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On the C-trace pseudospectrum in the matrix algebra

Sobre el pseudoespectro de C-traza en el álgebra matricial

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Abstract

In this paper we discuss theoretical properties of the C-trace pseudospectrum for an element in the matrix algebra. We also make several observations on the C-trace pseudospectrum.

Key words and phrases: Pseudospectrum, condition pseudospectrum, trace pseudospectrum.

Resumen

En este artículo discutimos las propiedades teóricas del pseudoespectro de C-traza para un elemento en el álgebra matricial. También hacemos varias observaciones sobre el pseudoespectro de C-trazas.

Palabras y frases clave: Pseudoespectro, condición pseudoespectro, pseudoespectro de traza.

1 Introduction

Let $\mathcal{M}_n(\mathbb{C})$ ($\mathcal{M}_n(\mathbb{R})$) denote the algebra of all $n \times n$ complex (real) matrices and by $\mathcal{U}_n(\mathbb{C})$ the group of all unitary matrices in $\mathcal{M}_n(\mathbb{C})$. I denotes the $n \times n$ identity matrix and the conjugate transpose of T is denoted by T^*. Let $T \in \mathcal{M}_n(\mathbb{C})$, then the eigenvalues of the matrix T is denoted by $\sigma(T)$ and is defined as

$$\sigma(T) = \Big\{\lambda \in \mathbb{C} : \ \lambda I - T \text{ is not invertible}\Big\},$$

and its spectral radius by

$$r(T) = \sup\Big\{|\lambda| : \lambda \in \sigma(T)\Big\}.$$

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Now, let $\lambda \in \mathbb{C}$ and

$$\mathrm{s}_n(\lambda I - T) \leq \ldots \leq \mathrm{s}_2(\lambda I - T) \leq \mathrm{s}_1(\lambda I - T)$$

be the singular values of the matrix $\lambda I - T$ where $\mathrm{s}_1(\lambda I - T)$ is the smallest and $\mathrm{s}_n(\lambda I - T)$ is largest singular values of the matrix. For an $n \times n$ complex matrix T and a non-negative real number ε, the pseudospectrum of the matrix T is defined as the following closed set in the complex plane

$$\sigma_\varepsilon(T) = \Big\{\lambda \in \mathbb{C} : \ \mathrm{s}_n(\lambda I - T) \leq \varepsilon\Big\}.$$

Let $T \in \mathcal{M}_n(\mathbb{C})$ and $0 < \varepsilon < 1$. The condition pseudospectrum of the matrix T is denoted by $\Sigma_\varepsilon(T)$ and is defined as

$$\Sigma_\varepsilon(T) = \Big\{\lambda \in \mathbb{C} : \mathrm{s}_n(\lambda I - T) \leq \varepsilon \ \mathrm{s}_1(\lambda I - T)\Big\}.$$

For more information on various details on the above concepts, properties and applications of pseudospectrum [1, 7], condition spectrum [2, 3] and the interested reader may consult the remarkable books [5, 6]. In [4], A. Ammar, A. Jeribi and K. Mahfoudhi defined the notion of trace pseudospectrum for an element in the matrix algebra $\mathcal{M}_n(\mathbb{C})$, for every $T \in \mathcal{M}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$ by

$$\mathrm{Tr}_\varepsilon(T) = \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda I - T)| \leq \varepsilon\big\},$$

where $\mathrm{Tr}(\cdot)$ denotes the trace of a matrix.

In this paper, we are interested in another generalization of eigenvalues called C-trace pseudospectrum for an element in the matrix algebra to give more information about matrix T. Let $C \in \mathcal{M}_n(\mathbb{C})$, $U \in \mathcal{U}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$ and $\varepsilon > 0$. Then, the C-trace pseudospectrum of $T \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\mathrm{Tr}_\varepsilon^C(T)$ and is defined as

$$\mathrm{Tr}_\varepsilon^C(T) = \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(CU(\lambda I - T)U^*)| \leq \varepsilon\big\}.$$

The C-trace pseudoresolvent of T is denoted by $\mathrm{Tr}^{\mathrm{C}}\rho_\varepsilon(T)$ and is defined as

$$\mathrm{Tr}^{\mathrm{C}}\rho_\varepsilon(T) = \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(CU(\lambda I - T)U^*)| > \varepsilon\big\}$$

while the C-trace pseudospectral radius of T is defined as

$$\mathrm{Trr}_\varepsilon^C(T) := \sup\Big\{|\lambda| : \lambda \in \mathrm{Tr}_\varepsilon^C(T)\Big\}.$$

Remark 1.1. Let $T, C \in \mathcal{M}_n(\mathbb{C})$ and $U \in \mathcal{U}_n(\mathbb{C})$. Then, for $C = U = U^* = I$, the C-trace pseudospectrum coincides with the trace pseudospectrum, i.e., $\mathrm{Tr}_\varepsilon^C(T) = \mathrm{Tr}_\varepsilon(T)$.

The C-trace pseudospectrum of a matrix may be used to draw surprisingly strong conclusions about the spectrum and the combinatorial structure of a matrix. In this paper, we will develop some results on the C-trace pseudospectrum for an element in the matrix algebra, which directly relate its shape to intrinsic properties of the matrix, and thus provide means of detecting such properties.

2 C-trace pseudospectrum.

We begin by a simple example in which the C-trace pseudospectrum can be obtained analytically:

Example 2.1. *Let $\varepsilon > 0$ and consider the matrices*

$$T = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} \in \mathcal{M}_2(\mathbb{C}), \quad C = \begin{pmatrix} \alpha_1 & \gamma_1 \\ 0 & \beta_1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{C})$$

and

$$U = \begin{pmatrix} 0 & e^{i\phi} \\ -e^{-i\phi} & 0 \end{pmatrix} \in \mathcal{U}_2(\mathbb{C})$$

where, $\alpha, \beta, \gamma, \alpha_1, \beta_1$ and $\gamma_1 \in \mathbb{C}$. Now, we compute that

$$CU(\lambda I - T)U^* = \begin{pmatrix} -\gamma\gamma_1 e^{-i\phi} - \alpha_1(\lambda - \beta) & \gamma(\lambda - \alpha) \\ -\gamma\beta_1 e^{-2i\phi} & \beta_1(\lambda - \alpha) \end{pmatrix}.$$

Thus

$$\begin{aligned} \mathrm{Tr}_\varepsilon^C(T) &= \big\{\lambda \in \mathbb{C} : |-\gamma\gamma_1 e^{-i\phi} - \alpha_1(\lambda - \beta) + \beta_1(\lambda - \alpha)| \leq \varepsilon\big\} \\ &= \big\{\lambda \in \mathbb{C} : \big||\gamma\gamma_1| - |\alpha_1(\lambda - \beta)| - |\beta_1(\lambda - \alpha)|\big| \leq \varepsilon\big\}. \end{aligned}$$

Consequently, $\gamma = 0$ $(\gamma_1 = 0)$ if and only if

$$\mathrm{Tr}_\varepsilon^C(T) = \big\{\lambda \in \mathbb{C} : |\alpha_1(\lambda - \beta)| + |\beta_1(\lambda - \alpha)| \leq \varepsilon\big\}.$$

If $\gamma_1 = 0$, $\alpha = \beta$ and $\alpha_1 = \beta_1$, then

$$\mathrm{Tr}_\varepsilon^C(T) = \left\{\lambda \in \mathbb{C} : |\lambda - \alpha| \leq \frac{\varepsilon}{2|\alpha_1|}\right\}.$$

The following properties of the C-trace pseudospectrum are easy to check from the definition of the C-trace pseudospectrum.

Theorem 2.1. *Let $T, C \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,*

(1) $\mathrm{Tr}_0^C(T) = \bigcap_{\varepsilon > 0} \mathrm{Tr}_\varepsilon^C(T)$.

(2) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\mathrm{Tr}_{\varepsilon_1}^C(T) \subset \mathrm{Tr}_{\varepsilon_2}^C(T)$.*

(3) *$\mathrm{Tr}_\varepsilon^C(T)$ is a non-empty compact subset of $\mathbb{C}$.*

(4) *If $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\{0\}$. Then, $\mathrm{Tr}_\varepsilon^C(\beta T + \alpha I) = \beta \mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}^C(T) + \alpha$.*

(5) *$\mathrm{Tr}_\varepsilon^C(\alpha I) = \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(C)||\lambda - \alpha| \leq \varepsilon\big\}$ for all $\lambda, \alpha \in \mathbb{C}$.*

(6) *$\mathrm{Tr}_\varepsilon^C(UTU^*) = \mathrm{Tr}_\varepsilon^C(T)$, for all unitary (resp. anti-unitary) U on $\mathcal{M}_n(\mathbb{C})$.*

(7) *$\mathrm{Tr}_\varepsilon^C(T^*) = \overline{\mathrm{Tr}_\varepsilon^C(T)}$.*

Proof. The first two items, (6) and (7) can be immediately checked from the definitions of C-trace pseudospectrum, so we only include the proof of item (3), (4) and (5).

(3) Using the continuity from $\mathbb{C}$ to $[0,\infty[$ of the map

$$\lambda \to |\mathrm{Tr}(CU(\lambda I - T)U^*)|,$$

we get that $\mathrm{Tr}_\varepsilon^C(T)$ is a compact set in the complex plane containing the eigenvalues of T.

(4) For $C \in \mathcal{M}_n(\mathbb{C})$ and $U \in \mathcal{U}_n(\mathbb{C})$, we have

$$\begin{aligned}
\mathrm{Tr}_\varepsilon^C(\beta T + \alpha I) &= \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(CU(\lambda I - \beta T - \alpha I)U^*)| \le \varepsilon\Big\} \\
&= \Big\{\lambda \in \mathbb{C} : |\beta|\Big|\mathrm{Tr}\Big(CU(\frac{\lambda-\alpha}{\beta}I - T)U^*\Big)\Big| \le \varepsilon\Big\} \\
&= \Big\{\lambda \in \mathbb{C} : \Big|\mathrm{Tr}\Big(CU(\frac{\lambda-\alpha}{\beta}I - T)U^*\Big)\Big| \le \frac{\varepsilon}{|\beta|}\Big\}.
\end{aligned}$$

Then, $\lambda \in \mathrm{Tr}_\varepsilon^C(\beta T + \alpha I)$. Hence, $\dfrac{\lambda-\alpha}{\beta} \in \mathrm{Tr}^C_{\frac{\varepsilon}{|\beta|}}(T)$. Thus, $\lambda \in \beta \mathrm{Tr}^C_{\frac{\varepsilon}{|\beta|}}(T) + \alpha$.

(5) Let $\lambda \in \mathrm{Tr}_\varepsilon^C(\alpha I)$, then

$$\begin{aligned}
|\mathrm{Tr}(CU(\lambda I - \alpha I)U^*)| &= |\lambda - \alpha||\mathrm{Tr}(CUU^*)| \\
&= |\mathrm{Tr}(C)||\lambda - \alpha| \\
&\le \varepsilon.
\end{aligned}$$

which yields $\mathrm{Tr}_\varepsilon^C(\alpha I) = \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(C)||\lambda - \alpha| \le \varepsilon\big\}$ for all $\lambda, \alpha \in \mathbb{C}$. ☐

Next, we give characterization of the C-trace pseudospectrum $\mathrm{Tr}_\varepsilon^C(\cdot)$.

Theorem 2.2. *Let $T, C \in \mathcal{M}_n(\mathbb{C})$, $U \in \mathcal{U}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If there is $D \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(CUDU^*)| \le \varepsilon$ and $\mathrm{Tr}(CU(\lambda I - T - D)U^*) = 0$ if, and only if $\lambda \in \mathrm{Tr}_\varepsilon^C(T)$.*

Proof. The "if " part. We assume that there exists $D \in \mathcal{M}_n(\mathbb{C})$ such that

$$|\mathrm{Tr}(CUDU^*)| \le \varepsilon \quad \text{and} \quad \mathrm{Tr}(CU(\lambda I - T - D)U^*) = 0$$

for all $C \in \mathcal{M}_n(\mathbb{C})$, $U \in \mathcal{U}_n(\mathbb{C})$ and $\lambda \in \mathbb{C}$. Then, for all $C \in \mathcal{M}_n(\mathbb{C})$ and $U \in \mathcal{U}_n(\mathbb{C})$, we have

$$|\mathrm{Tr}(CU(\lambda I - T)U^*)| = |\mathrm{Tr}(CUDU^*)| \le \varepsilon.$$

Thus, $\lambda \in \mathrm{Tr}_\varepsilon^C(T)$.

The "only if " part. Suppose $\lambda \in \mathrm{Tr}_\varepsilon^C(T)$. There are two cases:

<u>1^{st} case</u> : If $\lambda \in \mathrm{Tr}_0^C(T)$, then it is sufficient to take the matrix zero ($D = 0_{n\times n}$).

<u>2^{nd} case</u> : If $\lambda \in \mathrm{Tr}_\varepsilon^C(T)\backslash\mathrm{Tr}_0^C(T)$. Then, for all $C \in \mathcal{M}_n(\mathbb{C})$ and $U \in \mathcal{U}_n(\mathbb{C})$,

$$|\mathrm{Tr}(CU(\lambda I - T)U^*)| \le \varepsilon.$$

We now consider a square matrix D by

$$D = \frac{\mathrm{Tr}(CU(\lambda I - T)U^*)}{\mathrm{Tr}(C)}\ I$$

where, C is not a scalar matrix and $\mathrm{Tr}(C) \neq 0$. Then, D is well defined and, as is easily verified, $D \in \mathcal{M}_n(\mathbb{C})$ and

$$\begin{aligned} |\mathrm{Tr}(CUDU^*)| &= \left|\mathrm{Tr}\Big(CU\Big(\frac{\mathrm{Tr}(CU(\lambda I - T)U^*)}{\mathrm{Tr}(C)}\ I\Big)U^*\Big)\right| \\ &= \frac{|\mathrm{Tr}(CU(\lambda I - T)U^*)|}{|\mathrm{Tr}(C)|}\ |\mathrm{Tr}(CUIU^*)| \leq \varepsilon. \end{aligned}$$

Also, we have

$$\mathrm{Tr}(CU(\lambda I - T - D)U^*) = \mathrm{Tr}\Big(CU\big(\lambda I - T - \frac{\mathrm{Tr}(CU(\lambda I - T)U^*)}{\mathrm{Tr}(C)}\ I\big)U^*\Big) = 0.$$

So, the proof is complete. □

Theorem 2.3. *$T, C \in \mathcal{M}_n(\mathbb{C})$, and $\varepsilon > 0$. Then,*

$$\mathrm{Tr}^C_\delta(T) + \mathcal{O}_\varepsilon \subseteq \mathrm{Tr}^C_{\varepsilon+\delta}(T), \tag{1}$$

holds for $\delta, \varepsilon > 0$ with $\mathcal{O}_\varepsilon$, denoting the closed disk in the complex plane centered at the origin with radius $\frac{\varepsilon}{|\mathrm{Tr}(C)|}$. If we take $\delta = 0$, we obtain an inner bound for $\mathrm{Tr}^C_\varepsilon(T)$, namely

$$\mathrm{Tr}^C_0(T) + \mathcal{O}_\varepsilon \subseteq \mathrm{Tr}^C_\varepsilon(T). \tag{2}$$

Proof. Let $\lambda \in \mathrm{Tr}^C_\delta(T) + \mathcal{O}_\varepsilon$. Then, there exists $\lambda_1 \in \mathrm{Tr}^C_\delta(T)$ and $\lambda_2 \in \mathcal{O}_\varepsilon$ such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$|\mathrm{Tr}(CU(\lambda_1 I - T)U^*)| \leq \delta \text{ and } |\lambda_2| \leq \frac{\varepsilon}{|\mathrm{Tr}(C)|}$$

for all $C \in \mathcal{M}_n(\mathbb{C})$ and $U \in \mathcal{U}_n(\mathbb{C})$. Now, we have for all $C \in \mathcal{M}_n(\mathbb{C})$ and $U \in \mathcal{U}_n(\mathbb{C})$ that

$$\begin{aligned} |\mathrm{Tr}(CU(\lambda I - T)U^*)| &= |\mathrm{Tr}((CU(\lambda_1 + \lambda_2)I - T)U^*)| \\ &= |\mathrm{Tr}(CU\lambda_2 U^* + CU(\lambda_1 - T)U^*)| \\ &\leq |\lambda_2||\mathrm{Tr}(CUU^*)| + |\mathrm{Tr}(CU(\lambda_1 I - T)U^*)| \\ &\leq |\mathrm{Tr}(C)||\lambda_2| + |\mathrm{Tr}(CU(\lambda_1 I - T)U^*)| \\ &\leq \varepsilon + \delta, \end{aligned}$$

so that (1) holds. Finally, let $\delta = 0$, then the desired inclusion (2) is obtained. □

Theorem 2.4. *Let T, B and $C \in \mathcal{M}_n(\mathbb{C})$ such that $TB = BT$ and $\varepsilon > 0$. If T is normal (i.e. $T^*T = TT^*$), then*

$$\mathrm{Tr}^C_\varepsilon(T + B) \subseteq \sigma(T) + \mathrm{Tr}^C_\varepsilon(B).$$

Proof. Let T is normal, so there exists a unitary matrix $Z \in \mathcal{M}_n(\mathbb{C})$ such that

$$Z^*TZ = \lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2} \oplus \ldots \oplus \lambda_k I_{n_k}.$$

The condition $TB = BT$ implies that

$$Z^*BZ = T_1 \oplus T_2 \ldots \oplus T_k$$

where, $T_i \in \mathcal{M}_{n_k}(\mathbb{C})$, $i = 1, \ldots, k$. From Property (4) and (6) in Theorems 2.1 we obtain that

$$\begin{aligned} \mathrm{Tr}^C_\varepsilon(T+B) &= \mathrm{Tr}^C_\varepsilon(Z^*TZ + Z^*BZ) \\ &= \mathrm{Tr}^C_\varepsilon((\lambda_1 I_{n_1} + T_1) \oplus \ldots \oplus (\lambda_k I_{n_k} + T_k)) \\ &= \bigcup_{i=1}^{k} \mathrm{Tr}^C_\varepsilon(\lambda_i I_{n_i} + T_i) \\ &= \bigcup_{i=1}^{k} \lambda_i + \mathrm{Tr}^C_\varepsilon(T_i) \\ &\subseteq \sigma(T) + \mathrm{Tr}^C_\varepsilon(B). \end{aligned}$$

This is what we wanted to prove. □

Now, the following should be obvious.

Corollary 2.1. *If $B = 0_{n\times n}$, then*

$$\mathrm{Tr}^C_\varepsilon(T) \subseteq \sigma(T) + \left\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{\varepsilon}{\mathrm{Tr}(C)}\right\}.$$

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