Divulgaciones Matemáticas Vol. 20, No. 2 (2019), pp. 45–62

Composition operators from Sobolev spaces into Lebesgue spaces

Operadores de composición desde espacios de Sobolev en espacios de Lebesgue

Wadie Aziz (wadie@ula.ve)
Universidad de los Andes
Departamento de Física y Matemáticas
Trujillo-Venezuela

Abstract

In this paper, we shall obtain a compactness of weighted Sobolev embeddings and use it to get a composition operators from Sobolev spaces into Lebesgue spaces. Applying these results we shall study the multiplicity for singular asymptotically linear p–Laplacian problems.

Key words and phrases: Sobolev embeddings, composition operators, eigenvalues, p–Laplacian, Multiplicity of solutions.

Resumen

En este artículo, obtenemos una compacidad de inmersiones de Sobolev ponderadas y lo usamos para tener operadores de composición del espacio de Sobolev en espacios de Lebesgue. Aplicando estos resultados estudiaremos la multiplicidad para problemas p–laplacianos.

Palabras y frases clave: Inmersión en espacio de Sobolev, operadores de composición, autovalores, p–laplaciano, multiplicidad de soluciones.

1 Introduction

The compactness of Sobolev embeddings in [1, 5] have been extended in [10, 21, 23]. The results in [21] are very general but not quite convenient to be applied to study partial differental equations. In this paper we obtain a result on the compactness of weighted Sobolev embeddings (see Theorem 2.1). Our results include cases of Poincaré-Sobolev's embeddings and Hardy–Sobolev's embeddings.

The Composition Operators studied in [4, 8, 12, 15, 17, 20, 22, 25] usually act between spaces of same types, for example: Bounded variation spaces, Lebesgue spaces, Sobolev spaces and Orlicz-Sobolev spaces. In this paper, we prove results on Nemytskii operators from Sobolev spaces into Lebesgue spaces (see Theorems 2.2 and 2.3). This idea has appeared in [14]. Our results fit to study problem in partial differential equations. For example, we can study the multiplicity of asymptotically p–laplacian problem (4.1) with singular condition (4.3). The embbeding in [14] may be not compact, therefore it is not convenient to study the asymptotically p–laplacian

Received 21/07/2019. Revised 30/10/2019. Accepted 18/12/2019.
MSC (2010): Primary 46E35; Secondary 35J20; 35P30.
Corresponding author: Wadie Aziz

problem nor the multiplicity of solutions. We get the compactness of the embeddings in Theorem 2.2.

We prove the weighted Sobolev embeddings and the Nemytskii Operators from Sobolev spaces into Lebesgue spaces in section 2, extending the results presented in [4, 12, 15, 19] for measurable functions $\omega \in \Omega$. In the third section we improve by theorem 3.2 some results on eigenvalues of p–laplacian in [3, 6]. Finally, we apply these esults to study the multiplicity of asymptotically p–laplacian problem in the fourth section, illustrating by an example (Example 4.1) the application of Main Theorem 4.1.

2 A weighted Sobolev embeddings and Nemytskii operators

Let N be an integer ≥ 3, Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and r be in the interval $[1, N)$. Let $W_0^{1,p}(\Omega)$ be the usual Sobolev space with the following norm

$$\|u\|_{1,p} = \left\{ \int_\Omega |\nabla u|^p dx \right\}^{\frac{1}{p}} \qquad \forall u \in W_0^{1,p}(\Omega).$$

Let σ be a measurable function on Ω. We put

$$T_\sigma(v) = \sigma v \qquad \forall v \in W_0^{1,r}(\Omega).$$

We have the following result.

Theorem 2.1. *Let s be in $\left[1, \frac{Nr}{N}\right)$. α be in $(0,1)$, ω and θ be measurable functions on Ω such that $|\theta| \leq |\omega|^\alpha$ and T_ω is a continuous mapping from $W^{1,r}(\Omega)$ into $L^s(\Omega)$. Then T_θ is a linear compact continuous mapping from $W^{1,r}(\Omega)$ into $L^s(\Omega)$.*

Proof. Since T_ω is linear and continuous from $W^{1,r}(\Omega)$ into $L^s(\Omega)$, there is a positive real number C and

$$\left\{ \int_\Omega |u|^s |\omega|^s dx \right\}^{\frac{1}{s}} \leq C\|u\|_{1,r} \qquad \forall u \in W_0^{1,r}(\Omega). \tag{2.1}$$

Thus T_θ is a linear and continuous mapping from $W^{1,r}(\Omega)$ to $L^s(\Omega)$. Let M be a positive real number and $\{u_n\}$ be a sequence in $W_0^{1,r}(\Omega)$, such that $\|u_n\|_{1,r} \leq M$ for any n. By Rellich–Kondrachov's theorem (Theorem 9.16 in [5]), $\{u_n\}$ has a subsequence $\{u_{n_k}\}$ converging to u in $L^s(\Omega)$ and $\{u_{n_k}\}$ converging weakly to u in $W_0^{1,r}(\Omega)$, therefore $\|u\|_{1,r} \leq \liminf_{k\to+\infty} \|u_{n_k}\|_{1,r} \leq M$. We shall prove $\{T_\theta(u_{n_k})\}$ converges to $T_\theta(u)$ in $L^s(\Omega)$.

Let ε be a positive real number. Choose a positive real number μ such that

$$(2CM)^s \mu^{(1-\alpha)s} < \frac{\varepsilon^s}{2}. \tag{2.2}$$

Put $\Omega' = \{x \in \Omega : \omega(x) > \mu\}$. We have by (2.1) and (2.2)

$$\begin{aligned}
\int_\Omega |u_{n_k} - u|^s |\theta|^s dx &\leq \int_{\Omega'} |u_{n_k} - u|^s |\omega|^{\alpha s} dx + \int_{\Omega\setminus\Omega'} |u_{n_k} - u|^s |\omega|^{\alpha s} dx \\
&\leq \mu^{(1-\alpha)s} \int_{\Omega'} |u_{n_k} - u|^s |\omega|^{\alpha s} dx + \mu^{\alpha s} \int_{\Omega\setminus\Omega'} |u_{n_k} - u|^s dx \\
&\leq \mu^{(1-\alpha)s} \int_{\Omega} |u_{n_k} - u|^s |\omega|^{\alpha s} dx + \mu^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx \\
&\leq \mu^{(1-\alpha)s} (C\|u_{n_k} - u\|_{1,r})^s + \mu^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx \\
&\leq \mu^{(1-\alpha)s} (2CM)^s + \mu^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx \\
&\leq \frac{\varepsilon^s}{2} + \mu^{\alpha s} \int_{\Omega} |u_{n_k} - u|^s dx. \qquad (2.3)
\end{aligned}$$

Since $\{u_{n_k}\}$ converges in $L^s(\Omega)$, there is an integer k_0 such that

$$\int_\Omega |u_{n_k} - u|^s dx \leq \mu^{-\alpha s} \frac{\varepsilon^s}{2} \qquad \forall k \geq k_0. \qquad (2.4)$$

Combining (2.3) and (2.4), we get the theorem. □

Remark 2.1. Let $\rho(x)$ be the distance from x to the boundary $\partial\Omega$ of Ω for any x in Ω. If $s = r$, $\omega = \rho^{-1}$ and $\theta = \rho^{-\alpha}$ with α in $(0, 1)$, the inequality (2.1) is Theorem 8.4 in [16], and (ii) of Theorem 2.1 is proved in [23]. We have Poincaré–Sobolev embeddings in this case.

If 0 is in Ω, $s = r$ and $\omega(x) = \dfrac{1}{|x|}$ for any x in Ω, we have Hardy's embeddings (see [5]). There are other examples of ω in [10, 16, 17, 19, 21, 23].

Theorem 2.2. *Let s and ω be as in Theorem 2.1 and β be in $(0, 1]$. Let t be in $[1, +\infty)$ and g be a Caratheodory function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$. Assume that there are a positive real number c, a real number β in $(0, 1]$ and a function $b \in L^t(\Omega)$; such that*

$$|g(x, z)| \leq c|\omega(x)|^{\frac{\beta s}{t}} |z|^{\frac{s}{t}} + b(x) \qquad \forall (x, z) \in \Omega \times \mathbb{R}. \qquad (2.5)$$

Put

$$N_g(v)(x) = g(x, v(x)) \qquad \forall v \in W_0^{1,r}(\Omega),\, x \in \Omega. \qquad (2.6)$$

We have

i) N_g is a continuous mapping from $W_0^{1,r}(\Omega)$ into $L^t(\Omega)$.

ii) If A is a bounded subset in $W_0^{1,r}(\Omega)$, then $N_g(A)$ is bounded in $L^t(\Omega)$.

iii) If A is a bounded subset in $W_0^{1,r}(\Omega)$ and $\beta < 1$, then $\overline{N_g(A)}$ is compact in $L^t(\Omega)$.

Proof. i) Put

$$g_1(x, \zeta) = g(x, |\omega(x)|^{-\beta}\zeta) \qquad \forall (x, \zeta) \in \Omega \times \mathbb{R}.$$

By (2.5), we have

$$|g_1(x, \zeta)| \leq c|\zeta|^{\frac{s}{t}} + b(x) \qquad \forall (x, \zeta) \in \Omega \times \mathbb{R}. \qquad (2.7)$$

On the other hand

$$N_g(v) = N_{g_1} \circ T_{|\omega|^\beta}(v) \qquad v \in W^{1,r}(\Omega). \tag{2.8}$$

Applying Theorem 2.3 in [12] and Theorem 2.1, we get the theorem. □

Theorem 2.3. *Let g be as in Theorem 2.2 with $s \in \left(1, \dfrac{Nr}{N-r}\right)$ and $t = \dfrac{s}{s-1}$. Put*

$$\Psi_g(u) = \int_\Omega \int_0^{u(x)} g(x,\xi) d\xi dx \qquad \forall u \in W_0^{1,r}(\Omega). \tag{2.9}$$

We have

i) Ψ_g *is continuously Fréchet differentiable mapping from* $W_0^{1,r}(\Omega)$ *into* $\mathbb{R}$ *and*

$$D\Psi_g(u)(\phi) = \int_\Omega g(x,\xi)\phi dx \qquad \forall u, \phi \in W_0^{1,r}(\Omega). \tag{2.10}$$

ii) If A is a bounded subset in $W_0^{1,r}(\Omega)$, then there is a positive real number M such that

$$|\Psi_g(v)| + \|D\Psi_g(v)\| \le M \qquad \forall v \in A.$$

Proof. Let g_1 be as in the proof of Theorem 2.2. Put

$$\Psi_{g_1}(u) = \int_\Omega \int_0^{u(x)} g_1(x,\xi) d\xi dx \qquad \forall u \in W_0^{1,r}(\Omega).$$

By Theorem 2.8 in [12], then Ψ_{g_1} is continuously Fréchet differentiable mapping from $L^s(\Omega)$ into $\mathbb{R}$. We see that $\Psi_g = \Psi_{g_1} \circ T_{|\omega|^\beta}$. By Theorem 2.1, we get the theorem. □

Remark 2.2. If $\omega = 1$, Theorems 2.2 and 2.3 were proved in [4, 12, 15, 19].

3 Eigenvalues for p–laplacian

Let $r = s = p$ be in $[2, N)$, ω and α be as in Theorem 2.1 such that $\omega \neq 0$. Put $\sigma = |\omega|^{\alpha p}$. By Theorem 2.1, there is a real number λ such that

$$\lambda = \inf \left\{ \int_\Omega |\nabla u|^p dx : \, u \in W_0^{1,r}(\Omega), \, \int_\Omega \sigma |u|^p dx = 1 \right\}. \tag{3.1}$$

We have the following result.

Theorem 3.1. *There are v_1 and v_2 in $W_0^{1,p}(\Omega)$ such that $v_1 \ge 0$ and $v_2 \le 0$ and*

$$\int_\Omega |\nabla v_i|^{p-2} \nabla v_i \nabla \phi dx = \lambda \int_\Omega \sigma |v_i|^{p-2} v_i \phi dx \qquad \forall i = 1, 2; \; \varphi \in W_0^{1,r}(\Omega). \tag{3.2}$$

Proof. Put

$$\begin{aligned} E(u) &= \int_\Omega |\nabla u|^p dx \qquad \forall u \in W_0^{1,r}(\Omega), \\ S(u) &= \int_\Omega \sigma |u|^p dx \qquad \forall u \in W_0^{1,r}(\Omega), \\ S_1(u) &= \int_\Omega |u|^p dx \qquad \forall u \in W_0^{1,r}(\Omega), \\ M &= \left\{ u \in W_0^{1,r}(\Omega) : \ S_1(u) = 1 \right\}. \end{aligned}$$

By Theorem 2.1, M is weakly closed in $W_0^{1,p}(\Omega)$. We see that E is coercive and weakly lower semi–continuous on $W_0^{1,p}(\Omega)$. By Theorem 1.2 in [24], there is w in M such that

$$E(w) = \min\{E(u) : \ u \in M\}. \tag{3.3}$$

Hence

$$\int_\Omega |\nabla w|^p dx = \lambda \int_\Omega \sigma |w|^p dx. \tag{3.4}$$

Put $v_1 = |w|$ and $v_2 = -|w|$. By Lemma 7.6 in [13], v_1 and v_2 belong to $W_0^{1,p}(\Omega)$ and $E(v_1) = E(v_2) = E(w)$. By Theorem 9 in [7], E is continuously Fréchet differentiable on $W^{1,p}(\Omega)$. Since S_1 is continuously Fréchet differentiable on $W^{1,p}(\Omega)$ and $S = S_1 \circ T_{|w|^\alpha}$, by Theorem 2.1, S is continuously Fréchet differentiable on $W^{1,p}(\Omega)$. Thus by Theorem 43.D in [26], there are real numbers λ_1' and λ_2' such that

$$DE(v_i)\varphi = \lambda_i' DS(v_i)\varphi \qquad \forall i = 1, 2; \ \varphi \in W_0^{1,r}(\Omega) \ \text{ or}$$

$$\int_\Omega |\nabla v_i|^{p-2} \nabla v_i \cdot \nabla \varphi dx = \lambda_i' \int_\Omega \sigma |v_i|^{p-2} v_i \varphi dx \qquad \forall i = 1, 2; \ \varphi \in W_0^{1,r}(\Omega). \tag{3.5}$$

Taking $\varphi = v_i$ in (3.5), we see that $\lambda_i' = \lambda$ for any $i = 1, 2$ and we get the theorem.

□

Put $\gamma(\sigma) = \lambda$ and $\phi(\sigma) = \|v_1\|^{-1} v_1$ where θ, λ and v_1 are as in Theorem 2.3. If $\sigma = 1$, we denote $\gamma(\sigma)$ by λ_1. We have the following results.

Theorem 3.2. *Let α be in $(0, 1)$, σ_1 and σ_2 be two measurable functions on such that $0 < \sigma_1 < \sigma_2$ and $\sigma_2^p < |\omega|^{\alpha p}$. Then $0 < \gamma(\sigma_2) \leq \gamma(\sigma_1)$.*

Proof. By Poincaré's inequality, $\int_\Omega |\nabla u|^p dx > 0$ when $\int_\Omega \sigma_2 |u|^p dx = 1$. Thus by the proof of Theorem 2.3, we get $0 < \gamma(\sigma_2)$.

Since $\int_\Omega \sigma_1 |u|^p dx \leq \int_\Omega \sigma_2 |u|^p dx$, we see that $\gamma(\sigma_2) \leq \gamma(\sigma_1)$.

□

Remark 3.1. If σ, σ_1 and σ_2 are in $L^s(\Omega)$ with some s in $\left(\frac{N}{p}, +\infty\right)$, Theorems 2.3 and 3.2 were proved in [3, 6].

4 Multiplicity of asymtotically linear p–laplacian problems

Let p be in $[2, N)$ and q be in $(1, p)$, f be the real Caratheodory function on $\Omega \times \mathbb{R}$, and h be a real measurable function on Ω. We consider the following p–Laplacian problem:

$$\begin{cases} -\Delta_p u = h(x)|u|^{q-2}u + f(x,u) \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{4.1}$$

Our main result is

Theorem 4.1. *Suppose*

(F1) There exist $\alpha \in (0,1)$, $\alpha' = \alpha^{\frac{p-1}{p}}$, $C \in (0, +\infty)$, a measurable real function V_0 on Ω, positive measurable real functions V_1, V_2 and η on Ω such that $T_{|\eta|}$ is a continuous mapping from $W_0^{1,p}(\Omega)$ into $L^p(\Omega)$ and

$$\left(\int_\Omega |\eta v|^p dx\right)^{\frac{1}{p}} \leq C\|v\|_{1,p} \qquad \forall v \in W_0^{1,p}(\Omega). \tag{4.2}$$

$$|f(x,s)| \leq \eta^{\alpha(p-1)}(x)|s|^{p-1} = \eta^{\alpha' p}(x)|s|^{p-1} \qquad \forall x \in \Omega, \forall s \in \mathbb{R}. \tag{4.3}$$

$$\lim_{s\to 0} \frac{f(x,s)}{|s|^{p-2}s} = V_0(x) \qquad \forall x \in \Omega. \tag{4.4}$$

$$\lim_{s\to -\infty} \frac{f(x,s)}{|s|^{p-2}s} = V_1(x) \qquad \forall x \in \Omega. \tag{4.5}$$

$$\lim_{s\to +\infty} \frac{f(x,s)}{|s|^{p-2}s} = V_2(x) \qquad \forall x \in \Omega. \tag{4.6}$$

(V1) There exists a positive constant c_0 such that

$$\int_\Omega (|\nabla u|^p - V_0|u|^p)\, dx \geq c_0\|u\|_{1,p}^p \qquad \forall u \in W_0^{1,p}(\Omega). \tag{4.7}$$

(V2) $V_i \not\equiv 0$ and $0 < \gamma(V_i) < 1$ for any $i = 1, 2$.

(V3) $V_0 \geq 0$ for e.a. $x \in \Omega$.

(H1) $|h|^{\frac{p}{p-q}}\eta^{-\frac{\alpha pq}{p-q}}$ is integrable on Ω.

(H2) There is a sufficiently small positive number τ such that $\left\||h|^{\frac{p}{p-q}}\eta^{-\frac{\alpha pq}{p-q}}\right\|_{L^1} \leq \tau$.

Then the problem (4.1) has at least two non-trivial solutions u_1 and u_2 such that $u_1 \geq 0$; $u_2 \leq 0$; $J(u_1) > 0$ and $J(u_2) < 0$, where

$$F(x,s) = \int_0^s f(x,t)dt \qquad \forall (x,s) \in \Omega \times \mathbb{R} \text{ and} \tag{4.8}$$

$$J(u) = \int_\Omega \frac{1}{p}|\nabla u|^p dx - \frac{1}{q}\int_\Omega h(x)|u|^q dx - \int_\Omega F(x,u)dx. \tag{4.9}$$

Corollary 4.1. *Theorem 4.1 still holds if we replace (V3) and (H2) by the following condition:*

(H3) $h \leq 0$ *a.e on* Ω.

Corollary 4.2. *Corollary 4.1 still holds if we replace (V1) and (H3) by the following conditions*

(V1')

$$\int_\Omega (|\nabla u|^p - V_0|u|^p)\,dx \geq 0 \qquad \forall u \in W_0^{1,p}(\Omega).$$

(H4) $h < 0$ *a.e on* Ω.

Theorem 4.2. *Suppose that conditions (F1), (V1), (V2), (V3), (H1) and (H2) hold. Assume also that (H5) There exists* $v \in W_0^{1,p}(\Omega)$ *such that*

$$\int_\Omega h(x)|v^+|^q dx > 0.$$

Then the problem (4.1) has at least four non-trivial solutions u_1, u_2, u_3 *and* u_4 *such that* $u_1 \geq 0$, $u_2 \geq 0$, $u_3 \leq 0$, $u_4 \leq 0$ *and* $J(u_1) > 0$, $J(u_3) > 0$, $J(u_2) < 0$ *and* $J(u_4) < 0$.

To prove these results we need the following notations. Let u be in $W_0^{1,p}(\Omega)$, F and J be as in (4.8) and (4.9). Put

$$u^+ = \max\{u, 0\}, \qquad u^- = \min\{u, 0\},$$

$$J^\pm = \int_\Omega \frac{1}{p}|\nabla u|^p dx - \frac{1}{q}\int_\Omega h(x)|u^\pm|^q dx - \int_\Omega F(x, u^\pm)dx. \tag{4.10}$$

By Theorem 9 in [7], (F1) and Theorem 2.2 (with $r = s = p$, $t = \frac{p}{p-1}$ and $\beta = \alpha$), the functionals J and $J^\pm$ belong to $C^1(W_0^{1,p}(\Omega), \mathbb{R})$. Moreover, for every u and v in $W_0^{1,p}(\Omega)$,

$$\langle DJ(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx - \int_\Omega h(x)|u|^{q-2}uv dx - \int_\Omega f(x, u)v dx \tag{4.11}$$

$$\langle DJ^+(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx - \int_\Omega h(x)(u^+)^{q-2}u^+ v dx - \int_\Omega f(x, u^+)v dx \tag{4.12}$$

$$\langle DJ^-(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx - \int_\Omega h(x)|u^-|^{q-2}u^- v dx - \int_\Omega f(x, u^-)v dx \tag{4.13}$$

We have the following lemmas.

Lemma 4.1. *Under (F1), (V2) and (H1), the functionals* $J^\pm$ *satisfy the Palais–Smale condition.*

Proof. We prove J^+ satisfies the Palais–Smale condition, the other case is similar. Let $\{u_n\}$ be a sequence in $W_0^{1,p}(\Omega)$. Assume $\lim_{n\to+\infty} DJ^+(u_n) = 0$ and $|J^+(u_n)| \leq M$ for a real number M. We shall prove that $\{u_n\}$ has a convergent subsequence in $W_0^{1,p}(\Omega)$. By (4.10) and (4.12), we obtain

$$\left|\int_\Omega \frac{1}{p}|\nabla u_n|^p dx - \frac{1}{q}\int_\Omega h(x)(u_n^+)^q dx - \int_\Omega F(x, u_n^+)dx\right| \leq M, \tag{4.14}$$

$$\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \Delta v dx - \int_\Omega h(x)(u_n^+)^{q-2}u_n^+ v dx - \int_\Omega f(x, u_n^+)dx\right| \\ \leq \|DJ^+(u_n)\|\|v\|_{1,p} \qquad \forall v \in W_0^{1,p}(\Omega). \tag{4.15}$$

First, we prove $\{u_n^-\}$ is bounded, indeed, take $v = u_n^-$ in (4.15) we have

$$\|u_n^-\|_{1,p}^p = \left|\int_\Omega \frac{1}{p}|\nabla u_n^-|^p dx\right| \leq \|DJ^+(u_n)\|\|u_n^-\|_{1,p}.$$

Hence $\|u_n^-\|_{1,p}^{p-1} \leq \|DJ^+(u_n)\| \longrightarrow 0$, or $\|u_n^-\|_{1,p} \longrightarrow 0$.

Now we prove the boundedness of $\{u_n^+\}$. Suppose by contradiction that $\{u_n^+\}$ is unbounded, then there exists a subsequence of $\{u_n\}$ (also denoted by $\{u_n\}$) such that $a_n = \|u_n^+\|_{1,p} > 0$ for any integer n and $\lim_{n\to+\infty} a_n = +\infty$. By (4.12), we have

$$\begin{aligned}
&\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx - \int_\Omega h(x)(u_n^+)^{q-2}u_n^+ v dx - \int_\Omega f(x,u_n^+)dx\right| \\
&\quad \leq \|DJ^+(u_n)\|\|v\|_{1,p} + \left|\int_\Omega |\nabla u_n^-|^{p-2}\nabla u_n^- \nabla v dx\right| \\
&\quad \leq \|DJ^+(u_n)\|\|v\|_{1,p} + \left(\int_\Omega |\nabla u_n^-|^{(p-1)p'}dx\right)^{\frac{1}{p'}} \left(\int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}} \\
&\qquad \leq \|DJ^+(u_n)\|\|v\|_{1,p} + \|u_n^-\|_{1,p}^{p-1}\|v\|_{1,p} \qquad \forall v \in W_0^{1,p}(\Omega).
\end{aligned} \tag{4.16}$$

Put $w_n = \dfrac{u_n^+}{a_n}$. Divide both sides by a_n^{p-1}, we obtain

$$\begin{aligned}
&\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla v dx - \int_\Omega h(x)(u_n^+)^{q-2}u_n^+ v dx - \int_\Omega f(x,u_n^+)dx\right| \\
&\qquad \leq \frac{\|DJ^+(u_n)\|\|v\|_{1,p}}{a_n^{p-1}} + \frac{1}{a_n^{p-1}}\|u_n^-\|_{1,p}^{p-1}\|v\|_{1,p}.
\end{aligned} \tag{4.17}$$

Note that $\|w_n\|_{1,p} = 1$. By Theorem 9.16 and Theorem 4.4 in [5], (4.2), (4.3) and Theorem 2.1, there exist a subsequence of $\{w_n\}$ (also denoted by $\{w_n\}$), a function $w \in W_0^{1,p}(\Omega)$ and $k \in L^1(\Omega)$ such that:

- (i) $\{w_n\}$ weakly (resp. strongly, pointwisely) converges to w in $W_0^{1,p}(\Omega)$ (resp. in $L^p(\Omega)$, on Ω),
- (ii) $\{w_n\eta^\alpha\}$ strongly (resp. pointwisely) converges to $w\eta^\alpha$ in $L^p(\Omega)$ (resp. on Ω), and
- (iii) $|w_n|^p + |w_n|^p\eta^{\alpha p} \leq k$.

By (H1), we have

$$\begin{aligned}
&\left|\int_\Omega h(x)|w_n(x)|^{q-1}v(x)dx\right| = \left|\int_\Omega h(x)\eta^{\alpha q}|w_n(x)|^{q-1}v(x)\eta^\alpha dx\right| \\
&\leq \left(\int_\Omega |h(x)|^{\frac{p}{p-q}}\eta^{-\frac{\alpha pq}{p-q}}dx\right)^{\frac{p-q}{p}} \left(\int_\Omega k_1 dx\right)^{\frac{q-1}{p}} \left(\int_\Omega |v(x)|^p\eta^{\alpha p}dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$\lim_{n\to+\infty} a_n^{q-p}\int_\Omega h(x)|w(x)|^{q-1}v(x)dx = 0. \tag{4.18}$$

By (4.3)

$$\begin{aligned}
\left|\frac{f(x,u_n^+(x))}{a_n^{p-1}}v(x)\right| &= \begin{cases} 0 & \text{if } u_n^+(x) = 0 \\ \left|\dfrac{f(x,u_n^+(x))}{u_n^+(x)^{p-1}}w_n(x)^{p-1}v(x)\right| & \text{if } u_n^+(x) \neq 0 \end{cases} \\
&\leq \eta^{\alpha(p-1)}(x)|w_n(x)|^{p-1}|v(x)| \leq k^{\frac{p-1}{p}}|v(x)|.
\end{aligned} \tag{4.19}$$

Let $\Omega^0 = \{x \in \Omega : \ w(x) = 0\}$ and $\Omega^+ = \{x \in \Omega : \ w(x) > 0\}$. Since $w(x) \geq 0$ for all $x \in \Omega$, result that $\Omega = \Omega^0 \cup \Omega^+$. Let x be in Ω^+, we have $\lim_{n\to+\infty} u_n(x) = +\infty$. By (4.6) and (4.19), we have

$$\lim_{n\to+\infty} \frac{f(x, u_n^+(x))}{a_n^{p-1}} v(x) = \begin{cases} 0 & \text{if } w(x) = 0 \\ V_2(x) w(x)^{p-1} v(x) & \text{if } w(x) \neq 0. \end{cases}$$

By (4.6), (4.3) and (4.2), we have

$$\int_\Omega |V_2 w^{p-1} v| dx \leq \int_\Omega \eta^{\alpha(p-1)} w^{p-1} |v| dx \leq \left(\int_\Omega |\eta^\alpha w|^p dx \right)^{\frac{p-1}{p}} \left(\int_\Omega |v|^p dx \right)^{\frac{1}{p}} < \infty.$$

Since $k \in L^1(\Omega)$ and $v \in L^p(\Omega)$, by Hölder's inequality, $k^{\frac{p-1}{p}} v$ belongs to $L^1(\Omega)$. Applying Lebesgue's Dominated Convergence Theorem, we have

$$\lim_{n\to+\infty} \int_\Omega \frac{f(x, u_n^+(x))}{a_n^{p-1}} v(x) dx = \int_\Omega V_2(x) w(x)^{p-1} v(x) dx \qquad \forall v \in W_0^{1,p}(\Omega). \tag{4.20}$$

We shall prove that $\{w_n\}$ converges to w in $W_0^{1,p}(\Omega)$. Let T be the operator $-\Delta_p$ from $W_0^{1,p}(\Omega)$ into $W_0^{1,p'}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. By Theorem 10 in [8], T is of class S_+, that is, $\{\psi_n\}$ strongly converges to ψ in $W_0^{1,p}(\Omega)$ if $\{\psi_n\}$ weakly converges to ψ in $W_0^{1,p}(\Omega)$ and $\limsup_{m\to\infty} T(\psi_m)(\psi_m - \psi) \leq 0$. Note that

$$T(\phi)(\theta) = \int_\Omega |\nabla\phi|^{p-2} \nabla\phi \nabla\theta dx, \qquad \forall \phi, \theta \in W_0^{1,p}(\Omega).$$

Using a similar argument as above for $w_m - w$ instead of v, we obtain the following results

$$\begin{aligned} &\Bigg| \int_\Omega |\nabla w_m|^{p-2} \nabla w_m \nabla (w_m - w) dx \\ &- \frac{1}{a_n^{p-q}} \int_\Omega h(x) |w_m|^{q-1} (w_m - w) dx - \int_\Omega \frac{f(x, u_m^+)}{|u_m^+|^{p-1}} w_m^{p-1} (w_m - w) dx \Bigg| \\ &\leq \frac{1}{a_n^{p-1}} \|DJ^+(u_m)\| \|w_m - w\|_{1,p} + \\ &+ \frac{1}{a_n^{p-1}} \|u_m^-\|_{1,p}^{p-1} \|w_m - w\|_{1,p}, \ \forall n \in \mathbb{N}, \end{aligned} \tag{4.21}$$

$$\lim_{m\to+\infty} \frac{1}{a_n^{p-q}} \int_\Omega h(x) |w_m|^{q-1} (w_m - w) dx = 0 \tag{4.22}$$

$$\lim_{m\to+\infty} \int_\Omega \frac{f(x, u_m^+)}{|u_m^+|^{p-1}} w_m^{p-1} (w_m - w) dx = 0. \tag{4.23}$$

Since $\{\|w_m - w\|_{1,p}\}$ is bounded and $\|u_m^-\|_{1,p} \longrightarrow 0$, by (4.22), (4.23) and (4.21), we have

$$\limsup_{m\to+\infty} T(w_m)(w_m - w) \leq 0,$$

which implies that $\{w_m\}$ strongly converges to w in $W_0^{1,p}(\Omega)$ and thus

$$\lim_{m\to+\infty} \int_\Omega |\nabla w_m|^{p-2} \nabla w_m \nabla v dx = \int_\Omega |\nabla w|^{p-2} \nabla w \nabla v dx \quad \forall v \in W_0^{1,p}(\Omega). \tag{4.24}$$

(c.f. Theorem 9, [8]). Combining (4.17), (4.18), (4.20) and (4.24), we get

$$\int_\Omega |\nabla w|^{p-2}\nabla w \nabla v dx - \int_\Omega V_2(x) w^{p-1} v dx \qquad \forall v \in W_0^{1,p}(\Omega). \tag{4.25}$$

Note that $w \geq 0$ and $w \not\equiv 0$, by (4.25) we have $w \equiv \phi(V_2)$ and $\gamma(V_2) = 1$, which is a contradiction to (V2). This contradiction gives us the conclusion that $\{\|u_m^+\|\}$ is bounded.

Therefore $\{u_m\}$ is bounded in $W_0^{1,p}(\Omega)$. By Lemma 6.2 in [12], $\{u_m\}$ is strongly convergent in $W_0^{1,p}(\Omega)$.

□

Lemma 4.2. *Suppose conditions (4.4), (F1), (V1), (H1) and (H3) hold. Then there exist positive real numbers R and C such that*

$$J^{\pm}(u) \geq C\|u\|_{1,p}^p, \qquad \forall u \in B_R(0).$$

Proof. If we have

$$J(u) \geq K\|u\|_{1,p}^p, \qquad \forall u \in B_R(0) \setminus \{0\}.$$

then the result follows. Indeed,

$$J^{\pm}(u) = J(u^{\pm}) + \int_\Omega \frac{1}{p}|\nabla u^{\mp}|^p dx \geq K\|u^{\pm}\|_{1,p}^p + \frac{1}{p}\|u^{\mp}\|_{1,p}^p \geq \min\{K, \frac{1}{p}\}\|u\|_{1,p}^p.$$

Suppose by contradiction that for each $n \in \mathbb{N}$ there exists $u_n \in B_{\frac{1}{n}}(0) \setminus \{0\}$ such that $J(u_n) < \frac{1}{n}\|u_n\|_{1,p}^p$, that is

$$\int_\Omega \frac{1}{p}|\nabla u|^p dx - \frac{1}{q}\int_\Omega h(x)|u_n|^q dx - \int_\Omega F(x, u_n) dx < \frac{1}{n}\|u_n\|_{1,p}^p. \tag{4.26}$$

Put $a_n = \|u_n\|_{1,p} \in \left(0, \frac{1}{n}\right)$, and $w_n = \frac{u_n}{a_n}$. Divide both sides of (4.26) by a_n^p, we obtain

$$\int_\Omega \frac{1}{p}|\nabla w_n|^p dx - \frac{1}{q}\int_\Omega \frac{h(x)|w_n|^q}{a_n^{p-q}} - \int_\Omega \frac{F(x, a_n w_n)}{a_n^p} dx < \frac{1}{n}.$$

Since $h \leq 0$ a.e. on Ω and $a_n > 0$, we have

$$\int_\Omega \frac{1}{p}|\nabla w_n|^p dx - \frac{F(x, a_n w_n)}{a_n^p} dx < \frac{1}{n}. \tag{4.27}$$

Since $\|w_n\|_{1,p} = 1$, as in the proof of Lemma 4.1 and (4.19), there are $w \in W_0^{1,p}(\Omega)$ and $k \in L^1(\Omega)$ such that for any x in

$$\left|\frac{f(x, s a_n w_n(x))}{a_n^p} a_n w_n(x)\right| \leq \eta^{\alpha(p-1)} s^{p-1}|w_n(x)|^{p-1} \leq s^{p-1} k \quad s \in (0,1)\, n \in \mathbb{N}.$$

Thus by *Lebesgue's Dominated Convergence Theorem, Fubini's theorem* and (4.4), we have

$$\lim_{n\to+\infty} \int_\Omega \frac{f(x, a_n w_n(x))}{a_n^p} dx = \lim_{n\to+\infty} \int_\Omega \int_0^1 \frac{f(x, s a_n w_n(x))}{a_n^p} a_n w_n(x) ds dx$$
$$\lim_{n\to+\infty} \int_\Omega \int_0^1 \frac{f(x, s a_n w_n(x))}{|s a_n w_n(x)|^{p-1}} s^{p-1}|w_n(x)|^{p-1} w_n(x) ds dx = \frac{1}{p}\int_\Omega V_0 |w|^{p-1} w dx.$$

Replace w_n by w, we have a similar result

$$\lim_{n\to+\infty}\int_\Omega \frac{f(x,a_nw_n(x))}{a_n^p}dx=\frac{1}{p}\int_\Omega V_0|w|^{p-1}wdx. \tag{4.28}$$

By (V1) (actually, we only need (V1')), we have

$$\int_\Omega \frac{1}{p}|\nabla w|^p dx \geq \frac{1}{p}\int_\Omega V_0|w|^p dx.$$

It follows that for any $\varepsilon>0$, there exist $N_\varepsilon\in\mathbb{N}$ such that,

$$\int_\Omega \frac{1}{p}|\nabla w|^p dx-\int_\Omega \frac{F(x,a_nw(x))}{a_n^p}dx\geq\varepsilon, \qquad \forall n\in\mathbb{N}. \tag{4.29}$$

From (4.27) and (4.29), for $n\geq N_\varepsilon$, we have

$$\int_\Omega \frac{1}{p}|\nabla w_n|^p dx-\int_\Omega \frac{F(x,a_nw_n(x))}{a_n^p}dx-\int_\Omega \frac{1}{p}|\nabla w|^p dx+\int_\Omega \frac{F(x,a_nw(x))}{a_n^p}dx<\frac{1}{n}+\varepsilon.$$

This implies

$$\limsup_{n\to+\infty}\left[\int_\Omega \frac{1}{p}|\nabla w_n|^p dx-\int_\Omega \frac{1}{p}|\nabla w|^p dx\right]\leq\varepsilon, \qquad \forall\varepsilon>0.$$

Hence

$$\limsup_{n\to+\infty}\left[\int_\Omega \frac{1}{p}|\nabla w_n|^p dx-\int_\Omega \frac{1}{p}|\nabla w|^p dx\right]\leq 0.$$

By Theorem 6 in [7], the space $(W_0^{1,p}(\Omega),\|\cdot\|_{1,p})$ is uniformly convex, then by Proposition 3.32 in [5], we have $w_n\to w$ in $W_0^{1,p}(\Omega)$ and therefore, $\|w\|_{1,p}=1$. On the one hand, from (V1), we have

$$\int_\Omega \frac{1}{p}|\nabla w|^p dx-\frac{1}{p}\int_\Omega V_0|w|^p dx\geq c_0\|w\|_{1,p}=c_0>0.$$

On the other hand, let $n\to+\infty$ in (4.27), we obtain

$$\int_\Omega \frac{1}{p}|\nabla w|^p dx-\frac{1}{p}\int_\Omega V_0|w|^p dx\leq 0$$

This is a contradiction. Thus we get the lemma. □

Lemma 4.3. *Suppose conditions (4.4), (F1) (V1'), (H1), and (H4) hold. Then there exist positive real numbers R and C such that*

$$J^{\pm}(u)\geq C\|u\|_{1,p}^p, \qquad \forall u\in B_R(0).$$

Proof. Arguing as in Lemma 4.2, we only need to show

$$J(u)\geq K\|u\|_{1,p}^p, \qquad \forall u\in B_R(0)\setminus\{0\}.$$

Suppose by contradiction that for each $n\in\mathbb{N}$ there exists un $u_n\in B_{\frac{1}{n}}(0)\setminus\{0\}$ such that $J(u_n)<\frac{1}{n}\|u_n\|_{1,p}^p$, that is

$$\int_\Omega \frac{1}{p}|\nabla u_n|^p dx-\frac{1}{q}\int_\Omega h(x)|u_n|^q dx-\int_\Omega F(x,u_n)dx<\frac{1}{n}\|u_n\|_{1,p}^p. \tag{4.30}$$

Divide both sides of (4.30) by a_n^p, where $a_n = \|u_n\|_{1,p} \in (0, \frac{1}{n})$, and $w_n = \frac{u_n}{a_n}$, we obtain

$$\int_\Omega \frac{1}{p}|\nabla w_n|^p dx - \frac{1}{q}\int_\Omega \frac{h(x)|w_n|^q}{a_n^{p-q}} dx - \int_\Omega \frac{F(x, a_n w_n)}{a_n^p} dx < \frac{1}{n}. \tag{4.31}$$

Since $a_n \in (0,1)$ and $h < 0$ a.e. on Ω, then (4.31) becomes

$$\int_\Omega \frac{1}{p}|\nabla w_n|^p dx - \frac{1}{q}\int_\Omega h(x)|w_n|^q dx - \int_\Omega \frac{F(x, a_n w_n)}{a_n^p} dx < \frac{1}{n}. \tag{4.32}$$

Since $\|w_n\|_{1,p} = 1$ for any n, then we can assume $\{w_n\}$ weakly converges to w in $W_0^{1,p}(\Omega)$. Arguing as in the proof of Lemma 4.1, we obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\int_\Omega h(x)|w_n|^q dx = \int_\Omega \lim_{n\to+\infty} h(x)|w_n|^q dx = \int_\Omega h(x)|w|^q dx \\
&\lim_{n\to+\infty}\int_\Omega \frac{F(x, a_n w_n)}{a_n^p} dx = \frac{1}{p}\int_\Omega V_0|w|^p dx \quad \text{and} \\
&\int_\Omega \frac{1}{p}|\nabla w|^p dx - \frac{1}{q}\int_\Omega h(x)|w|^q dx - \int_\Omega V_0|w|^p dx \le 0.
\end{aligned}$$

By (V1'), we get

$$-\int_\Omega h(x)|w|^q dx = 0.$$

Thus by (H4), $w = 0$ a.e. on Ω, which contradicts to $\|w\|_{1,p} = 1$. The lemma is proved. □

Lemma 4.4. *Suppose conditions (4.4), (F1), (V1), (V3) and (H1) hold. Then there exists a constant $\tau > 0$ having the following properties: for all measurable function h on Ω with $\left\| |h_n|^{\frac{p}{p-q}} \eta^{-\frac{\alpha pq}{p-q}} \right\|_{L^1} \le \tau$, there exist positive numbers σ and μ such that $J^{\pm}(u) \ge \mu$ for all $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = \sigma$.*

Proof. We consider the case of J^+, the case of J^- is similar. Suppose by contradiction that: for all $\tau > 0$, there exists a measurable function h with $\left\| |h_n|^{\frac{p}{p-q}} \eta^{-\frac{\alpha pq}{p-q}} \right\|_{L^1} \le \tau$ such that, for all $\sigma > 0$ and $\eta > 0$, we have $u \in W_0^{1,p}(\Omega)$ with $\|u\|_{1,p} = \sigma$ and $J^+(u) < \eta$.

Choose $\tau = \frac{1}{n^{p-q+1}}$, $\sigma = \frac{1}{n}$, $\eta = \frac{1}{n^{p+1}}$ where $n \in \mathbb{N}$, we have: for any $n \in \mathbb{N}$, there exist a measurable function h_n and $u_n \in W_0^{1,p}(\Omega)$ such that $\left\| |h_n|^{\frac{p}{p-q}} \eta^{-\frac{\alpha pq}{p-q}} \right\|_{L^1}^{\frac{p-q}{p}} \le \frac{1}{n^{p-q+1}}$, $\|u_n\|_{1,p} = \frac{1}{n}$ and

$$J^+(u_n) = \int_\Omega \frac{1}{p}|\nabla u_n|^p dx - \frac{1}{q}\int_\Omega h(x)(u_n^+)^q dx - \int_\Omega F(x, u_n^+) dx < \frac{1}{n^{p+1}}. \tag{4.33}$$

Divide both sides of (4.33) by a_n^p, where $a_n = \|u_n\|_{1,p} = \frac{1}{n}$, and $w_n = \frac{u_n}{a_n}$, we obtain

$$\int_\Omega \frac{1}{p}|\nabla w_n|^p dx - \frac{1}{q}\int_\Omega \frac{h_n(x)(w_n^+)^q}{a_n^{p-q}} dx - \int_\Omega \frac{F(x, a_n w_n^+)}{a_n^p} dx < \frac{1}{n}, \tag{4.34}$$

and $\|w_n\|_{1,p} = 1$, then we can assume $\{w_n\}$ weakly converges to w in $W_0^{1,p}(\Omega)$. Arguing as in the proof of Lemma 4.2 we obtain

$$\lim_{n\to+\infty} \frac{F(x, a_n w_n^+)}{a_n^p} dx = \frac{1}{p}\int_\Omega V_0|w^+|^p dx. \tag{4.35}$$

By Hölder inequality and (4.2), we have

$$\begin{aligned}\left|\int_\Omega \frac{h_n(x)|w_n^+|^q}{a_n^{p-q}}dx\right| &\le n^{p-q}\left(\int_\Omega |h_n|^{\frac{p}{p-q}}\eta^{-\frac{\alpha pq}{p-q}}\right)^{\frac{p-q}{p}}\left(\int_\Omega |w_n^+|^p\eta^{\alpha p}\right)^{\frac{q}{p}}\\ &\le \frac{n^{p-q}C}{n^{p-q+1}}\|w_n\|_{1,p}^q \le \frac{n^{p-q}C}{n^{p-q+1}} = \frac{C}{n}.\end{aligned}$$

Therefore

$$\lim_{n\to+\infty}\int_\Omega \frac{h_n(x)|w_n^+|^q}{a_n^{p-q}}dx = 0. \tag{4.36}$$

From (4.34), (4.35), (4.36), (V3) and (V1), we get

$$1 \le \int_\Omega V_0|w^+|^p dx \le \int_\Omega V_0|w|^p dx \le \int_\Omega |\nabla w|^p dx.$$

By Theorem 6 in [7], the space $(W_0^{1,p}(\Omega), \|\cdot\|_{1,p})$ is uniformly convex, then by Proposition 3.32 in [5], we have $w_n \to w$ in $W_0^{1,p}(\Omega)$ and therefore, $\|w\|_{1,p} = 1$. From (4.35), (4.36) and letting $n \to +\infty$ in (4.34), we have

$$\int_\Omega \frac{1}{p}|\nabla w|^p dx - \frac{1}{p}\int_\Omega V_0|w|^p dx \le 0,$$

which contradicts to (V1). Therefore we get the lemma. □

Lemma 4.5. *Under conditions (V2), (F1), (V2) and (H1), we have*

$$\lim_{t\to+\infty}\frac{J^+(t\phi(V_2))}{t^p} < 0 \quad \text{and} \quad \lim_{t\to+\infty}\frac{J^-(-t\phi(V_2))}{t^p} < 0$$

Proof. Put $v = \phi_1(V_2)$, then $\|v\|_{1,p} = 1$, $v > 0$ and

$$\int_\Omega |\nabla v|^{p-2}\nabla v\nabla w dx = \gamma(V_2)\int_\Omega V_2(x)|v|^p v w dx \qquad \forall w \in W_0^{1,p}(\Omega). \tag{4.37}$$

Thus

$$\int_\Omega |\nabla v|^p dx = \gamma(V_2)\int_\Omega V_2(x)|v|^p dx. \tag{4.38}$$

By (4.10) we have

$$J^+(tv) = \int_\Omega \frac{1}{p}|\nabla(tv)|^p dx - \frac{1}{q}\int_\Omega h(x)(tv^+)^q dx - \int_\Omega F(x, tv^+)dx.$$

It implies that

$$\frac{J^+(tv)}{t^p} = \frac{1}{p}\int_\Omega |\nabla v|^p dx - \frac{1}{q t^{p-q}}\int_\Omega h(x)|v|^q dx - \frac{1}{t^p}\int_\Omega F(x, tv)dx. \tag{4.39}$$

By *Hölder's inequality*, (H1) and (4.2), we have

$$\int_\Omega |h||v|^q dx = \int_\Omega |h|\eta^{-q\alpha}|v|\eta^{q\alpha}dx \le \left(\int_\Omega |h|^{\frac{p}{p-q}}\eta^{-\frac{pq\alpha}{p-q}}dx\right)^{\frac{p-q}{p}}\left(\int_\Omega |v|^p\eta^{p\alpha}dx\right)^{\frac{q}{p}} < \infty.$$

Hence

$$\lim_{t\to+\infty}\frac{1}{q^{t^{p-q}}}\int_\Omega h(x)|v|^q dx=0. \tag{4.40}$$

As in (4.28), we get

$$\lim_{t\to+\infty}\frac{1}{t^p}\int_\Omega F(x,tv)dx=\frac{1}{p}\int_\Omega V_2(x)|v(x)|^p dx. \tag{4.41}$$

Combining (4.38), (4.39), (4.40) and (4.41), we have

$$\begin{aligned}\lim_{t\to+\infty}\frac{J^+(tv)}{t^p} &= \frac{1}{p}\int_\Omega|\nabla v|^p dx-\frac{1}{p}\int_\Omega V_2|v|^p dx\\ &= \frac{1}{p}\int_\Omega|\nabla v|^p dx-\frac{1}{p\gamma(V_2)}\int_\Omega|\nabla v|^p dx\\ &= \frac{1}{p}\left(1-\frac{1}{\gamma(V_2)}\right)<0,\end{aligned}$$

which contradicts to (V2) and that completes the proof of Lemma. □

Lemma 4.6. *Under conditions (F1), (H1) and (H5), we have*

$$\lim_{t\to+\infty}\frac{J^\pm(\pm tv)}{t^p}<0,$$

where v is given in (H5).

Proof. Let $\{t_n\}$ be a sequence of positive real numbers such that $t_n\to 0$ as $n\to+\infty$. First, we consider the case of J^+. We have

$$\begin{aligned}\frac{J^+(t_nv)}{t_n^q} &= \frac{1}{t_n^q}\left[\int_\Omega\frac{1}{p}|\nabla t_nv|^p dx-\frac{1}{q}\int_\Omega h(x)(t_nv^+)^q dx-\int_\Omega F(x,t_nv^+)dx\right]\\ &= \frac{t_n^{p-q}}{p}\int_\Omega|\nabla v|^p dx-\frac{1}{q}\int_\Omega h(x)(v^+)^q dx-\frac{1}{t_n^q}\int_\Omega F(x,v^+)dx. \end{aligned}\tag{4.42}$$

By (F1), we have

$$\begin{aligned}&\left|\int_\Omega F(x,v^+)dx\right|=\left|\int_\Omega\int_0^{t_nv^+}f(x,\tau)d\tau dx\right|=\left|\int_\Omega\int_0^1 f(x,\xi t_nv^+)d\xi dx\right|\\ &\le\int_\Omega\int_0^1\left|\eta^{\alpha(p-1)}\left(|\xi t_nv^+|\right)^{p-1}t_nv^+\right|d\xi dx=\int_\Omega\int_0^1\left|\eta^{\alpha(p-1)}\xi^{p-1}t_n^p(v^+)^p\right|d\xi dx\\ &\le\frac{1}{p}t_n^p\int_\Omega\eta^{\alpha(p-1)}|v|^p dx.\end{aligned}$$

It follows that

$$\begin{aligned}&\lim_{n\to+\infty}\frac{1}{t_n^q}\int_\Omega F(x,t_nv^+)dx=0\quad\text{and}\\ &\lim_{t\to0^+}\frac{J^+(tv)}{t^q}=-\frac{1}{q}\int_\Omega h(x)(v^+)^q dx<0.\end{aligned}$$

In case of J^-, note that $-v^+ = (-v)^-$ then $(-t_n v)^- = -t_n v^+ = t_n v$, we have

$$\begin{aligned}\frac{J^-(t_n v)}{t_n^q} &= \frac{1}{t_n^q}\left[\int_\Omega \frac{1}{p}|\nabla(-t_n v)|^p dx - \frac{1}{q}\int_\Omega h(x)|(-t_n v)^-|^q dx - \int_\Omega F(x,(-t_n v)^-)dx\right] \\ &= \frac{t_n^{p-q}}{p}\int_\Omega |\nabla v|^p dx - \frac{1}{q}\int_\Omega h(x)v^q dx - \frac{1}{t_n^q}\int_\Omega F(x,-t_n v)dx.\end{aligned}$$

Arguing as in the case J^+, we also have

$$\lim_{t\to 0^+}\frac{J^-(-tv)}{t^q} = -\frac{1}{q}\int_\Omega h(x)|v^+|^q dx < 0.$$

□

Proof of Theorem 4.1 We have $J^+(0) = 0$. By Mountain Pass Theorem in [2], Lemma 4.1, Lemma 4.4 and Lemma 4.5, there exists a critical point u_1 of J^+ with $J^+(u_1) \geq \mu > 0$ (μ is given in Lemma 4.4). We prove that $u_1 \geq 0$. Since u_1 is the critical point of J^+, by (4.12), for any $v \in W_0^{1,p}(\Omega)$

$$\int_\Omega |\nabla u_1|^{p-2}\nabla u_1 \nabla v dx - \int_\Omega h(x)(u_1^+)^{q-2}u_1^+ v dx - \int_\Omega f(x,u_1^+)dx = 0. \tag{4.43}$$

Take $v = u_1^-$ in (4.43), we obtain

$$\int_\Omega |\nabla u_1^-|^p dx = 0.$$

Hence $u_1^- = 0$ a.e. on Ω, this implies $u_1 \geq 0$ a.e. on Ω. Since $J^+(u_1) > 0$, then $u_1 \not\equiv 0$, therefore u_1 is a non-trivial nonnegative weak solution of the problem (4.1) such that $J(u_1) = J^+(u_1) > 0$.

Arguing similarly for J^-, we obtain a non-trivial non-positive weak solution u_2 of (4.1) such that $J(u_2) > 0$.

Proof of Corollary 4.1 Using Mountain Pass Theorem in [2], Lemma 4.1, Lemma 4.2 and Lemma 4.5, and arguing as in Proof of Theorem 4.1, we get the corollary.

Proof of Corollary 4.2 Using Mountain Pass Theorem in [2], Lemma 4.1, Lemma 4.3 and Lemma 4.5, and arguing as in Proof of Theorem 4.1, we get the corollary.

Proof of Theorem 4.2 By Theorem 4.1, we have two nontrivial weak solutions for (4.1), one solution is non–negative and one solution is non–positive, but both of them have positive energy. Therefore, we only need to find two nontrivial weak solutions for (4.1) which have negative energy. The first one is found by J^+, and the second is found by a similar argument for J^-.

J is lower semi–continuous by its differentiability. Let τ, σ and μ as in Lemma 4.4. Put $B = \{u \in W_0^{1,p}(\Omega) : \|u\|_{1,p} \leq \sigma\}$. Arguing as in (4.39) and using (ii) of Theorem 2.4, we see that $J^+(B)$ is bounded. Put $c = inf_B J^+$, then by Lemma 4.6, we have $c < 0$.

For $n \in \mathbb{N}$, let $u_n \in B$ such that

$$c \leq J^+(u_n) \leq c + \frac{1}{n^2}. \tag{4.44}$$

Let $e \in W_0^{1,p}(\Omega)$ with $\|e\|_{1,p} = 1$. Apply Ekeland's Variational Principle in [11] with $\varepsilon = \frac{1}{n^2}$ and $\delta = \frac{1}{n}$, there exists $v_n \in B$ such that

$$\begin{aligned}J^+(v_n) = J^+_{v_n}(v_n) \leq J^+_{v_n}(v_n + te) = J^+(v_n + te) + \frac{1}{n}|t|\|e\|_{1,p} \quad \forall t \in \mathbb{R}\setminus\{0\} \\ J^+(v_n) \leq J^+(u_n).\end{aligned} \tag{4.45}$$

From (4.45), we have

$$\frac{J^+(v_n+te)-J^+(v_n)}{|t|} \geq -\frac{1}{n}.$$

Thus

$$\begin{cases} \dfrac{J^+(v_n+te)-J^+(v_n)}{t} \geq -\dfrac{1}{n} & \text{if } t>0, \\ \dfrac{J^+(v_n+te)-J^+(v_n)}{-t} \geq -\dfrac{1}{n} & \text{if } t<0. \end{cases}$$

Therefore

$$\left|\frac{J^+(v_n+te)-J^+(v_n)}{t}\right| \leq \frac{1}{n}.$$

Letting $t \to 0$, we have

$$|DJ^+(v_n)(e)| \leq \frac{1}{n}, \qquad \forall e \in W_0^{1,p}(\Omega),\ \|e\|_{1,p}=1.$$

Therefore

$$\lim_{n\to+\infty} \left\|DJ^+(v_n)\right\|_{(W_0^{1,p})^*} = \lim_{n\to+\infty}\left(\sup_{\|e\|_{1,p}=1} |DJ^+(v_n)(e)|\right) \leq \lim_{n\to+\infty}\frac{1}{n}=0.$$

Since $c = inf_B J^+$, by (4.44) and (4.45), we have

$$c \leq J^+(v_n) \leq c + \frac{1}{n^2}. \tag{4.46}$$

Therefore, $J^+(v_n) \to c$ as $n \to +\infty$. It follows that $\{v_n\}$ is a Palais–Smale sequence. By (4.5), we can assume that $v_n \to v_0$ in $W_0^{1,p}(\Omega)$. Hence,

$$DJ^+(v_0) = \lim_{n\to+\infty} DJ^+(v_n) = 0 \quad \text{and} \quad J^+(v_0) = \lim_{n\to+\infty} J^+(v_n) = c < 0.$$

Therefore, v_0 is a critical point of J^+ with negative energy. Arguing as in Proof of Theorem 4.1 we get have $v_0 \geq 0$ and Theorem 4.2.

Remark 4.1. If η in (4.3) is constant, the results in this section have been proved in [9].

Example 4.1. *Let $p=2$, Ω be the unit ball $B_1(0)$ in $\mathbb{R}^N$ ($N \geq 3$) and $\eta(x) = (2\lambda_1)^{\frac{2}{3}}(1-|x|)^{-1}$ for any x in Ω. By Theorem 8.4 in [16], there is a real number $c \geq 1$ such that*

$$\int_\Omega |\eta u|^2 dx \leq c\|u\|_{1,2}^2 \qquad \forall u \in W_0^{1,2}(\Omega). \tag{4.47}$$

Let δ be in $\left(0, \dfrac{2\lambda_1}{c(1+2\lambda_1)}\right)$ and f be a real C^1–function on $\Omega \times \mathbb{R}$ such that

$$\begin{cases} f(x,s) = \delta(1-|x|^2)^{-\frac{2}{3}} s & \text{if } |s| \leq \dfrac{1}{2},\ \forall x \in \Omega, \\ |f(x,s)| \in \left[0, 2\lambda_1(1-|x|^2)^{-\frac{2}{3}}|s|\right] & \text{if } |s| \in \left[\frac{1}{2}, 1\right],\ \forall x \in \Omega, \\ f(x,s) = \dfrac{2\lambda_1 s^3(1-|x|^2)^4}{1+s^2(1-|x|^2)^4}(1-|x|^2)^{-\frac{3}{2}} & \text{if } |s| \geq 1,\ \forall x \in \Omega. \end{cases} \tag{4.48}$$

(i) *We have*

$$V_0(x) = \lim_{s\to 0} \frac{f(x,s)}{s} = \delta(1-|x|)^{-\frac{2}{3}}, \qquad \forall x \in \Omega.$$

We get (V2), (F1) with $\alpha = \frac{2}{3}$ *and*

$$\int_\Omega \left(|\nabla u|^p - V_0|u|^2\right) dx \geq (1-c\delta)\|u\|^2_{W_0^{1,2}} \qquad \forall u \in W_0^{1,2}(\Omega).$$

Thus (V1) is fulfilled.

(ii) *Put* $V_1(x) = V_2(x) = 2\lambda_1(1-|x|)^{-\frac{2}{3}}$ *for every* x *in* Ω*. Since* $V_1(x) = V_2(x) > \lambda_1$ *for every* x *in* Ω*, by Theorem 3.2,* $\lambda(V_1) = \lambda(V_2) < 1$ *and we get the condition (V2).*

(iii) *Let* $q = \frac{3}{2}$ *and* $h = \varepsilon\eta^{\frac{\alpha q}{p}} = \varepsilon\eta^{\frac{9}{8}}$ *with a sufficiently small positive real number* ε*, then we get (H1).*

Thus we can apply Theorem 4.1 for f *and* h*, but the results in [9] do not work in this case.*

5 Acknowledgements

The author would like to thank to Prof. Dr. Jürgen Appell's for their valuable comments, suggestions, review and approval about of the paper. Also, the author would like to express their sincere gratitude to the referee of this paper for a very careful reading of it and for all the insightful comments and suggestions, which help us to improve the quality of this manuscript.

References

[1] R. Adam and J. Fournier, *Sobolev spaces*. Academic Press, New York 2005.

[2] Z. W. Birnbaum and W. Orlicz. *Über die verallgemeinerung des berriffes der zueinander konjugierten potenzen*. Stu. Math., **30** (1968), 21–42.

[3] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*. J. Funct. Anal. **14** (1973), 349–381.

[4] A. Anane, *Etude des valeurs propres et de la résonnance pour l'opérateur p-laplacien*. C. R. Ac. Sc. Paris, **305** (1987), 725–728.

[5] J. Appell and P. Zabreiko, *Nonlinear superposition operators*. Cambridge University Press, 2008.

[6] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer, 2011.

[7] M. Cuesta. *Eigenvalue problems for the p–laplacian with indefinite weights*. Electronic Journal of Differential Equations. **2001**(33) (2001), 1–9.

[8] G. Dinca, P. Jebelean and J. Mawhin. *Variational and topological methods for Dirichlet problems with p–laplacian*. Portugaliae Mathematica **58** (2001), 339–378.

[9] T. K. Donaldson and N. S. Trudinger. *Orlicz-Sobolev spaces and imbedding theorems.* J. Funct. Anal. **8** (1971), 52–75.

[10] D. M. Duc and N. Q. Huy. *Non-uniformly asymptotically linear p–Laplacian problems.* Nonlinear Analysis **92** (2013), 183–197.

[11] D. M. Duc. *Nonlinear singular elliptic equations.* J. London Math. Soc. **40**(2) (1989), 420–440.

[12] I. Ekeland, *Nonconvex minimization problems*, Bull. Amer. Math. Soc. **1** (1979), 443–474.

[13] D. G. De Figueiredo. *Lectures on the Ekeland variational principle with applications and detours.* Tata Institute of Fundational research, Bombay, 1989.

[14] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, Springer, Berlin, 2001.

[15] B. J. Jaye, V. G. Mazya and I. E. Verbitsky. *Quasilinear elliptic equations and weighted Sobolev-Poincaré inequalities with distributional weights*, Advances in Mathematics **232** (2013), 513–542.

[16] M. A. Kranosel'skii. *Topological methods in the theory of nonlinear integral equations.* Macmillan, New York, 1964.

[17] A. Kufner. *Weighted Sobolev spaces.* Wiley, New York, 1985.

[18] A. Kufner, O. John and S. Fucik. *Function spaces.* Noordhoff, Leyden 1977.

[19] S. Fucik and A. Kufner. *Nonlinear Differential Equations.* Vol. 2, Elsevier, 1980.

[20] M. Marcus and V. J. Mizel. *Every superposition operator mapping one Sobolev space into another is continuous*, J. Funct. Anal. 33 (1979), 217–229.

[21] M. Marcus and V. J. Mizel. *Complete characterization of functions which act, via superposition, on Sobolev spaces.* Transactions of AMS **251** (1979), 187–218.

[22] J. Matkwoski. *Functional equation and Nemytskii operators.* Funkc. Ekvac. **25** (1982) 127–132.

[23] V. G. Mazja. *Sobolev spaces*, Springer, Berlin, 1985.

[24] B. Opic and A. Kufner. *Remark on compactness of imbeddings in weighted spaces.* Math. Nachr. **133** (1987), 63–70.

[25] T. Runst and W. Sickel, *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations.* Series: De Gruyter Series in Nonlinear Analysis and Applications **3**, 2011.

[26] M. Struve. *Variational methods.* Springer, Berlin, 2008.

[27] E. Zeidler. *Nonlinear functional analysis and its applications III: Variational methods and optimization.* Springer, Berlín, 1985.