportionally to its own concentration; but in the variation of carbon dioxide, it is added proportionally to its concentration and decreased proportionally to the concentration of oxygen.

In order to formulate the model using a system of differentiable equations, the following variables will be introduced:
$\tilde{x}_1$ is the total concentration of oxygen in the lungs at the moment t.
$\tilde{x}_2$ is is the total concentration of carbon dioxide in the lungs at the moment t.
In addition, $\bar{x}_1$ and $\bar{x}_2$ the optimal values of oxygen and carbon dioxide in the lungs respectively.

Here the variables will be introduced x_1 and x_2 defined as follows:
$x_1 = \tilde{x}_1 - \bar{x}_1$ and $x_2 = \tilde{x}_2 - \bar{x}_2$ so if $\bar{x}_1 \to 0$ and $\bar{x}_2 \to 0$ the following conditions would be met $\tilde{x}_1 \to \bar{x}_1$ and $\tilde{x}_2 \to \bar{x}_2$, which would constitute the main objective of this work. So, the model will be given by the following system of equations

$$\begin{cases} x_1' = -a_1x_1 + a_2x_2 + X_1(x_1, x_2) \\ x_2' = -a_3x_1 + a_4x_2 + X_2(x_1, x_2) \end{cases} \tag{1}$$

Where the series $X_i(x_1, x_2)$, $i = 1, 2$ are disturbances of the system and have the following expressions:

$$X_i(x_1, x_2) = \sum_{|p|\geq 2} X_i^{(p)} x_1^{(p_1)} x_2^{(p_2)}, \quad |p| = p_1 + p_2, \quad (i = 1, 2).$$

The characteristic equation of the matrix of the linear part of the system (1) has the following form,

$$\operatorname{Det} \begin{pmatrix} -a_1 - \lambda & a_2 \\ -a_3 & a_4 - \lambda \end{pmatrix} = 0$$

This expression is equivalent to,

$$\lambda^2 + (a_1 - a_4)\lambda + a_2 a_3 - a_1 a_4 = 0 \tag{2}$$

In this case, applying the first approximation method, the following result is obtained.

Teorem 2.1. *The null solution of the system (1) is asymptotically stable if and only if the following conditions are met: $a_1 > a_4$ and $a_2 a_3 > a_1 a_4$, otherwise, it is unstable.*

The proof is a direct consequence of the conditions of the Hurwitz theorem, because in this case the main minors of the Hurwitz matrix are positive.

In this model it is absolutely possible the case where $a_2 a_3 = a_1 a_4$ and $a_1 > a_4$; this causes the matrix of the linear part of the system (1) to have a zero eigenvalue and a negative one; this constitutes a critical case, that is to say a case that cannot be solved by applying the method of first approximation. In that case, by means of a non-degenerate transformation $X = SY$, the system (1) can be transformed into the system,

$$\begin{cases} y_1' = Y_1(y_1, y_2) \\ y_2' = \lambda y_2 + Y_2(y_1, y_2) \end{cases} \tag{3}$$

Where $\lambda = -(a_1 + a_2) < 0$, as the system (3) constitutes a critical case, for which the first approximation method cannot be applied, in this case the second Lyapunov method will be applied once this system is reduced to the quasi-normal form.

Teorem 2.2. *The exchange of variables*

$$\begin{cases} y_1 = z_1 + h_1(z_1) + h^0(z_1, z_2) \\ y_2 = z_2 + h_2(z_1) \end{cases} \tag{4}$$

transforms the system (3) into a quasi-normal form,

$$\begin{cases} z_1' = Z_1(z_1) \\ z_2' = \lambda z_2 + Z_2(z_1, z_2) \end{cases} \tag{5}$$

Where $h^0(z_1, z_2)$ and $Z_2(z_1, z_2)$ cancel each other out $z_2 = 0$.

Proof. Deriving the transformation (4) along the trajectories of the systems (3) and (5) the system of equations is obtained,

$$\begin{cases} p_2 \lambda h^0(z_1, z_2) = Y_1(z_1, z_2) - \dfrac{dh_1}{dz_1} Z_1(z_1) - \dfrac{\partial h^0}{\partial z_1} Z_1(z_1) \dfrac{\partial h^0}{\partial z_2} Z_2(z_1, z_2) \\ \lambda h_2(z_1) + Z_2(z_1, z_2) = Y_2(z_1, z_2) - \dfrac{dh_2}{dz_1} Z_1(z_1) \end{cases} \tag{6}$$

To determine the series that intervene in the systems and the transformation, we will separate the coefficients from the powers of degree $p = (p_1, p_2)$ in the following two cases:
Case I) Making in the system (6) $z_2 = 0$, is to say for the vector $p = (p_1, 0)$ results the system.

$$\begin{cases} Z_1(z_1) = Y_1(z_1 + h_1(z_1), h_2(z_1)) - \dfrac{dh_1}{dz_1} Z_1(z_1) \\ \lambda h_2(z_1) = Y_2(z_1 + h_1(z_1), h_2(z_1)) - \dfrac{dh_2}{dz_1} Z_1(z_1) \end{cases} \tag{7}$$

The system (7) allows determining the series coefficients, $Z_1(z_1)$, $h_1(z_1)$ and $h_2(z_1)$, where for being the resonant case $h_1(z_1) = 0$, and the remaining series are determined in a unique way.
Case II) For the case when $z_2 \neq 0$ of the system (6) it follows that,

$$\begin{cases} p_2 \lambda h^0 = Y_1(z_1, z_2 + h_2(z_1)) - \dfrac{\partial h^0}{\partial z_1} Z_1(z_1) - \dfrac{\partial h^0}{\partial z_2} Z_2(z_1, z_2) \\ Y_2(z_1 + h_1(z_1), z_2 + h_2(z_1)) = Z_2(z_1, z_2) \end{cases} \tag{8}$$

Because the system series (5) are known expressions, the system (8) allows you to calculate the series $h^0(z_1, z_2)$ and $Z_2(z_1, z_2)$. This proves the existence of variable exchange. In system (5) the function $Z_1(z_1)$ admits the following development in power series:

$$Z_1(z_1) = \alpha z_1^s + \dots$$

Where α is the first non-zero coefficient and s is the corresponding power. □

Teorem 2.3. *If $\alpha < 0$ and s is odd, so the trajectories of the system (5) are asymptotically stable, otherwise they are unstable.*

Proof. Consider the Lyapunov function defined positive,

$$V(z_1, z_2) = \frac{1}{2}(z_1^2 + z_2^2)$$

The function $V(z_1, z_2)$ is such that its derivative along the trajectories of the system (5) has the following expression,

$$\alpha z_1^{s+1} + \lambda z_2^2 + R(z_1, z_2)$$

It can be seen that the derivative $V(z_1, z_2)$ is defined as negative, because with respect to z_1 in function $R(z_1, z_2)$ you only have terms of a degree greater than $s + 1$, however with respect z_2 the terms in this function are of a greater degree to the second. This coupled with that $V(z_1, z_2)$ is positive and guarantees the asymptotic stability of the null solution of the system (5). □

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