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Locally defined operators in the space of functions of bounded Λ-variation

Operadores localmente definidos en espacios de funciones de Λ-variación acotada

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Abstract

We prove that every locally defined operator mapping the space of continuous and bounded Λ-variation functions into itself is a Nemytskii composition operator.

Key words and phrases: Function of bounded Λ-variation, local operator, Nemytskii operator, continuous function.

Resumen

Se demuestra que cada operador localmente definido actuando entre espacios de funciones reales definidas en un intervalo de Λ-variación acotada es un operador de composición Nemytskii.

Palabras y frases clave: Funciones de Λ-variación acotada, operador local, operador Nemystkii, función continua.

1 Introduction

Let I be a closed interval of the real line $\mathbb{R}$ and let $\mathcal{G} = \mathcal{G}(I), \mathcal{H} = \mathcal{H}(I)$ be function spaces $f : I \longrightarrow \mathbb{R}$. An operator $K : \mathcal{G} \longrightarrow \mathcal{H}$ is called a *locally defined*, or $(\mathcal{G}, \mathcal{H})$-*local operator*, briefly, a *local operator*, if for every open interval $J \subset \mathbb{R}$ and for all functions $f, g \in \mathcal{G}$, the equality $f\big|_{J\cap I} = g\big|_{J\cap I}$ implies $K(f)\big|_{J\cap I} = K(g)\big|_{J\cap I}$ (where $f\big|_{J\cap I}$ denotes the restriction of f to $J \cap I$). The mappings of this type are sometimes called the *operators with memory*. The main result of this paper, Theorem 3.1, gives a representation formula for locally defined operators. We obtain the main result of [4] saying that the operator K must be of the for

$$K(f)(x) = h(x, f(x)), \quad x \in I,$$

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where h is uniquely determined function. In the present paper we are mainly interested in such that operators in the case $\mathcal{G} = \mathcal{H}$ is the class of continuous and Λ-bounded variation functions denoted by $C\Lambda BV(I)$.

In the spaces of the measurable functions, the acting of the locally defined operators were considered by many authors (cf. [2]). Local operators mapping the space $C^m(I)$ of m-times continuously differentiable functions in an interval $I \subset \mathbb{R}$ into $C^0(I)$ and $C^1(I)$ were considered in [4]. Subsequently, this result has been extended by several authors: [5], [6], [9] (for spaces Whitney differentiable functions), [7], [11] (for space of Hölder functions), [10] (for continuous and monotone functions), [12] (for functions of bounded φ-variation in the sense Wiener) and [3] (for functions of bounded p-variation in the sense Riesz). In the present paper we are interested on such operators in the context of bounded Λ-variation functions. In particular, we show that, if the operator K maps the space $\Lambda BV(I)$ into itself and is locally defined, then K is a Nemytskii composition operator.

2 Notation and Preliminaries

In this section we present some necessary notations and definitions and recall some known concerning the bounded Λ-variation.

In the sequel, $\mathbb{N}, \mathbb{N}_0, \mathbb{R}$ denote, respectively, the set of positive integers, nonnegative integers and the set of real numbers.

Let us start with the notion of Λ-bounded variation introduced by D. Waterman [8] in 1972. Let f be a real-valued function defined on an interval $I = [a, b] \subset \mathbb{R}$, $\{I_n\}$ be a sequence of non-overlapping intervals $I_n = [a_n, b_n] \subset I$ and let Λ denote a nondecreasing sequence of positive numbers λ_n such that $\displaystyle\sum_n \frac{1}{\lambda_n}$ is divergent. A function f is said to be Λ-bounded variation (ΛBV) if for every sequence $\{I_n\}$ we have

$$\sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} < \infty, \qquad \text{where} \quad f(I_n) = f(b_n) - f(a_n).$$

In the particular case when $\Lambda = \{n\}$, we say that f is of harmonic bounded variation. If f has a Λ-bounded variation, then the Λ-variation of f on an interval $[a, x]$ $(a \le x \le b)$ is defined in the following way

$$V_\Lambda(f) = V_\Lambda(f, [a, x]) := \sup\Big\{ \sum_{n=1}^{\infty} \frac{|f(I_n)|}{\lambda_n} \Big\},$$

where the supremum is taken over all sequences of non-overlapping intervals $\{I_n\}$ such that $\bigcup I_n \subseteq [a, x]$. We remark that ΛBV functions have some of the properties that BV (functions which have bounded variation in the sense of Jordan) have. For example, a ΛBV function is bounded and its discontinuities are at most denumerable ([8, p. 108]).

By $\Lambda BV(I) = \Lambda BV(I, \mathbb{R})$ we will denote the space of all functions $f : I \longrightarrow \mathbb{R}$ such that

$$V_\Lambda(f) = V_\Lambda(f, I) < \infty,$$

with the norm

$$\|f\|_\Lambda = |f(a)| + V_\Lambda(f), \quad f \in BV(I, \mathbb{R}),$$

$\Lambda BV(I)$ is a Banach space.

From now on, let $C\Lambda BV(I) = C(I) \bigcap \Lambda BV(I)$, where $C(I)$ stands for the space of continuous functions defined on I.

Lemma 2.1. *Let $I \subset \mathbb{R}$ be an interval and let $(x_0, y_0) \in I \times \mathbb{R},\ x_0 < \sup(I)$ be fixed. Then for every sequence $(x_k, y_k) \in I \times \mathbb{R}$ satisfying the condition*

$$x_{k+1} < x_k; \quad y_{k+1} < y_k,\ k \in \mathbb{N} \text{ and } \lim_{k \longrightarrow \infty} (x_k, y_k) = (x_0, y_0), \tag{2.1}$$

there exist a function $\gamma \in C\Lambda BV(I)$ such that, for all $k \in \mathbb{N}_0$,

$$\gamma(x_k) = y_k.$$

Proof. Take an arbitrary sequence $(x_k, y_k) \in I \times \mathbb{R}$ satisfying (2.1) and define a sequence of functions $\gamma_k : I \longrightarrow \mathbb{R},\ k \in \mathbb{N}$, by

$$\gamma_k(x) := \begin{cases} y_0 & \text{for} \quad x \in [a, x_0]; \\ \dfrac{y_k - y_0}{x_k - x_0}(x - x_0) + y_0 & \text{for} \quad x \in (x_0, x_k]; \\ \dfrac{y_i - y_{i-1}}{x_i - x_{i-1}}(x - x_i) + y_i & \text{for} \quad x \in (x_i, x_{i-1}], \quad i \in \{2, \dots, k\}; \\ y_1 & \text{for} \quad x \in (x_1, b]. \end{cases}$$

Let us observe that

$$\gamma_k(x_0) = y_0, \qquad \gamma_k(x_k) = \gamma_{k+\ell}(x_k) = y_k, \quad k, \ell \in \mathbb{N} \tag{2.2}$$

and for every $x \in I \backslash \{x_k : k \in \mathbb{N}_0\}$ there exist $k_0 \in \mathbb{N}$ such that

$$\gamma_k(x) = \gamma_{k_0}(x), \quad k \geq k_0,\ k \in \mathbb{N}. \tag{2.3}$$

Put

$$\gamma(x) = \lim_{k \longrightarrow \infty} \gamma_k(x), \quad x \in I. \tag{2.4}$$

From (2.2) and (2.3), the function γ is well defined. Moreover, γ is nondecreasing and

$$\gamma(x_k) = y_k, \quad \text{for all } \ k \in \mathbb{N}_0,$$

and by (2.4), for each $\epsilon > 0$, we obtain

$$|\gamma_k(x) - \gamma(x)| < \epsilon, \quad \text{for all } x \in I,$$

so $\|\gamma_k - \gamma\|_\infty \leq \epsilon$. Thus the sequence $(\gamma_k)_{k \in \mathbb{N}}$ tends uniformly to γ.

Since $\gamma_k \in C\Lambda BV(I)$ for all $k \in \mathbb{N}$, for any $\{I_n\},\ n = 1, \dots, N$, of non-overlapping intervals $[a_n, b_n] = I_n \subset I$, we get

$$\begin{aligned} \frac{|\gamma(b_n) - \gamma(a_n)|}{\lambda_n} &= \lim_{k \longrightarrow \infty} \frac{|\gamma_k(b_n) - \gamma_k(a_n)|}{\lambda_n} \\ &\leq \lim_{k \longrightarrow \infty} V_\Lambda(\gamma_k, I) < \infty, \end{aligned}$$

thus $V_\Lambda(\gamma, I) < \infty$ and therefore $\gamma \in C\Lambda BV(I)$. □

Remark 2.1. If $(x_0, y_0) \in I \times \mathbb{R}$ where $x_0 > \inf(I)$ and $(x_k, y_k) \in I \times \mathbb{R}$ is a sequence satisfying the condition

$$x_k < x_{k+1}; \quad y_k \leq y_{k+1}, \; k \in \mathbb{N} \; \text{ and } \; \lim_{k \longrightarrow \infty} (x_k, y_k) = (x_0, y_0),$$

then there exist a function $\gamma \in C\Lambda BV(I)$ such that, for all $k \in \mathbb{N}_0$,

$$\gamma(x_k) = y_k.$$

3 Locally defined operators

Now, we are in position to introduce the definition of the local defined operators of type $K : C\Lambda BV(I) \longrightarrow C(I)$.

Definition 3.1. An operator $K : C\Lambda BV(I) \longrightarrow C(I)$ is said to be locally defined, if for every two functions $f, \; g \in C\Lambda BV(I)$ and for every open interval $J \subset \mathbb{R}$,

$$f\big|_{J\cap I} = g\big|_{J\cap I} \quad \Rightarrow \quad K(f)\big|_{J\cap I} = K(g)\big|_{J\cap I}.$$

The main result reads as follows.

Theorem 3.1. *If a locally defined operator K maps $C\Lambda BV(I)$ into $C(I)$ then there exist a unique function $h : I \times \mathbb{R} \longrightarrow \mathbb{R}$ such that, for all $f \in C\Lambda BV(I)$,*

$$K(f)(x) = h(x, f(x)), \quad x \in I.$$

Proof. We begin by showing that for every $f, g \in C\Lambda BV(I)$ and for every $x_0 \in \mathrm{int}(I)$ the condition

$$f(x_0) = g(x_0) \tag{3.1}$$

implies that

$$K(f)(x_0) = K(g)(x_0).$$

To this end choose arbitrary $x_0 \in \mathrm{int}(I)$ and take an arbitrary pair of functions $f, g \in C\Lambda BV(I)$ which fulfil (3.1) (that is $f(x_0) = g(x_0)$). The function $\gamma : I \longrightarrow \mathbb{R}$ defined by

$$\gamma(x) = \begin{cases} f(x) & \text{for} \quad x \in [a, x_0]; \\ g(x) & \text{for} \quad x \in (x_0, b] \end{cases}$$

belongs to $C\Lambda BV(I)$. Indeed, define the functions $f_1, g_1 : I \longrightarrow \mathbb{R}$ by

$$f_1(x) = \begin{cases} f(x) - f(x_0) & \text{for} \quad x \in [a, x_0]; \\ 0 & \text{for} \quad x \in (x_0, b] \end{cases}$$

and

$$g_1(x) = \begin{cases} 0 & \text{for} \quad x \in [a, x_0]; \\ g(x) - g(x_0) & \text{for} \quad x \in (x_0, b]. \end{cases}$$

Since $f, g \in C\Lambda BV(I)$, f, g are continuous on I, $V_\Lambda(f) < \infty$ and $V_\Lambda(g) < \infty$. For any $\{I_n\}$, $n = 1, \ldots, N$, of non-overlapping intervals $[a_n, b_n] = I_n \subset I$ with $a_{n_0} < x_0 < b_{n_0}$ for some $1 < n_0 < N$, we have

$$\begin{aligned}\sum_{i=1}^{N} \frac{|f_1(b_i) - f_1(a_i)|}{\lambda_i} &\le \sum_{i=1}^{n_0-1} \frac{|f_1(b_i) - f_1(a_i)|}{\lambda_i} \\ &\quad + \frac{|f_1(x_0) - f_1(a_{n_0})|}{\lambda_{n_0}} + \frac{|f_1(b_{n_0}) - f_1(x_0)|}{\lambda_{n_0'}} \\ &\le V_\Lambda(f).\end{aligned}$$

Hence $V_\Lambda(f_1) < \infty$. By a similar reasoning, we have $V_\Lambda(g_1) < \infty$. It is clear that $f_1, g_1 \in C(I)$. Finally $f_1 + g_1 \in C\Lambda BV(I)$, as $C\Lambda BV(I)$ is a linear space. Thus

$$V_\Lambda(f_1 + g_1) < \infty. \tag{3.2}$$

Since, for all interval $\{I_n\} \subset I$

$$(f_1 + g_1)(b_n) - (f_1 + g_1)(a_n) = \gamma(b_n) - \gamma(a_n),$$

the condition (3.2) implies that $\gamma \in C\Lambda BV(I)$.

As

$$f\big|_{(-\infty,x_0)\cap I} = \gamma\big|_{(-\infty,x_0)\cap I} \quad \text{and} \quad g\big|_{(x_0,\infty)\cap I} = \gamma\big|_{(x_0,\infty)\cap I},$$

by definition of a local operator, we get

$$K(f)\big|_{(-\infty,x_0)\cap I} = K(\gamma)\big|_{(-\infty,x_0)\cap I} \quad \text{and} \quad K(g)\big|_{(x_0,\infty)\cap I} = K(\gamma)\big|_{(x_0,\infty)\cap I}.$$

Therefore, by the continuity of $K(f)$, $K(g)$ and $K(\gamma)$ at x_0, we obtain

$$K(f)(x_0) = K(\gamma)(x_0) = K(g)(x_0).$$

Suppose now that x_0 is the left endpoint of the interval I (i.e., $x_0 = a$). There exist a sequence $(x_k, y_k) \in I \times \mathbb{R}$ such that: $x_0 < x_{k+1} < x_k$, $y_0 \le y_{k+1} < y_k$, $k \in \mathbb{N}$, and by the continuity of f and g at x_0

$$\lim_{k \longrightarrow \infty} (x_k, y_k) = (x_0, y_0).$$

By Lemma 2.1 there exist a function $\gamma \in C\Lambda BV(I)$ such that $\gamma(x_k) = y_k$ for all $k \in \mathbb{N}_0$.

There is no loss of generality in supposing that

$$f(x_0) = g(x_0) = y_0, \ \gamma(x_{2k-1}) = y_{2k-1} = g(x_{2k-1})$$

and

$$\gamma(x_{2k}) = y_{2k} = f(x_{2k}), \quad k \in \mathbb{N}.$$

According to the first part of the proof, we have

$$K(\gamma)(x_{2k-1}) = K(g)(x_{2k-1}) \ \text{ and } \ K(\gamma)(x_{2k}) = K(f)(x_{2k}), \quad k \in \mathbb{N}.$$

Hence, by continuity of $K(\gamma),\ K(g)$ and $K(f)$ at x_0, letting $k \longrightarrow \infty$, we get

$$K(f)(x_0) = K(\gamma)(x_0) = K(g)(x_0).$$

When x_0 is the right endpoint of I, the argument is similar.

To define the function $h : I \times \mathbb{R} \longrightarrow \mathbb{R}$, fix arbitrarily an $y_0 \in \mathbb{R}$, let us define a function $P_{y_0} : I \longrightarrow \mathbb{R}$ by

$$P_{y_0}(x) := y_0, \quad x \in I. \tag{3.3}$$

Of course P_{y_0}, as a constant function, belongs to $C\Lambda BV(I)$. For $x_0 \in I,\ y_0 \in \mathbb{R}$, put

$$h(x_0, y_0) = K(P_{y_0})(x_0).$$

Since, by (3.3), for all functions f,

$$f(x_0) = P_{f(x_0)}(x_0),$$

according to what has already been proved, we have

$$K(f)(x_0) = K(P_{f(x_0)})(x_0) = h(x_0, f(x_0)). \tag{3.4}$$

To prove the uniqueness of h, assume that $\overline{h} : I \times \mathbb{R} \longrightarrow \mathbb{R}$ is such that

$$K(f)(x) = \overline{h}(x, f(x))$$

for all $f \in C\Lambda BV(I)$ and $x \in I$. To show that $h = \overline{h}$ let us fix arbitrarily $x \in I,\ y \in \mathbb{R}$ and take $f \in C\Lambda BV(I)$ with $f(x) = y$. From (3.4), we have

$$\overline{h}(x, y) = \overline{h}(x, f(x)) = K(f)(x) = h(x, f(x)) = h(x, y),$$

which proves the uniqueness of h. □

Definition 3.2. Let $X \subset \mathbb{R}$ and a function $h : X \times \mathbb{R} \longrightarrow \mathbb{R}$ be fixed. The mapping $H : \mathbb{R}^X \longrightarrow \mathbb{R}^X$ given by

$$H(f)(x) := h(x, f(x)), \quad f \in \mathbb{R}^X, \quad x \in X,$$

is said to be composition (Nemytskii or superposition) operator. The function h is referred to as the generator of the operator H.

As an immediate consequence of Theorema 3.1 we get

Corollary 3.1. *If a local operator K maps $C\Lambda BV(I)$ into $C(I)$, then it is a Nemytskii operator.*

Note, that if a local operator K maps $C\Lambda BV(I)$ into itself then, obviously, K maps $C\Lambda BV(I)$ into $C(I)$. Therefore, by Theorem 3.1, we get

Theorem 3.2. *If a local operator K maps $C\Lambda BV(I)$ into itself, then there exist a unique function $h : I \times \mathbb{R} \longrightarrow \mathbb{R}$ such that, for all $f \in C\Lambda BV(I)$,*

$$K(f)(x) = h(x, f(x)), \quad x \in I.$$

Corollary 3.2. *If a local operator K maps $C\Lambda BV(I)$ into itself, then it is a Nemytskii operator.*

Under the additional assumption that the locally defined operator is uniformly continuous, we get a complete characterization of its generating function h. Namely, we have the following

Theorem 3.3. *If a local operator $K : C\Lambda BV(I) \longrightarrow C\Lambda BV(I)$ is uniformly continuous, then there exist $f_1, f_2 \in C\Lambda BV(I)$ such that*

$$K(f)(x) = f_1(x)f(x) + f_2(x), \quad f \in C\Lambda BV(I), \quad x \in I.$$

Proof. From Theorem 3.2 there exist a unique function $h : I \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $K(f)(x) = h(x, f(x))$ for all $f \in C\Lambda BV(I),\ x \in I$. Fix $(x_0, y_0) \in I \times \mathbb{R}$, take an arbitrary sequence $x_n \in I$ with $x_n \longrightarrow x_0$ and let $P_{y_0} : I \longrightarrow \mathbb{R}$ defined by $P_{y_0}(x) = y_0,\ x \in I$. Since $h(x_0, y_0) = K(P_{y_0})(x_0)$,

$$\begin{aligned} \big|h(x_n, y_0) - h(x_0, y_0)\big| &= \big|h(x_n, P_{y_0}(x_n)) - h(x_0, P_{y_0}(x_0))\big| \\ &= \big|K(P_{y_0})(x_n)) - K(P_{y_0})(x_0))\big|, \end{aligned}$$

applying the continuity of $K(P_{y_0})$ at x_0, we get the continuity of h with respect to the first variable. Thus, by [1, Theorem 1],

$$h(x, y) = f_1(x)y + f_2(x), \quad x \in I, \quad y \in \mathbb{R},$$

for some $f_1, f_2 : I \longrightarrow \mathbb{R}$.

Since $h(\cdot, y_0) = K(P_{y_0})(\cdot) \in C\Lambda BV(I)$ and $f_2(x) = h(x, 0), \quad f_1(x) = h(x, 1) - f_2(x)$, the functions $f_1, f_2 \in C\Lambda BV(I)$. □

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