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On some interesting properties of p–laplacian equation

Sobre algunas propiedades interesantes de la ecuación p–laplaciana

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Abstract

In the present paper we establish, on the one hand, some singular solutions concerning to the 1–laplacian equation. On the other hand, we give some properties related to the weak solutions of p–lapalcian equation

Key words and phrases: Singular solution, p–laplacian equation, p–harmonic function.

Resumen

En el presente artículo establecemos, por una parte, algunas soluciones singulares concernientes a la ecuación 1–lapaciana. Por otro lado, damos algunas propiedades relacionadas a la debil solución de la ecuación p–laplaciana.

Palabras y frases clave: Solución singular, ecuación p–laplaciana, función p–armónica.

1 Introduction

In this paper, we are investigating singular solutions and properties to the following equation which we shall call the p–Laplace equation [1-6, 9-11].

$$div(\mid \nabla u \mid^{p-2} \nabla u) = 0, \tag{1.1}$$

where p satisfies $1 \leq p \leq \infty$. The p–laplacian operator is defined as

$$\begin{aligned} \Delta_p u &= div(\mid \nabla u \mid^{p-2} \nabla u) \\ &= \mid \nabla u \mid^{p-4} \left(\mid \nabla u \mid^2 \Delta u + (p-2) \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} \right) \end{aligned} \tag{1.2}$$

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There are several noteworthy values of p:

a) $p = \infty$. As $p \to \infty$ one encounters the infinity Laplacian equation

$$\sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \tag{1.3}$$

in $\mathbb{R}^n$, which some singular solutions are given by

$$a\sqrt{x_1^2 + \cdots + x_k^2} + b \quad (1 \le k \le n) \tag{1.4}$$

$$a_1 x_1 + \cdots + a_n x_n + b \tag{1.5}$$

$$a_1 x_1^{4/3} + \cdots + a_n x_n^{4/3} \quad \Big(\sum_{j=1}^{n} a_j^3 = 0\Big) \tag{1.6}$$

b) $p = 2$. In this case we have the Laplace equation

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0 \tag{1.7}$$

c) $p = 1$. In this case we obtain the 1-laplacian equation

$$div\Big(\frac{\nabla u}{\mid \nabla u \mid}\Big) = 0 \tag{1.8}$$

For $x \in \mathbb{R}^3$ and under the assumption that $|\nabla u| \neq 0$, it then follows from (1.8)

$$\begin{aligned} |\nabla u|^2 \cdot \Delta u - \Bigg\{ &\Big(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x \partial z}\Big)\frac{\partial u}{\partial x} + \\ &+ \Big(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y \partial z}\Big)\frac{\partial u}{\partial y} + \\ &+ \Big(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial z \partial y} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\Big)\frac{\partial u}{\partial z} \Bigg\} = 0 \end{aligned} \tag{1.9}$$

The purpose of this paper is to obtain nontrivial singular solutions of (1.8) and some properties of weak solutions concerning the equation (1.1). The paper has been organized as follows: in section 2, we briefly review the basic definitions used in our subsequent discussions, next, the preliminary results are established in section 3. Section 4 present the main results.

2 Basic definitions

Here, we give some definitions used in our subsequent discussions. For more details, see [7-8, 12]

Definition 2.1. We denote by $C_0(\Omega)$ the space of all continuous functions on Ω with compact support.

Other interesting notations are

$$C_0^k(\Omega) = C^k(\Omega) \cap C_0(\Omega) \tag{2.1}$$

Definition 2.2. The Sobolev space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \Big\{ u \in L^p \Big| \exists g_1, g_2, \cdots, g_N \in L^p \text{ such that}$$
$$\textstyle\int_\Omega u \dfrac{\partial \varphi}{\partial x_i} = - \int_\Omega g_i \varphi, \quad \forall \varphi \in C_0^\infty(\Omega), \quad \forall i = 1, 2, \cdots, N \Big\}$$

Definition 2.3. Let Ω be a domain in $\mathbb{R}^n$. We say that $u \in W_{loc}^{1,p}(\Omega)$ is a weak solution of the p–harmonic equation (1.1) in Ω, if

$$\int |\nabla u|^{p-2} \nabla u \nabla \eta dx = 0 \quad \forall \eta \in C_0^\infty(\Omega). \tag{2.2}$$

If, in adition, u is continuous, then we say u is a p–harmonic function.

Definition 2.4. Let Ω be a domain in $\mathbb{R}^n$. We say that $u \in W_{loc}^{1,p}(\Omega)$ is a classical solution of (1.1), if u satisfies (1.1)

Definition 2.5. We say that $u \in W_{loc}^{1,p}(\Omega)$ is a weak supersolution of the p–harmonic equation (1.1) in Ω, if

$$\int_\Omega | \nabla u |^{p-2} \cdot \nabla u \cdot \nabla \eta dx \geq 0 \tag{2.3}$$

for all nonnegative $\eta \in C_0^\infty(\Omega)$. For weak subsolution, the inequality is reversed.

Definition 2.6. Let $\Omega \subset \mathbb{R}^n$ be an open set, a linear differential operator of second order $L : C^2(\Omega) \longrightarrow C(\Omega)$ is defined as

$$L(u) = - \sum_{j=1}^n \sum_{i=1}^n \frac{\partial}{\partial x_i} \Big(a_{ij} \frac{\partial u}{\partial x_j} \Big) + \sum_{i=1}^n a_i \frac{\partial u}{\partial x_j} + a_0 u \tag{2.4}$$

where $a_{ij} \in C^1(\Omega), a_i \in C(\Omega), \forall i, j = 1, \cdots, n$ y $a_0 \in C(\bar{\Omega})$.

Definition 2.7. Let V an K-vectorial space. A function $g : V \times V \longrightarrow K$ is called a *billinear form* if

i) $g(u + v, w) = g(u, w) + g(v, w), \quad \forall u, v, w \in V,$

ii) $g(\lambda v, w) = \lambda g(v, w), \quad \forall \lambda \in K, \forall v, w \in V,$

iii) $g(u, v + w) = g(u, v) + g(u, w), \quad \forall u, v, w \in V,$

iv) $g(v, \lambda w) = \lambda g(v, w), \quad \forall \lambda \in K, \forall v, w \in V.$

Definition 2.8. Let H be a Hilbert space, we say that a billinear form $g : H \times H \longrightarrow \mathbb{R}$ is

a) continuous if there exists a constant $C > 0$ such that

$$|g(u, v)| \leq C \|u\|_H \cdot \|v\|_H \quad \forall u, v \in H. \tag{2.5}$$

b) coersive if there exists a constant $\theta > 0$ such that

$$g(u,v) \geq \theta \|u\|_H^2 \quad \forall u \in H. \tag{2.6}$$

Definition 2.9. The billinear form $g : H_0^1(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ conected with the operator L is defined as

$$g(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^{N} \int_\Omega a_i \frac{\partial u}{\partial x_i} \cdot v + \int_\Omega a_0 uv \tag{2.7}$$

for all $u, v \in H_0^1(\Omega)$.

3 Preliminary results

Theorem 3.1. *Let suposse that the billinear form (2.7) is coersive with the constant of coersity θ. Then the bilinear form (2.7) is continuous and furthermore exist $\alpha, \gamma \geq 0$ such that [8]*

$$\alpha \cdot \|u\|^2_{H_0^1(\Omega)} \leq g(u,u) + \gamma \cdot \|u\|^2_{L^2(\Omega)} \tag{3.1}$$

for all $u \in H_0^1(\Omega)$. Where

$$\alpha = \frac{\theta}{2} \quad y \quad \gamma = \frac{1}{4\varepsilon} \sum_{i=1}^{n} \|a_i\|_{L^\infty(\Omega)} + \|a_0\|_{L^\infty(\Omega)}. \tag{3.2}$$

Theorem 3.2. *(Caccioppoli) If u is a weak solution of (1.1) in Ω, then*

$$\int_\Omega \xi^p \cdot |\nabla u|^p dx \leq p^p \cdot \int_\Omega |u|^p \cdot |\nabla \xi|^p dx \tag{3.3}$$

for al $\xi \in C_c^\infty(\Omega), \xi \geq 0$.

Proof. Use

$$\eta = \xi^p u$$
$$\nabla \eta = \xi^p \nabla u + p\xi^{p-1} u \nabla \xi$$

By the equation (2.2) and Holder's inequality

$$\begin{aligned} \int_\Omega \xi^p |\nabla u|^p dx &= -p \int_\Omega \xi^{p-1} \cdot u \langle |\nabla u|^{p-2} \nabla u, \nabla \xi \rangle dx \\ &\leq p \int_\Omega |\xi \cdot \nabla u|^{p-1} \cdot |u \cdot \nabla \xi| dx \\ &\leq p \Big\{ \int_\Omega \xi^p |\nabla u|^p dx \Big\}^{1-\frac{1}{p}} \cdot \Big\{ \int_\Omega |u|^p \cdot |\nabla \xi|^p \Big\}^{\frac{1}{p}} \end{aligned}$$

The estimate follows. □

Theorem 3.3. *If $v > 0$ is a weak supersolution of (1.1) in Ω, then*

$$\int_\Omega \xi^p \cdot |\nabla \log v|^p dx \le \Big(\frac{p}{p-1}\Big)^p \cdot \int_\Omega |\nabla \xi|^p dx \tag{3.4}$$

whenever $\xi \in C_c^\infty(\Omega), \xi \ge 0$.

Proof. One may add constants to the weak supersolutions. First, prove the estimate for $v(x)+\epsilon$ in place of $v(x)$. Then let $\epsilon \to 0$ in

$$\int_\Omega \frac{\xi^p \cdot |\nabla v|^p}{(v+\epsilon)^p} dx \le \Big(\frac{p}{p-1}\Big)^p \cdot \int_\Omega |\nabla \xi|^p dx \tag{3.5}$$

Hence we may assume that $v(x) \ge \epsilon > 0$. Next use the test function $\eta = \xi^p v^{1-p}$. Then

$$\nabla \eta = p\xi^{p-1} v^{1-p} \nabla \xi - (p-1)\xi^p v^{-p} \nabla v \tag{3.6}$$

and we obtain

$$\begin{aligned}(p-1)\int_\Omega \xi^p v^{-p} |\nabla v|^p dx &\le p \int_\Omega \xi^{p-1} \cdot v^{1-p} \big\langle |\nabla v|^{p-2} \nabla v, \nabla \xi \big\rangle dx \\ &\le p \int_\Omega \xi^{p-1} \cdot v^{1-p} \cdot |\nabla v|^{p-1} \cdot |\nabla \xi| dx \\ &\le p \Big\{ \int_\Omega \xi^p \cdot v^{-p} |\nabla v|^p dx \Big\}^{1-\frac{1}{p}} \cdot \Big\{ \int_\Omega |\nabla \xi|^p \Big\}^{\frac{1}{p}}\end{aligned}$$

from which the result follows. □

Theorem 3.4. *Suposse that $1 \le p \le \infty$ and Ω is a bounded open set. Then exists a constant C (depending on Ω and p) such that [7]*

$$\|u\|_{L^p} \le C \|\nabla u\|_{L^p}, \quad \forall u \in W_0^{1,p}(\Omega)(1 \le p \le \infty)$$

In particular, the expression $\|\nabla u\|_{L^p}$ is a norm on $W_0^{1,p}(\Omega)$, and it is equivalent to the norm $\|u\|_{W^{1,p}}$

4 Main Results

4.1 Some results of weak solutions

Theorem 4.1. *A C^2 function u that satisfies (1.1) is a weak solution of (1.1)*

Proof. Multiply (1.1) by $\eta \in C_0^\infty(\Omega)$ and integrate by parts; we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \eta dx = 0 \quad \forall \eta \in C_0^\infty(\Omega)$$

as required. □

Theorem 4.2. *Let $u \in W^{1,p}(\Omega)$ be a weak solution of p-harmonic equation (1.1) in Ω, then*

$$\|\xi \cdot |\nabla u|\|_{L^p} \leq p\big(\|u\|^2_{L^{2p}} + \|\nabla \xi\|^2_{L^{2p}}\big) \quad \forall \xi \in C_0^\infty(\Omega), \xi \geq 0.$$

Proof. From Theorem 3.2, we have

$$\int_\Omega \xi^p \cdot |\nabla u|^p dx \leq p^p \cdot \int_\Omega |u|^p \cdot |\nabla \xi|^p dx \tag{4.1}$$

In terms of (4.1), it then follows that

$$\|\xi \cdot |\nabla u|\|_{L^p} \leq p \cdot \||u| \cdot |\nabla \xi|\|_{L^p} \tag{4.2}$$

To continue we need the Young inequality

$$a \cdot b \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2, \qquad a, b \in \mathbb{R}, \varepsilon \in \mathbb{R}_+ \tag{4.3}$$

We obtain

$$\begin{aligned}
\|\xi \mid \nabla u \mid \|_{L^p} &\leq p\| \mid u \mid\mid \nabla \xi \mid \|_{L^p} \\
&\leq p\|\varepsilon \mid u \mid^2 + \frac{1}{4\varepsilon} \mid \nabla \xi \mid^2 \|_{L^p} \\
&\leq p(\varepsilon\| \mid u \mid^2 \|_{L^p} + \frac{1}{4\varepsilon}\| \mid \nabla \xi \mid^2 \|_{L^p})
\end{aligned} \tag{4.4}$$

We insert $\varepsilon \in [\frac{1}{4}, 1]$ into the inequality (4.4). This yields

$$\begin{aligned}
\|\xi \mid \nabla u \mid \|_{L^p} &< p[\varepsilon\| \mid u \mid^2 \|_{L^p} + (1-\varepsilon)\| \mid u \mid^2 \|_{L^p} \\
&\quad + \frac{1}{4\varepsilon}\| \mid \nabla \xi \mid^2 \|_{L^p} + \frac{4\varepsilon - 1}{4\varepsilon}\| \mid \nabla \xi \mid^2 \|_{L^p}] \\
&= p[\| \mid u \mid^2 \|_{L^p} + \| \mid \nabla \xi \mid^2 \|_{L^p}] \\
&= p(\|u\|^2_{L^{2p}} + \|\nabla \xi\|^2_{L^{2p}}),
\end{aligned}$$

as required. □

Theorem 4.3. *Let suposse that the constant of coersity of the billinear form (2.7) is θ. Let also $v \in W^{1,p}(\Omega)$ be a positive ($v > 0$) weak supersolution of the p-harmonic equation (1.1) in Ω, then there exist constants $\beta > 0$ and $\gamma \geq 0$ such that*

$$\beta\|u \cdot |\nabla \log v|\|^2_{L^2(\Omega)} \leq g(u, u) + \gamma \cdot \|u\|^2_{L^2(\Omega)} \tag{4.5}$$

for all $u \in C_0^\infty(\Omega), u \geq 0$, with γ and $g(u, u)$ given by (3.2) and (2.7) respectively.

Proof. Let $p = 2$. By Theorem 3.3, we obtain inequality

$$\|u \cdot |\nabla \log v|\|^2_{L^2(\Omega)} \leq 4 \cdot \|\nabla u\|^2_{L^2(\Omega)}$$

which can be written as

$$\|u \cdot |\nabla \log v|\|^2_{L^2(\Omega)} \leq 4 \cdot \|\nabla u\|^2_{L^2(\Omega)} = 4 \cdot \|u\|^2_{H_0^1(\Omega)} \tag{4.6}$$

Combining the inequality (3.1) with the estimate (4.6), we obtain

$$\alpha\|u \cdot |\nabla \log v|\|^2_{L^2(\Omega)} \leq 4 \cdot g(u,u) + 4 \cdot \gamma \cdot \|u\|^2_{L^2(\Omega)} \tag{4.7}$$

Divide out the common factor. We arrive at

$$\beta\|u \cdot |\nabla \log v|\|^2_{L^2(\Omega)} \leq g(u,u) + \gamma \cdot \|u\|^2_{L^2(\Omega)}, \tag{4.8}$$

with the constant $\beta = \dfrac{\alpha}{4}$. This concludes the proof. □

4.2 Singular solutions of 1-Laplacian equation

The purpose of this section is to prove that the functions

$$u = (x_1 + x_2 + \cdots + x_n + d_0)e^{x_1+x_2+\cdots+x_n+d_1} + d_2 \tag{4.9}$$
$$u = e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d} + d_0 \tag{4.10}$$
$$u = \ln(a_1x_1 + a_2x_2 + \cdots + a_nx_n + d) + d_0 \tag{4.11}$$
$$u = \ln(e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_0} + d_1) + d_2 \tag{4.12}$$
$$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_0 + e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_1} \tag{4.13}$$
$$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_0 + \ln(a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_1) \tag{4.14}$$
$$u = a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_0 + \ln(e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_2} + d_1) \tag{4.15}$$
$$u = e^{a_1x_1+a_2x_2+\cdots+a_nx_n+d_0} + \ln(a_1x_1 + a_2x_2 + \cdots + a_nx_n + d_1) + d_2 \tag{4.16}$$

are singular solutions of the equation (1.8). Where $a_i(i = 0, \cdots, n), a, b, c, d, d_0$ $,d_1, d_2, d_3$ are real numbers. For $x \in \mathbb{R}^3$, we shall have

$$u = (x + y + z + d_0)e^{x+y+z+d_1} + d_2 \tag{4.17}$$
$$u = e^{ax+by+cz+d} + d_0 \tag{4.18}$$
$$u = \ln(ax + by + cz + d) + d_0 \tag{4.19}$$
$$u = \ln(e^{ax+by+cz+d_0} + d_1) + d_2 \tag{4.20}$$
$$u = ax + by + cz + d_0 + e^{ax+by+cz+d_1} \tag{4.21}$$
$$u = ax + by + cz + d_0 + \ln(ax + by + cz + d_1) \tag{4.22}$$
$$u = ax + by + cz + d_0 + \ln(e^{ax+by+cz+d_2} + d_1) \tag{4.23}$$
$$u = e^{ax+by+cz+d_0} + \ln(ax + by + cz + d_1) + d_2 \tag{4.24}$$

Theorem 4.4. *The function (4.17) is singular solution of the equation (1.9)*

Proof. Noticing that

$$|\nabla u|^2 \cdot \Delta u = 9 \cdot (x+y+z+d_0+1)^2 \cdot (x+y+z + d_0+2) \cdot e^{3(x+y+z+d_1)} \tag{4.25}$$

$$\Big(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial x \partial z}\Big)\frac{\partial u}{\partial x} = 3 \cdot (x+y+z+d_0+1)^2 \cdot (x+y+z + d_0+2) \cdot e^{3(x+y+z+d_1)} \tag{4.26}$$

$$\Big(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y \partial z}\Big)\frac{\partial u}{\partial y} = 3 \cdot (x+y+z+d_0+1)^2 \cdot (x+y+z + d_0+2) \cdot e^{3(x+y+z+d_1)} \tag{4.27}$$

$$\Big(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial z \partial x} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial z \partial y} + \frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\Big)\frac{\partial u}{\partial z} = 3 \cdot (x+y+z+d_0+1)^2 \cdot (x+y+z + d_0+2) \cdot e^{3(x+y+z+d_1)} \tag{4.28}$$

and inserting (4.25)-(4.28) into the equation (1.9), we obtain

$$0 \cdot (x+y+z+d_0+1)^2 \cdot (x+y+z+d_0+2) \cdot e^{3(x+y+z+d_1)} = 0, \tag{4.29}$$

from which the proof follows. □

Similar to the proof of the foregoing theorem, we have

Theorem 4.5. *Functions (4.18)-(4.24) are singular solutions of the equation (1.9).*

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