Divulgaciones Matemáticas Vol. 23-24, No. 1-2 (2022-2023), pp. 1–11
https://produccioncientificaluz.org/index.php/divulgaciones/
DOI: https://doi.org/10.5281/zenodo.11515886

e-ISSN 2731-2437
p-ISSN 1315-2068

Super quasi-topological and paratopological vector spaces versus topological vector spaces

Super casi-topológicos y paratopológicos espacios vectoriales versus espacios vectoriales topológicos

Madhu Ram (madhuram0502@gmail.com)
ORCID: https://orcid.org/0000-0001-6583-0978
Department of Mathematics
University of Jammu
Jammu-180006, Jammu & Kashmir, India.

Bijan Davvaz (davvaz@yazd.ac.ir)
Department of Mathematics
Yazd University
Iran.

Abstract

In this paper, we introduce the idea of super quasi-topological vector space which is an extension of the concept of topological vector space and investigate some of its basic properties. We extend the existing notion of quasi-topological vector space to all complex vector spaces and investigate the relationship of super quasi-topological vector spaces with paratopological and quasi-topological vector spaces.

Palabras y frases clave: Topological vector space, paratopological vector space, quasi-topological vector space, super quasi-topological vector space, quotient space.

Resumen

En este artículo, presentamos la idea del espacio vectorial supercuasitopológico, que es una extensión del concepto de espacio vectorial topológico, e investigamos algunas de sus propiedades básicas. Extendemos la noción existente de espacio vectorial cuasi-topológico a todos los espacios vectoriales complejos e investigamos la relación de los espacios vectoriales súper cuasi-topológicos con los espacios vectoriales paratopológicos y cuasi-topológicos.

Key words and phrases: Espacio vectorial topológico, espacio vectorial paratopológico, espacio vectorial cuasi-topológico, espacio vectorial supercuasi-topológico, espacio cociente.

1 Introduction

Recall that a paratopological group is a group G with a topology such that the group operation of G is continuous. If in addition, the inversion map in a paratopological group is continuous, then it is called a topological group.

Recibido 11/03/2022. Revisado 7/04/2022. Aceptado 21/09/2022.
MSC (2010): Primary 57N17; Secondary , 57N99.
Autor de correspondencia: Madhu Ram

According to [2], a real vector space L endowed with a topology τ such that $(L,\ +,\ \tau)$ is a paratopological group, is called:

(1) paratopological vector space if for each neighborhood U of λx with $x \in L$ and $\lambda \in \mathbb{R}^+$ (the set of non-negative real numbers), there exist a neighborhood V of x and an $\epsilon > 0$ such that $[\lambda,\ \lambda + \epsilon[.V \subseteq U$.

(2) quasi-topological vector space if the function $H_r \colon L \to L$ defined by $H_r(x) = rx$ with $r \in \mathbb{R}^+$, is continuous.

Hence, all translations and dilations of a paratopological (resp. quasi-topological) vector space are homeomorphisms. For more details, see [1] and [2]. Paratopological vector spaces were discussed and many results have been obtained (for example, see [1], [2], [3] and [4]).

Lemma 1.1. *(cf. [2]) For a real vector space L with a topology τ, the following conditions are equivalent.*

I. $(L,\ \tau)$ is a paratopological vector space.

II. There exists a local basis $\mathcal{B}$ at 0 of L satisfying the following conditions:

(a) for every $U, V \in \mathcal{B}$, there exists $W \in \mathcal{B}$ such that $W \subseteq U \cap V$;

(b) for each $U \in \mathcal{B}$, there exists $V \in \mathcal{B}$ such that $V + V \subseteq U$;

(c) for each $U \in \mathcal{B}$ and for each $x \in U$, there exists $V \in \mathcal{B}$ such that $x + V \subseteq U$;

(d) for each $U \in \mathcal{B}$ and for each $r > 0$, $rU \in \mathcal{B}$;

(e) each $U \in \mathcal{B}$ is absorbent and quasi-balanced.

Motivated by the papers [2] and [3], the aim of this paper is to introduce and study the super quasi-topological vector spaces. Relationship of super quasi-topological vector spaces with paratopological, quasi-topological and topological vector spaces is investigated.

In the following, all vector spaces are over the field $F \in \{\mathbb{R},\ \mathbb{C}\}$. For any undefined concepts and terminologies, refer to [8].

2 Relationship among various classes of topological vector spaces

In this section, we define super quasi-topological vector space and extend the definition of paratopological and quasi-topological vector space to all complex vector spaces. Then we investigate the relation between super quasi-topological, quasi-topological, paratopological and topological vector spaces.

Definition 2.1. Let L be a vector space that is equipped with a topology τ such that $(L,\ +,\ \tau)$ is a paratopological group. We say that $(L,\ \tau)$ is

1. paratopological vector space if for each neighborhood U of rx with $x \in L$ and $r \in \mathbb{R}^+$ (the set of non-negative real numbers), there exist a neighborhood V of x and an $\epsilon > 0$ such that $[r,\ r + \epsilon[.V \subseteq U$;

2. quasi-topological vector space if the function $\varphi_r\colon L \to L$ defined by $\varphi_r(x) = rx$ with $r \in \mathbb{R}^+$, is continuous;

3. super quasi-topological vector space if the function $\varphi_r\colon L \to L$ defined by $\varphi_r(x) = rx$ with $r \in \mathbb{R}$, is continuous.

Proposition 2.1. *There is a first countable locally connected quasi-topological vector space which is not a super quasi-topological vector space.*

Proof. Suppose that the complex vector space $\mathbb{C} \times \mathbb{C}$ is endowed with the topology which has a base of the sets of the form $D_r \times D_s$ where $D_r = \{\frac{1}{\sqrt{2}}(x-y)+\frac{i}{\sqrt{2}}(x+y)\colon x,\ y \in \mathbb{R},\ x \geq r,\ i^2 = -1\}$, $D_s = \{s + iy\colon y \in \mathbb{R},\ i^2 = -1\}$ and $r,\ s \in \mathbb{R}$. Then $\mathbb{C} \times \mathbb{C}$ is a first countable locally connected quasi-topological vector space but it is not a super quasi-topological vector space. Furthermore, $\mathbb{C} \times \mathbb{C}$ is not a paratopological vector space. Also, it is neither a second countable nor a lindelof space. □

Proposition 2.2. *There is a first countable non-connected quasi-topological vector space which is not a paratopological vector space.*

Proof. Endow the complex vector space $\mathbb{C}$ with the topology generated by the family of sets of the form $D_r = \{\frac{1}{\sqrt{2}}(x-r) + \frac{i}{\sqrt{2}}(x+r)\colon x \in \mathbb{R},\ i^2 = -1\}$, with $r \in \mathbb{R}$. Then $\mathbb{C}$ is first countable non-connected quasi-topological vector space. Observe that $\mathbb{C}$ is not a paratopological vector space. □

Proposition 2.3. *There is a first countable connected paratopological vector space which is not a topological vector space.*

Proof. Consider the topology on the complex vector space $\mathbb{C} \times \mathbb{C}$ which has a base of the sets of the form $P_r \times Q_s$, where $P_r = \{\frac{1}{\sqrt{2}}(x-y) + \frac{i}{\sqrt{2}}(x+y)\colon x,\ y \in \mathbb{R},\ x > r,\ i^2 = -1\}$, $Q_s = \{x + iy\colon x,\ y \in \mathbb{R},\ y > s,\ i^2 = -1\}$ and $r,\ s \in \mathbb{R}$. Then $\mathbb{C} \times \mathbb{C}$ with this topology is a first countable connected paratopological vector space which is not a topological vector space. Moreover, it is second countable as well as lindelof space. □

Proposition 2.4. *There is a first countable non-connected super quasi-topological vector space which is not a paratopological vector space.*

Proof. Obtain the topology on the complex vector space $\mathbb{C}$ by the family of sets of the form $Q_r = \{\frac{1}{2}(r - \sqrt{3}y) + \frac{i}{2}(\sqrt{3}r + y)\colon y \in \mathbb{R},\ i^2 = -1\}$, with $r \in \mathbb{R}$. Then $\mathbb{C}$ with this topology is a first countable super quasi-topological vector space, but it is not a paratopological vector space. □

Proposition 2.5. *There is a first countable connected real quasi-topological vector space which is not a super quasi-topological vector space.*

Proof. Consider the topology on the real vector space $\mathbb{R}$ generated by the family of sets of the form $[a,\ +\infty)$, with $a \in \mathbb{R}$. Then $\mathbb{R}$ with this topology is a first countable connected quasi-topological vector space which is not a super quasi-topological vector space. □

Proposition 2.6. *Let $(L,\ \tau)$ be a complex paratopological vector space. Then $(L,\ \tau_\theta)$ is also a paratopological vector space where $\tau_\theta = \{e^{i\theta}U\colon U \in \tau,\ 0 \leq \theta \leq 2\pi\}$.*

Proof. Let x and y be any two elements of L, and $e^{i\theta}D$ an open neighborhood of $x+y$ (with respect to the topology τ_θ). Then there exist a neighborhood U of $e^{-i\theta}x$ and a neighborhood V of $e^{-i\theta}y$ (with respect to the topology τ) such that $U+V\subseteq D$. As $e^{-i\theta}x\in U$ and $e^{-i\theta}y\in V$, we have $x\in e^{i\theta}U$ and $y\in e^{i\theta}V$. This gives

$$x+y\in e^{i\theta}(U+V)\subseteq e^{i\theta}D.$$

Let r be any non-negative real number and $e^{i\theta}U$ an open neighborhood of rx (with respect to the topology τ_θ). Then there exist a neighborhood V of $e^{-i\theta}x$ (with respect to the topology τ) and an $\epsilon>0$ such that $[r,\ r+\epsilon[.V\subseteq U$ which implies that $rx\in[r,\ r+\epsilon[.e^{i\theta}V\subseteq[r,\ r+\epsilon[.e^{i\theta}U$. Thus $(L,\ \tau_\theta)$ is a paratopological vector space. □

Proposition 2.7. *Let $(L,\ \tau)$ be a complex quasi-topological vector space. Then $(L,\ \tau_\theta)$ is also a quasi-topological vector space where $\tau_\theta=\{e^{i\theta}U\colon U\in\tau,\ 0\le\theta\le2\pi\}$.*

Proof. Follows in a similar way as the proof of Proposition 2.6. □

Proposition 2.8. *Let $(L,\ \tau)$ be a complex super quasi-topological vector space. Then $(L,\ \tau_\theta)$ is also a super quasi-topological vector space where $\tau_\theta=\{e^{i\theta}U\colon U\in\tau,\ 0\le\theta\le2\pi\}$.*

Proof. Follows in a similar way as the proof of Proposition 2.6. □

Definition 2.2. We say that a quasi-topological vector space $(L,\ \tau)$ is strong if it satisfies the following conditions:

1. there exists a topology $\Im$ on L such that $(L,\ \Im)$ is a topological vector space with $\Im\subseteq\tau$, and

2. there exists a local base $\mathcal{B}$ at the zero vector of the quasi-topological vector space $(L,\ \tau)$ such that $V\backslash\{0\}$ is open in $(L,\ \Im)$ for every $V\in\mathcal{B}$.

Proposition 2.9. *There exists a first countable non-connected strong quasi-topological vector space which is not second countable.*

Proof. Consider the real vector space $\mathbb{R}$ endowed with the topology τ which has a base of the sets of the form $(a,\ b)$ and $[c,\ +\infty)$, where $a,\ b$ and c are real numbers. Then $(\mathbb{R},\ \tau)$ is a first countable strong quasi-topological vector space. Clearly, it is neither a connected space nor a second countable space. □

Proposition 2.10. *There exists a first countable non-connected quasi-topological vector space which is not strong.*

Proof. Consider the complex plane $\mathbb{C}$ endowed with the topology τ which has a base of the sets of the form $D(z,\ r)$ and D_t where $D(z,\ r)$ denotes the open disk with center z and radius r, and $D_t=\{z\in\mathbb{C}\colon Re(z)\ge t,\ t\in\mathbb{R}\}$. Then $(\mathbb{C},\ \tau)$ is a quasi-topological vector space which is not strong. □

Proposition 2.11. *There exists a regular super quasi-topological vector space which is not strong.*

Proof. Let $\mathbb{C}$ and τ be as in Proposition 2.5. Then $\mathbb{C}$ is not a strong quasi-topological vector space. □

Proposition 2.12. *There exists a Hausdorff strong quasi-topological vector space which is not a super quasi-topological vector space.*

Proof. Let $\mathbb{R}$ and τ be as in Proposition 2.11. Then $\mathbb{R}$ is not a super quasi-topological vector space. □

The following result collects the above information and shows that the class of paratopological vector spaces and the class of quasi-topological vector spaces are sufficiently wide.

Theorem 2.1. *The following statements are valid.*

1. *The class of quasi-topological vector spaces contains the class of super quasi-topological, strong quasi-topological, paratopological and topological vector spaces.*
2. *The class of super quasi-topological vector spaces contains the class of topological vector spaces.*
3. *The class of super quasi-topological vector spaces is independent of the class of paratopological vector spaces.*

3 Basic properties of super topological vector spaces

In this section, we investigate some basic properties of super quasi-topological vector spaces. By definition, every topological vector space is a super quasi-topological vector space, so our results on a super quasi-topological vector space can be viewed as either improvements or extensions of results in topological vector spaces. When we say that a topology τ is a super quasi-topology on a vector space L, we mean that $(L,\ \tau)$ is a super quasi-topological vector space.

Theorem 3.1. *For a super quasi-topology τ on a vector space L, $x \in L$ and a non-zero real r, the following hold:*

1. *the function $T_x\colon L \to L$ defined by $T_x(y) = x + y$ is a homeomorphism;*
2. *the function $H_r\colon L \to L$ defined by $H_r(x) = rx$ is a homeomorphism.*

Consequently for any subset P of L, we have $Cl(x+P) = x + Cl(P)$; $Int(x+P) = x + Int(P)$; $Cl(rP) = rCl(P)$; $Int(rP) = rInt(P)$ and for any open (closed) subset Q of L, $x+Q$ and rQ are open (closed).

Corollary 3.1. *Every super quasi-topological vector space is a homogeneous space.*

A subset A of a super quasi-topological vector space L is called semi-balanced if for each $x \in A$, $\lambda x \in A$ whenever $-1 \leq \lambda \leq 1$. It is semi-absorbent if for each $x \in L$, there is a real $r > 0$ such that $\lambda x \in A$ for each real λ satisfying $-r < \lambda < r$. Moreover, A is called bounded if for every neighborhood U of 0, there is a real $t > 0$ such that $A \subseteq sU$ for all reals s satisfying $|s| \geq t$.

As a consequence of Theorem 3.1, it can be shown in a similar way to that of topological vector spaces, the following result:

Theorem 3.2. *Suppose that $(L,\ \tau)$ is a super quasi-topological vector space, $x \in L$, $0 \neq r \in \mathbb{R}$ and A, B are subsets of L. The following assertions are valid:*

1. *A is open if and only if $x + A$ and rA are open;*
2. *A is closed if and only if $x + A$ and rA are closed;*

3. *A is compact if and only if $x + A$ and rA are compact;*
4. *if A is convex, then so are $Cl(A)$ and $Int(A)$;*
5. *if A is semi-balanced, then so is $Cl(A)$;*
6. *if A and B are compact, then $A + B$ is compact;*
7. *if A and B are connected, then $A + B$ is connected;*
8. *if A and B are bounded, then so are $Cl(A)$ and $A \cup B$;*
9. *any finite subset of L is bounded.*

Theorem 3.3. *Let τ be a super quasi-topology on a vector space L. There exists a local base $\mathcal{B}$ at the origin satisfying the following conditions:*

1. *for every $U, V \in \mathcal{B}$, there is $W \in \mathcal{B}$ such that $W \subseteq U \cap V$;*
2. *for each $U \in \mathcal{B}$, there is $V \in \mathcal{B}$ such that $V + V \subseteq U$;*
3. *for each $U \in \mathcal{B}$, there is a symmetric $V \in \mathcal{B}$ such that $V + V \subseteq U$;*
4. *for each $U \in \mathcal{B}$ and for each $x \in U$, there is $V \in \mathcal{B}$ such that $x + V \subseteq U$;*
5. *for each $U \in \mathcal{B}$ and $r \in \mathbb{R}$, there is $V \in \mathcal{B}$ such that $rV \subseteq U$ and $Vr \subseteq U$.*

Conversely, let L be a vector space and let $\mathcal{B}$ be a family of subsets of L satisfying (1)-(5) and that each member of $\mathcal{B}$ contains the origin. Then there is a super quasi-topology on L with $\mathcal{B}$ as a base of neighborhoods of the origin.

Proof. From Definition 2.1, and Theorem 3.1, it is easy to check that conditions (1)-(5) hold.

To prove the converse part, let $\mathcal{B}$ be a family of subsets of L satisfying the conditions (1)-(5) and that each member of $\mathcal{B}$ contains 0. Let $\Im = \{W \subseteq L\colon$ for every $x \in W$, there exists $U \in \mathcal{B}$ such that $x + U \subseteq W\}$.

Claim 1. $\Im$ is a topology on L.

Clearly, $L \in \Im$ and $\emptyset \in \Im$. It is also easy to see that $\Im$ is closed under unions. To show that $\Im$ is closed under finite intersections, let $P, Q \in \Im$ and let $x \in P \cap Q$. Then there exist $U, V \in \mathcal{B}$ such that $x + U \subseteq P$ and $x + V \subseteq Q$. From condition (1), it follows that there exists $O \in \mathcal{B}$ such that $O \subseteq U \cap V$. Then $x + O \subseteq P \cap Q$. Hence $P \cap Q \in \Im$, and $\Im$ is a topology on L.

Claim 2. If $W \in \mathcal{B}$ and $x \in L$, then $x + W \in \Im$.

Let $y \in x + W$ be an arbitrary element. Then $-x + y \in W$. From condition (4), it follows that there exists $U \in \mathcal{B}$ such that $-x + y + U \subseteq W$. This means that $y + U \subseteq x + W$. Hence $x + W \in \Im$.

Claim 3. The family $\mathbb{T}_{\mathcal{B}} = \{x + U \colon x \in L,\ U \in \mathcal{B}\}$ is a base for the topology $\Im$ on L.

Obviously, it follows from Claim 2.

Claim 4. The vector addition mapping in L is continuous with respect to the topology $\Im$.

Let $x,\ y$ be arbitrary elements of L and let W be an element of $\Im$ such that $x + y \in W$. Then there exists $U \in \mathcal{B}$ such that $x + y + U \subseteq W$. For U, there is $V \in \mathcal{B}$ such that $V + V \subseteq U$ by condition (2). Then $x + V$ and $y + B$ be two elements of $\mathbb{T}_{\mathcal{B}}$ containing x and y, respectively such that

$$(x + V) + (y + V) \subseteq x + y + V + V \subseteq x + y + U \subseteq W.$$

This ends claim 4.

Claim 5. The function $H_r : L \to L$ defined by $H_r(x) = rx$ is continuous with $r \in \mathbb{R}$.

Let W be an element of $\Im$ containing rx with $x \in L$. Then there exists $U \in \mathcal{B}$ such that $rx + U \subseteq W$. By condition (5), there is $V \in \mathcal{B}$ such that $rV \subseteq U$. Then $r(x + V) = rx + rV \subseteq rx + U \subseteq W$. This shows that H_r is continuous. □

Theorem 3.4. *Let $(L,\ \tau)$ be a super quasi-topological vector space. If $\mathcal{V}$ is the neighborhood filter of the origin, then for each $x \in L$, $\mathcal{F}(x) = \{x + V : V \in \mathcal{V}\}$ is the neighborhood filter of the point x. Consequently, a topology of a super quasi-topological vector space is completely determined by the neighborhood filter of the origin.*

Theorem 3.5. *Let $(L,\ \tau)$ be a super quasi-topological vector space. If $\mathcal{N}$ is the neighborhood filter of the origin, then for every $A \subseteq L$, $Cl(A) = \bigcap\{A + U : U \in \mathcal{N}\}$.*

Proof. Suppose that $x \in U + A$ for each $U \in \mathcal{N}$, and let W be a neighborhood of x. By Theorem 3.4, there is a symmetric $V \in \mathcal{N}$ such that $x + V \subseteq W$. By assumption, there is some $a \in A$ such that $x \in a + V$. Since V is symmetric, $a \in A \cap (x + V)$. Thus, $x \in Cl(A)$.

Conversely, if $x \in Cl(A)$, then every neighborhood $U + x$, $U \in \mathcal{N}$, contains a point of A, so for some $u \in U$, $x + u \in A$. Without loss of generality, we assume that U is symmetric. Then $x \in A + U$. It ends the proof. □

Theorem 3.6. *Let $(L,\ \tau)$ be a super quasi-topological vector space and $\mathcal{N}$ the neighborhood filter of zero in L.*

1. *The open symmetric neighborhoods of the origin form a fundamental system of neighborhoods of the origin.*
2. *The closed symmetric neighborhoods of the origin form a fundamental system of neighborhoods of the origin.*

Proof. (1) Simple.

(2) If V is a neighborhood of zero, then there is $U \in \mathcal{N}$ such that $U + U \subseteq V$. By Theorem 3.6, $Cl(U) \subseteq U + U$. Thus, V contains a closed neighborhood of zero. If P is a closed neighborhood of zero, $P \cap (-P)$ is a closed symmetric neighborhood of zero contained in V by Theorem 3.1. □

Example 3.1. *Consider the real vector space $\mathbb{C} = \{x + iy : x,\ y \in \mathbb{R},\ i^2 = -1\}$ where the addition and multiplication operation of $\mathbb{C}$ are the usual addition and multiplication of complex numbers. Endow $\mathbb{C}$ with the topology which has a base of the sets of the form $D_r = \{r + ix : y \in \mathbb{R}, i^2 = -1\}$, with $r \in \mathbb{R}$ (the set of real numbers). Then $\mathbb{C}$ with this topology is a super quasi-topological vector space which is neither a paratopological vector space nor a topological vector space.*

Theorem 3.7. *Let $(L,\ \tau)$ be a super quasi-topological vector space. Then the following conditions are equivalent:*

1. *$\{0\}$ is closed;*
2. *$\{0\}$ is the intersection of neighborhoods of the origin;*
3. *L is Hausdorff.*

Proof. By Theorem 3.6, (1) and (2) are equivalent. (3) $\Rightarrow$ (2) is obvious. Let $x,\ y$ be two elements of L such that $x \neq y$. Then $x - y \neq 0$. By part (2), there is a neighborhood V of 0 such that $x - y \notin U$. By Theorem 3.4, there is a symmetric neighborhood V of 0 such that $V + V \subseteq U$. Then it is easy to check that $x + V$ and $y + V$ are disjoint neighborhoods of x and y, respectively. It ends the proof. □

Example 3.2. *Consider the vector space $\mathbb{C}$ as in Example 3.8. For each $z_0 \in \mathbb{C}$, with $y_0 = Im(z_0)$, denote by $L_{y_0} = \{x + iy_0 \colon x \in \mathbb{R},\ i^2 = -1\}$, the horizontal line passing through y_0, and $B_\epsilon(z_0)$, the open ball with center z_0 and radius ϵ. Let*

$$U_{y_0,\ z_0,\ \epsilon} = L_{y_0} \cap B_\epsilon(z_0) \tag{3.1}$$

Obtain the topology on $\mathbb{C}$ generated by the family of sets of the form (3.1). Then $\mathbb{C}$ is a Hausdorff super quasi-topological vector space which is not a paratopological vector space.

Example 3.3. *Let L be the vector space of all continuous functions on $(0,\ 1)$. For $\varphi \in L$ and $\epsilon > 0$, let $U(\varphi,\ \epsilon) = \{h \in L \colon |h(x) - \varphi(x)| < \epsilon,\ for\ all\ x \in (0,\ 1)\}$. Obtain the topology on L that these sets $U(\varphi,\ \epsilon)$ generate. Then L with this topology is a super quasi-topological vector space, but not a topological vector space.*

Theorem 3.8. *If M is a subspace of a super quasi-topological vector space L, then $Cl(M)$ is a vector subspace of L over the field of reals. Furthermore, if L is a dense vector subspace of a super quasi-topological vector space E and if M is a vector subspace of L, then the closure of M in E is a vector subspace of E over the field of reals.*

Proof. Follows from Theorem 3.1. □

Theorem 3.9. *Let $(L,\ \tau)$ be a super quasi-topological vector space. If C is the connected component of the origin and r a non-zero real, then*

1. *$x + C$ and rC are connected for each $x \in L$;*
2. *C is a vector subspace of L over the field of reals.*

Proof. Straightforward. □

A topological space X is totally disconnected if for each $x \in X$, the singleton $\{x\}$ is connected component of X. By Theorem 3.6, a super quasi-topological vector space is totally disconnected if and only if $\{0\}$ is the connected component of 0.

Theorem 3.10. *Let φ be a linear map from a super quasi-topological vector space L to a super quasi-topological vector space E, and let $\mathcal{V}$ be the neighborhood filter of the origin in L.*

1. *φ is continuous if and only if it is continuous at 0.*
2. *φ is open if and only if for every $V \in \mathcal{V}$, $\varphi(V)$ is a neighborhood of 0 in E.*

Proof. Follows from Theorem 3.1. □

Theorem 3.11. *If a vector subspace M of a super quasi-topological vector space L has an interior point, then M is open.*

Proof. Let x be an element of M and V a neighborhood of 0 in L such that $x + V \subseteq M$. Then for any $s \in M$, we have

$$s + V = (s - x) + (x + V) \subseteq M.$$

□

4 Quotients of super quasi-topological vector spaces

A super quasi-topology on vector space L clearly induces a topology on any vector subspace of L making it a super quasi-topological vector space, and unless the contrary is mentioned, we shall assume that a vector subspace of a super quasi-topological vector space is furnished with its induced topology.

Let M be a vector subspace of a super quasi-topological vector space L. Then there is the canonical map π of L onto L/M, which induces a topology on L/M, called the quotient topology. Given a vector subspace M of a super quasi-topological vector space L and $x \in L$, denote by $\pi(x)$ or $\tilde{x}$, the coset of M that contains x.

Theorem 4.1. *If M is a vector subspace of a super quasi-topological vector space L, then the quotient map π from L onto L/M is linear, continuous and open.*

Proof. The continuity and linearity of π are obvious. Let V be an open subset of L. Since the map $x \mapsto a + x$ from L to L, with $a \in L$ is a homeomorphism, $\pi^{-1}(\pi(V)) = V + M$, an open subset of L, so $\pi(V)$ is open in L/M. □

Theorem 4.2. *If M is a vector subspace of a super quasi-topological vector space L, then L/M is a super quasi-topological vector space.*

Proof. Let $\pi(x)$ and $\pi(y)$ be two elements of L/M, and let U be an open neighborhood of $\pi(x+y)$. Then $\pi^{-1}(U)$ is an open neighborhood of $x+y$ in L, so there exist open neighborhoods V_1 and V_2 of x and y, respectively in L such that $V_1 + V_2 \subseteq \pi^{-1}(U)$. Then $\pi(V_1) + \pi(V_2) \subseteq U$. By Theorem 4.1, $\pi(V_1)$ and $\pi(V_2)$ are open sets in L/M and hence the addition map $(\pi(x),\ \pi(y)) \mapsto \pi(x+y)$ from $L/M \times L/M$ to L/M is continuous.

Let r be any real number. We have to show that the map $\pi(x) \mapsto \pi(rx)$ from L/M to L/M is continuous. As L is a super quasi-topological vector space, so for any neighborhood U of $\pi(rx)$, there exists an open neighborhood V of x in L such that $rV \subseteq \pi^{-1}(U)$. Then $r\pi(V) \subseteq U$. It ends the proof. □

Theorem 4.3. *If $\mathcal{V}$ is the neighborhood filter of 0 in a super quasi-topological vector space L, and if M is a vector subspace of L, then $\pi(\mathcal{V})$ is the neighborhood filter of $\tilde{0}$ for the quotient topology of L/M.*

Proof. By Theorem 4.1, $\pi(V)$ is a neighborhood of $\tilde{0}$ in L/M for each $V \in \mathcal{V}$. Conversely, if U is a neighborhood of $\tilde{0}$ in L/M, then $\pi^{-1}(U)$ is a neighborhood of 0 in L; so there is $V \in \mathcal{V}$ such that $V \subseteq \pi^{-1}(U)$. Thus, $\pi(V) \subseteq U$. □

Theorem 4.4. *Let M be a vector subspace of a super quasi-topological vector space L.*

1. *L/M is Hausdorff if and only if M is closed.*

2. *L/M is discrete if and only if M is open.*

Proof. Straightforward. □

Theorem 4.5. *If M and N are vector subspaces of a super quasi-topological vector space L such that $N \subseteq M$, then the quotient topology of M/N is identical with the subspace topology of M/N.*

Proof. Since M is a vector subspace of L, it is a super quasi-topological vector space with the topology induced by the topology of L. Let φ and π be the canonical mappings from M to M/N and from L to L/N, respectively. Let U be open for the quotient topology of M/N. Then $\varphi^{-1}(U)$ is open in M, so $\varphi^{-1}(U) = M \cap V$ where V is an open subset of L.

Claim: $U = (M/N) \cap \pi(V)$.

Let $\eta \in (M/N) \cap \pi(V)$. Then $\eta = x + N$ for some $x \in M$ and $\eta = v + N$ for some $v \in V$. This implies that $v - x \in N$, so $v \in x + N \subseteq M + N = M$. Therefore, $v \in M \cap V = \varphi^{-1}(U)$, so $\eta = v + N \in U$. Clearly, $U \subseteq (M/N) \cap \pi(V)$ and the claim follows.

Now let A be open in M/N for the topology on M/N induced by the quotient topology of L/N. Then $A = (M/N) \cap B$ for some open subset B of L/N. Obviously, $\varphi^{-1}(A) = M \cap \pi^{-1}(B)$ is an open subset of M. This means that A is open for the quotient topology of M/N. □

Corollary 4.1. *If M and N are vector subspaces of a super quasi-topological vector space L, then the quotient topology on $(M + N)/N$ is identical with the topology on it induced by the quotient topology of L/N.*

Theorem 4.6. *Let f be a linear map from a super quasi-topological vector space L to a super quasi-topological vector space E, and let M be a vector subspace of L that is contained in the kernel of f. The linear map g from L/M to E satisfying $g \circ \pi = f$ is continuous (open) if and only if f is continuous (open).*

Proof. The necessity part follows from Theorem 4.1. Conversely, assume f is continuous. Let U be a neighborhood of 0 in E. Then $g^{-1}(U) = \pi \circ f^{-1}(U)$, so g is continuous at 0. By Theorem 3.14, g is continuous. □

Theorem 4.7. *If M is a vector subspace of a super quasi-topological vector space L, and if M and L/M are both Hausdorff, then L is Hausdorff.*

Proof. Let x be an element of L such that $x \neq 0$ and let $x \in U$ for each $U \in \mathcal{V}$, the neighborhood filter of 0 in L. Since M is Hausdorff, $x \notin M$. Then $x + M$ and M are two distinct elements of L/M. As L/M is Hausdorff, there are disjoint open sets A and B for the quotient topology of L/M containing $x + M$ and M, respectively. By Theorem 3.14, $\pi^{-1}(A)$ is a neighborhood of x and $\pi^{-1}(B)$ is a neighborhood of 0 in L. By assumption, $x \in \pi^{-1}(B)$, so $x \in \pi^{-1}(A) \cap \pi^{-1}(B)$, a contradiction. By Theorem 3.9, L is Hausdorff. □

Theorem 4.8. *If M is the connected component of zero in a super quasi-topological vector space L, and M a vector subspace, then L/M is totally disconnected.*

Proof. Let K be a closed subset of L/M such that $\pi^{-1}(K)$ is disconnected. We will show that K is disconnected. Let A and B be non-empty subsets of $\pi^{-1}(K)$ such that $A \cup B = \pi^{-1}(K)$ and $A \cap B = \emptyset$. As for each $x \in A$, $x + M$ is connected subset of $\pi^{-1}(K)$ and hence $A = A + M = \pi^{-1}(\pi(A))$.

Similarly, $B = \pi^{-1}(\pi(B))$.

Since $\pi(A) \cap \pi(B) = \pi(A \cap B) = \emptyset$ and $(L/M)\backslash\pi(A) = \pi(L\backslash A)$ which is open, so $\pi(A)$ is closed subset of L/M. Similarly, $\pi(B)$ is closed in L/M. As

$$\pi(A) \cup \pi(B) = \pi(A \cup B) = \pi(\pi^{-1}(K)) = K,$$

so K is disconnected. Now,

if C is the connected component of zero in L/M, and if there is a point $\pi(x)$ of L/M such that $\pi(x) \in C$ and $x \notin M$, then $\pi^{-1}(C)$ would be disconnected, which is a contradiction. It ends the proof. □

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