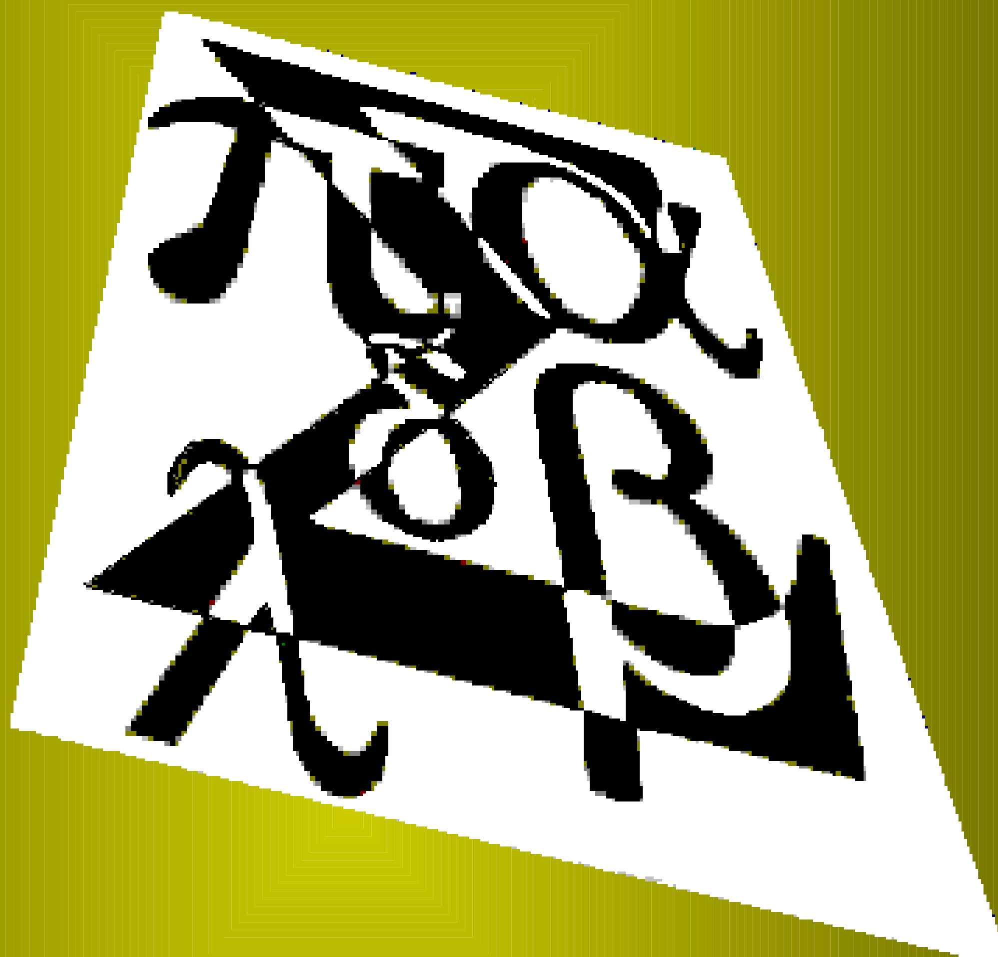
Divulgaciones Matemáticas

Universidad del Zulia



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DIVULGACIONES MATEMÁTICAS

Vol. 18 | 2017 | No. 2

Presentación

El Comité Editorial de ***Divulgaciones Matemáticas*** se complace en presentar el **Vol. 18**, **No. 2**, **2017**. En el presente número están contenidos los artículos procesados durante el segundo semestre del año **2017** que fueron evaluados y aceptados para su publicación.

Es importante resaltar que la revista recibió un total de 5 manuscritos, que fueron sometidos para su evaluación y posible publicación. Uno de estos trabajos no cumplía con el formato de la revista, por lo cual fue rechazado y se invitó al autor del mismo a escribir el artículo usando la plantilla de la revista, sin obtener respuesta. Un (1) manuscrito reprobó la evaluación de los árbitros y sólo tres (3) cumplieron con los requisitos que pide la revista y aprobaron la evaluación de los árbitros respectivos. Sin embargo, a pesar de la poca cantidad de artículos, el Comité Editorial decidió publicar este número para mostrar a la comunidad matemática que la revista sigue en funcionamiento, pese a las dificultades, tratando de matener el mejor nivel de calidad posible.

El trabajo editorial relacionado con este número es el resultado del esfuerzo de algunos miembros del Departamento de Matemática de la Facultad Experimental de Ciencias. Los Editores queremos expresar nuestro agradecimiento a todos aquellos que hicieron posible este número: a los autores de los trabajos que se presentan, que dieron su voto de confianza a la revista; a los árbitros que evaluaron los artículos, cuya labor desinteresada permitió satisfacer los estándares de calidad de la revista y mejorar sensiblemente la forma de los trabajos; al equipo editorial de ***Divulgaciones Matemáticas***; y en especial al Prof. José Heber Nieto por su aporte para la sección de ***Problemas y Soluciones***. A todos, mil gracias.

Por último, el Comité Editorial de ***Divulgaciones Matemáticas*** invita a la comunidad matemática venezolana e internacional a seguir dándonos su voto de confianza sometiendo sus trabajos en la revista para evaluación y posible publicación.

Dr. Alirio Peña [1]　　　　Dr. Tobías Rosas Soto.[2]

Dr. Vinicio Ríos[3]

[1]Editor en Jefe de *Divulgaciones Matemáticas*
[2]Editor del presente número y Editor Adjunto de *Divulgaciones Matemáticas*
[3]Editor Asociado y Miembro del Comité Editorial de *Divulgaciones Matemáticas*

Presentation

The Editorial Board of ***Divulgaciones Matemáticas*** is pleased to present the **Vol. 18**, **No. 2**, **2017**. Articles contained in this issue are those processed during the second semester of the year **2017** and were evaluated and accepted for publication.

It is important to note that the journal received a total of 5 manuscripts, which were submitted for evaluation and possible publication. One of these articles did not comply with the format of the journal, so it was rejected and the author was invited to write the article using the template of the journal, without obtaining a response. One (1) manuscript failed the evaluation of the referees and only three (3) complied with the requirements requested by the journal and approved the evaluation of the respective referees. However, in spite of the small number of articles, the Editorial Board decided to publish this issue to show the mathematical community that the journal is still working, despite the difficulties, is still active and makes efforts to keep the highest possible quality level.

The editorial work related to this issue is the result of the efforts of some members of the Department of Mathematics of the Experimental Faculty of Sciences. The Editors want to express their gratitude to all of those who made this issue possible: to the authors of the presented works, who gave their vote of confidence to the journal; to the referees, who evaluated the articles with selfless work, guaranteeing the quality standards of the journal and significantly improving the way of working; to the editorial team of ***Divulgaciones Matemáticas***; and especially to Professor José Heber Nieto, for his contribution to the ***Problems and Solutions*** section. To all of them, thanks a lot.

Finally, the Editorial Board of ***Divulgaciones Matemáticas*** invite the Venezuelan and international mathematical community to continue giving their support by submitting their articles to our journal for evaluation and possible publication.

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Dr. Tobías Rosas Soto.[5]

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[4]Chief Editor of *Divulgaciones Matemáticas*
[5]Editor of the present issue and Adjunct Editor of *Divulgaciones Matemáticas*
[6]Associate Editor and Member of the Editorial Board of *Divulgaciones Matemáticas*

DIVULGACIONES MATEMÁTICAS
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Explicit complete residue systems in a general quadratic field

Sistemas de residuos completos explícitos en un cuerpo cuadrático en general

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Abstract

Bergum explicitly determined three representations for a complete residue system in the quadratic field $\mathbb{Q}\left(\sqrt{-3}\right)$ extending two earlier results in $\mathbb{Q}\left(\sqrt{-1}\right)$ and $\mathbb{Q}\left(\sqrt{-2}\right)$. Among these three representations, the first is simplest to derive, while the third is minimal in the sense that the sum of their absolute values is minimal. Here, we extend these results by deriving explicit representations for a complete residue system in any general quadratic field. The first representation uses lattice points in a rectangle in the first quadrant of an appropriate plane, while the second representation uses lattice points in a parallelogram, and the third representation uses lattice points in a hexagon and possesses a minimality property for imaginary quadratic fields.

Key words and phrases: quadratic field, complete residue system, lattice point.

Resumen

Bergum determinó explícitamente tres representaciones para un sistema de residuo completo en el cuerpo cuadrático $\mathbb{Q}\left(\sqrt{-3}\right)$ extendiendo dos resultados anteriores en $\mathbb{Q}\left(\sqrt{-1}\right)$ y $\mathbb{Q}\left(\sqrt{-2}\right)$. Entre estas tres representaciones, la primera es más simple de obtener, mientras que la tercera es mínima en el sentido de que la suma de sus valores absolutos es mínimo. Aquí, ampliamos estos resultados obteniendo representaciones explícitas para un sistema completo de residuos en cualquier cuerpo cuadrático general. La primera representación usa puntos reticulares en un rectángulo en el primer cuadrante de un plano apropiado, mientras que la segunda representación utiliza puntos reticulares en un paralelogramo y la tercera representa puntos reticulares en un hexágono y posee una propiedad de minimalidad para cuerpos cuadráticos imaginarios.

Palabras y frases clave: cuerpos cuadráticos, sistema completo de residuos, punto reticular.

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MSC (2000): Primary 11A07; Secondary 11R04.

1 Introduction

The problem of explicitly determining complete residue systems in a general number field is nontrivial, useful and interesting. Apart from the simplest case of the rational number field [6, p. 57], not much is known for other algebraic number fields. Regarding the quadratic field, Jordan and Potratz [4] treated those in the Gaussian field $\mathbb{Q}(\sqrt{-1})$, Potratz [5] considered those in $\mathbb{Q}(\sqrt{-2})$, and Bergum [1] worked out those in $\mathbb{Q}(\sqrt{-3})$. The objective of this work is to extend these results by determining three representations of a complete residue system in any general quadratic field $\mathbb{Q}(\sqrt{m})$.

Throughout the entire paper, the following notation and terminology will be kept fixed.

1) m is a squarefree integer, $m \notin \{0, 1\}$;

2) $\sigma_m := \begin{cases} -\frac{1}{2} + \frac{\sqrt{m}}{2} & \text{if } m \equiv 1 \pmod 4 \\ \sqrt{m} & \text{if } m \not\equiv 1 \pmod 4; \end{cases}$

3) $\mathbb{Z}[\sigma_m] = \{a + b\sigma_m : \ a, b \in \mathbb{Z}\}$ is the ring of integers of $\mathbb{Q}(\sqrt{m})$;

4) $\gamma = a + b\sigma_m \in \mathbb{Z}[\sigma_m] \setminus \{0\}$ is a fixed element with (γ) being its principal ideal;

5) $N(\gamma) := \gamma\overline{\gamma} = \begin{cases} a^2 - ab + b^2(1-m)/4 & \text{if } m \equiv 1 \pmod 4 \\ a^2 - mb^2 & \text{if } m \not\equiv 1 \pmod 4 \end{cases}$ denotes the norm of γ;

6) by lattice points, we refer to the elements of $\mathbb{Z}[\sigma_m]$;

7) by a complete residue system modulo (γ) (or modulo γ), [3, Chapter IX], abbreviated by $CRS(\gamma)$, we mean a set of $|N(\gamma)|$ elements $\{\xi_1, \xi_2, \ldots, \xi_{|N(\gamma)|}\}$ such that
i) $\xi_i \not\equiv \xi_j \pmod \gamma$ for all $i, j \in \{1, 2, \ldots, |N(\gamma)|\}$ with $i \neq j$, and
ii) for each $\alpha \in \mathbb{Z}[\sigma_m]$, there is a unique $\xi_i \in CRS(\gamma)$ such that $\alpha \equiv \xi_i \pmod \gamma$.

Note that, in case $m \equiv 1 \pmod 4$, we have

$$\sigma_m^2 = -\sigma_m + \frac{m-1}{4}. \tag{1.1}$$

Our starting point is the following lemma which gives the least natural number divisible by γ; here and throughout divisibility refers to that in the ring $\mathbb{Z}[\sigma_m]$.

Lemma 1.1. *Let $\gamma = a + b\sigma_m \in \mathbb{Z}[\sigma_m] \setminus \{0\}$. If $d = \gcd(a, b) \in \mathbb{N}$ so that*

$$\gamma = d\mu, \quad \textit{where } \mu := a_1 + b_1\sigma_m \in \mathbb{Z}[\sigma_m], \ \gcd(a_1, b_1) = 1,$$

then $d\,|N(\mu)|$ is the least natural number divisible by γ.

Proof. Let $c \in \mathbb{N}$ be divisible by γ. Then there exists $\alpha = p + q\sigma_m \in \mathbb{Z}[\sigma_m]$ such that

$$c = \gamma\alpha = d(a_1 + b_1\sigma_m)(p + q\sigma_m). \tag{1.2}$$

Consider four possible cases depending on b_1 and q.

1. If $b_1 = 0$ and $q = 0$, since $\gcd(a_1, b_1) = 1$, we have $a_1 = \pm 1$, and (1.2) gives $c = \pm dp$, yielding $|c| = d\,|p| \geq d\,|N(\mu)|$.

2. If $b_1 = 0$ and $q \neq 0$, since $\gcd(a_1, b_1) = 1$, we have $a_1 = \pm 1$, and (1.2) gives $c = \pm(dp + dq\sigma_m)$, which is impossible because $q \neq 0$.

3. If $b_1 \neq 0$ and $q = 0$, from (1.2), we have $c = dpa_1 + dpb_1\sigma_m$, which implies that $p = 0$, yielding $c = 0$, a contradiction.

4. If $b_1 \neq 0$ and $q \neq 0$, there are two possible subcases depending on the value of $m \mod 4$. If $m \equiv 1 \pmod 4$, using (1.1) and (1.2), we have

$$c = d\left\{a_1 p - \left(\frac{1-m}{4}\right) b_1 q\right\} + d(a_1 q + b_1 p - b_1 q)\sigma_m \tag{1.3}$$

implying that

$$a_1 q + b_1 p - b_1 q = 0, \quad \text{i.e.,} \quad a_1 q = b_1\,(q - p)\,. \tag{1.4}$$

Thus, $b_1|q$, say, $q = b_1 l$, for some $l \in \mathbb{Z}$. Substituting into (1.4), we get $p = l(b_1 - a_1)$. Putting back into (1.3), we have $c = -ld\left(a_1^2 - a_1 b_1 + (1-m)b_1^2/4\right)$, and so $c = |-l|\,d\,|N(\mu)| \geq d\,|N(\mu)|$.

If $m \not\equiv 1 \pmod 4$, using (1.2), we have

$$c = d(a_1 p + b_1 q m) + d(a_1 q + b_1 p)\sqrt{m}. \tag{1.5}$$

implying that

$$a_1 q + b_1 p = 0, \quad \text{i.e.,} \quad a_1 q = b_1\,(-p)\,. \tag{1.6}$$

Thus, $b_1|q$, say, $q = b_1 l$, for some $l \in \mathbb{Z}$. Substituting into (1.6), we get $p = -a_1 l$. Putting back into (1.5), we have $c = -dl\left(a_1^2 - m b_1^2\right)$, and so $c = |-l|\,d\,|N(\mu)| \geq d\,|N(\mu)|$.

□

2 Representation I

Our first representation consists of lattice points in a rectangle in the first quadrant of the plane $\mathbb{R} \times \mathbb{R}\sqrt{m} = \{x + y\sqrt{m} : x, y \in \mathbb{R}\}$.

Theorem 2.1. *I. Keeping the notation of Lemma 1.1, consider the case* $m \equiv 1 \pmod 4$.
A) If d is even, let

$$T_1 := \left\{x + y\sqrt{m} : x, y \in \mathbb{Z},\ 0 \leq x \leq d\,|N(\mu)| - 1,\ 0 \leq y \leq \frac{d-2}{2}\right\}$$

$$T_2 := \left\{\left(x + \frac{1}{2}\right) + \left(y + \frac{1}{2}\right)\sqrt{m} : x, y \in \mathbb{Z},\ 0 \leq x \leq d\,|N(\mu)| - 1,\ 0 \leq y \leq \frac{d-2}{2}\right\},$$

then $T = T_1 \cup T_2$ *is a* $CRS(\gamma)$.
B) If d *is odd, let*

$$T_1 ;= \left\{ x + y\sqrt{m} : \ x, y \in \mathbb{Z}, \ 0 \leq x \leq d\,|N(\mu)| - 1, \ 0 \leq y \leq \frac{d-1}{2} \right\}$$
$$T_2 := \left\{ \left(x + \frac{1}{2}\right) + \left(y + \frac{1}{2}\right)\sqrt{m} : \ x, y \in \mathbb{Z}, \ 0 \leq x \leq d\,|N(\mu)| - 1, \ 0 \leq y \leq \frac{d-3}{2} \right\},$$

then $T = T_1 \cup T_2$ *is a* $CRS(\gamma)$.
II. For the case $m \not\equiv 1 \pmod 4$, *the set*

$$T := \left\{ x + y\sqrt{m} : x, y \in \mathbb{Z}, \ 0 \leq x \leq d\,|N(\mu)| - 1, \ 0 \leq y \leq d - 1 \right\},$$

is a $CRS(\gamma)$.

Proof. I. Let $m \equiv 1 \pmod 4$.
A) When d is even, we first show that the elements in T are distinct modulo γ. Let $\alpha_1, \alpha_2 \in T$ be such that $\alpha_1 \equiv \alpha_2 \pmod \gamma$. Then there exists $\delta = a_2 + b_2\sigma_m \in \mathbb{Z}[\sigma_m]$ such that

$$\alpha_1 - \alpha_2 = \gamma\delta = d(a_1 + b_1\sigma_m)(a_2 + b_2\sigma_m). \tag{2.1}$$

From (1.1) and (2.1), we have

$$\alpha_1 - \alpha_2 = \frac{d}{2}\left\{ \left(2a_1a_2 - a_1b_2 - a_2b_1 + \left(\frac{1+m}{2}\right)b_1b_2\right) + (a_1b_2 + a_2b_1 - b_1b_2)\sqrt{m} \right\}. \tag{2.2}$$

There are three possibilities.

Possibility 1: Both α_1 *and* α_2 *are elements of* T_1. Then they must be of the form

$$\alpha_i = x_i + y_i\sqrt{m} \qquad (i = 1, 2), \tag{2.3}$$

where $x_i, y_i \in \mathbb{Z}$, $0 \leq x_i \leq d\,|N(\mu)| - 1$ and $0 \leq y_i \leq \frac{d-2}{2}$. Substituting into (2.2) and equating the irrational parts, we get $y_1 - y_2 = \frac{d}{2}(a_1b_2 + a_2b_1 - b_1b_2)$, showing that $\frac{d}{2} \mid (y_1 - y_2)$. Since $0 \leq y_i \leq \frac{d-2}{2}$, we have $0 \leq |y_1 - y_2| \leq \frac{d-2}{2} < \frac{d}{2}$, which together with the last divisibility imply that $y_1 = y_2$. Thus, (2.1)-(2.3) yield $\gamma | (x_1 - x_2)$. Since $0 \leq x_i \leq d\,|N(\mu)| - 1$, we have $0 \leq |x_1 - x_2| \leq d\,|N(\mu)| - 1 < d\,|N(\mu)|$. Invoking upon Lemma 1.1, we deduce that $x_1 = x_2$, and so $\alpha_1 = \alpha_2$.

Possibility 2: Both α_1 *and* α_2 *are elements of* T_2. Then

$$\alpha_i = \left(x_i + \frac{1}{2}\right) + \left(y_i + \frac{1}{2}\right)\sqrt{m} \qquad (i = 1, 2), \tag{2.4}$$

where $x_i, y_i \in \mathbb{Z}$, $0 \leq x_i \leq d\,|N(\mu)| - 1$ and $0 \leq y_i \leq \frac{d-2}{2}$. Proceeding exactly as in Possibility 1, we deduce that $\alpha_1 = \alpha_2$.

Possibility 3: One of the α_i, *say,* $\alpha_1 \in T_1$, *while* $\alpha_2 \in T_2$. Then

$$\alpha_1 = x_1 + y_1\sqrt{m}, \ \ \alpha_2 = \left(x_2 + \frac{1}{2}\right) + \left(y_2 + \frac{1}{2}\right)\sqrt{m},$$

where $x_i, y_i \in \mathbb{Z}$, $0 \leq x_i \leq d\,|N(\mu)| - 1$, $0 \leq y_i \leq \frac{d-2}{2}$ $(i = 1, 2)$. Substituting into (2.2) and equating the irrational parts, we get $y_1 - y_2 - 1/2 = d\,(a_1b_2 + a_2b_1 - b_1b_2)\,/2$, which is a contradiction because the right-hand side is an integer while the left-hand side is not.

There remains to show that each element $\alpha = x + y\sigma_m \in \mathbb{Z}[\sigma_m]$ is congruent mod γ to an element of T_1 or T_2. By the Euclidean algorithm, there exist $q_1, r_1 \in \mathbb{Z}$ such that

$$y = dq_1 + r_1 \qquad (0 \leq r_1 < d).$$

Since $d = \gcd(a, b)$, there exist $u, v \in \mathbb{Z}$ such that $au + bv = dq_1$. These last two relations give

$$y = au + bv + r_1. \tag{2.5}$$

To finish the proof of this part, we treat two possible cases depending on the parity of r_1.

Case 1: r_1 is even, say, $r_1 = 2n_1$ $(n_1 \in \mathbb{N}_0)$. The next step involves a clever choosing of elements. By the Euclidean algorithm, there exist $q_2, n_2 \in \mathbb{Z}$ such that

$$x - n_1 - av - au + (1 - m)\,bu/4 = d\,|N(\mu)|\,q_2 + n_2, \quad 0 \leq n_2 < d\,|N(\mu)|\,,$$

and so

$$x = d\,|N(\mu)|\,q_2 + n_2 + n_1 + av + au - (1 - m)\,bu/4. \tag{2.6}$$

Using (2.5)-(2.6), we have

$$\begin{aligned}\alpha = x + y\sigma_m &= d\,|N(\mu)|\,q_2 + n_2 + n_1 + av + au - (1 - m)\,bu/4 + (au + bv + r_1)\sigma_m \\ &= d\,|N(\mu)|\,q_2 + (v + u(1 + \sigma_m))\gamma + n_2 + n_1\sqrt{m}.\end{aligned}$$

Since $d\,|N(\mu)| \equiv 0 \pmod{\gamma}$, we have

$$\alpha \equiv n_2 + n_1\sqrt{m} \pmod{\gamma}. \tag{2.7}$$

Since $0 \leq n_2 < d\,|N(\mu)|\,,\ 0 \leq r_1 = 2n_1 < d$, and d is even, we have $0 \leq n_2 \leq d\,|N(\mu)| - 1,\ 0 \leq n_1 \leq (d - 2)/2$. Thus, modulo γ, we have $\alpha \equiv n_2 + n_1\sqrt{m} \in\ T_1$.

Case 2: r_1 is odd, say, $r_1 = 2n_1 + 1$ $(n_1 \in \mathbb{N}_0)$. Proceeding in a manner similar to the previous case, there exist $q_2, n_2 \in \mathbb{Z}$ such that

$$x - n_1 - 1 - av - au + (1 - m)\,bu/4 = d\,|N(\mu)|\,q_2 + n_2 \qquad (0 \leq n_2 < d\,|N(\mu)|).$$

Then

$$\begin{aligned}\alpha = x + y\sigma_m &= d\,|N(\mu)|\,q_2 + n_2 + n_1 + 1 + av + au - (1 - m)\,bu/4 + (au + bv + r_1)\sigma_m \\ &= d\,|N(\mu)|\,q_2 + (v + u(1 + \sigma_m))\gamma + n_2 + 1/2 + (n_1 + 1/2)\,\sqrt{m} \\ &\equiv (n_2 + 1/2) + (n_1 + 1/2)\,\sqrt{m} \pmod{\gamma}.\end{aligned}$$

Since $0 \leq n_2 < d\,|N(\mu)|$ and $0 \leq n_1 = \frac{r_1 - 1}{2} \leq \frac{d-2}{2}$, we see that α is congruent mod γ to an element in T_2.

B) We proceed now to the case where d is odd. To show that the elements in T are distinct mod γ, let $\alpha_1, \alpha_2 \in T$ be such that $\alpha_1 \equiv \alpha_2 \pmod{\gamma}$. Then there exists $\delta = a_2 + b_2\sigma_m \in \mathbb{Z}[\sigma_m]$ such that

$$\alpha_1 - \alpha_2 = \gamma\delta = d(a_1 + b_1\sigma_m)(a_2 + b_2\sigma_m). \tag{2.8}$$

There are three possibilities.

Possibility 1: Both α_1 and α_2 are elements of T_1. Then

$$\alpha_i = x_i + y_i\sqrt{m} \qquad (i = 1, 2),$$

where $x_i, y_i \in \mathbb{Z},\ 0 \le x_i \le d\,|N(\mu)| - 1$ and $0 \le y_i \le \frac{d-1}{2}$. Substituting into (2.8) and multiplying by 2, we have

$$\begin{aligned}&2(x_1 - x_2) + 2(y_1 - y_2)\sqrt{m}\\&= d\left(\left(2a_1a_2 + \left(\frac{m+1}{2}\right)b_1b_2 - a_1b_2 - b_1a_2\right) + (a_1b_2 + b_1a_2 - b_1b_2)\sqrt{m}\right)\end{aligned}$$

Equating the irrational part, we get $2(y_1 - y_2) = d(a_1b_2 + b_1a_2 - b_1b_2)$, which shows that $d \mid 2(y_1 - y_2)$. Since $0 \le y_i \le (d-1)/2$ $(i = 1, 2)$, we deduce at once that $y_1 = y_2$, and consequently, $x_1 \equiv x_2 \pmod{\gamma}$. Since $0 \le x_i \le d\,|N(\mu)| - 1$ $(i = 1, 2)$, Lemma 1.1 shows immediately that $x_1 = x_2$, and so $\alpha_1 = \alpha_2$.

Possibility 2: Both α_1 and α_2 are elements in T_2. Then

$$\alpha_i = \left(x_i + \frac{1}{2}\right) + \left(y_i + \frac{1}{2}\right)\sqrt{m} \qquad (i = 1, 2),$$

where $x_i, y_i \in \mathbb{Z}, 0 \le x_i \le d\,|N(\mu)| - 1$ and $0 \le y_i \le \frac{d-3}{2}$. Proceeding exactly as in Possibility 1, we deduce that $\alpha_1 = \alpha_2$.

Possibility 3: One of the α_i, say, $\alpha_1 \in T_1$, while $\alpha_2 \in T_2$. Then

$$\alpha_1 = x_1 + y_1\sqrt{m}, \quad \alpha_2 = (x_2 + \frac{1}{2}) + \left(y_2 + \frac{1}{2}\right)\sqrt{m},$$

where $x_i, y_i \in \mathbb{Z},\ 0 \le x_i \le d\,|N(\mu)| - 1$ $(i = 1, 2),\ 0 \le y_1 \le \frac{d-1}{2}$ and $0 \le y_2 \le \frac{d-3}{2}$. Substituting into (2.8) and multiplying by 2, we have

$$\begin{aligned}&(2x_1 - 2x_2 - 1) + (2y_1 - 2y_2 - 1)\sqrt{m}\\&= d\left\{\left(2a_1a_2 + \frac{m+1}{2}b_1b_2 - a_1b_2 - b_1a_2\right) + (a_1b_2 + b_1a_2 - b_1b_2)\sqrt{m}\right\}.\end{aligned}$$

Equating the irrational part, we get $d \mid (2y_1 - 2y_2 - 1)$. Since $0 \le y_1 \le (d-1)/2$ and $0 \le y_2 \le (d-3)/2$, we deduce that $2y_1 = 2y_2 + 1$, which is a contradiction because the left-hand side is even, while the right-hand side is odd.

There remains to show that each element $\alpha = x + y\sigma_m \in \mathbb{Z}[\sigma_m]$ is congruent mod γ to an element of T. By the Euclidean algorithm, there exist $q_1, r_1 \in \mathbb{Z}$ such that $y = dq_1 + r_1,\ \ 0 \le r_1 < d$. Since $d = \gcd(a, b)$, there exist $u, v \in \mathbb{Z}$ such that $au + bv = dq_1$, and so $y = au + bv + r_1$. We treat three possible cases.

Case 1: r_1 is even, say $r_1 = 2n_1$ $(n_1 \in \mathbb{N}_0)$. Then there exist $q_2, n_2 \in \mathbb{Z}$ such that

$$x - n_1 - av - au + (1-m)\,bu/4 = d\,|N(\mu)|\,q_2 + n_2, \quad 0 \le n_2 < d\,|N(\mu)|,$$

and so

$$\begin{aligned}\alpha &= x + y\sigma_m \\ &= d\,|N(\mu)|\,q_2 + n_2 + n_1 + av + au - \left(\frac{1-m}{4}\right) bu + (au + bv + r_1)\left(\frac{-1}{2} + \frac{\sqrt{m}}{2}\right) \\ &= d\,|N(\mu)|\,q_2 + (v + u(1+\sigma_m))\gamma + n_2 + n_1\sqrt{m} \\ &\equiv n_2 + n_1\sqrt{m} \pmod{\gamma}.\end{aligned}$$

Since $0 \le n_2 < d\,|N(\mu)|\,,\ 0 \le n_1 = r_1/2 \le (d-1)/2$, we have $n_2 + n_1\sqrt{m} \in T_1$.

Case 2: r_1 is odd, say, $r_1 = 2n_1 + 1$ $(n_1 \in \mathbb{N}_0)$. Then there exist $q_2, n_2 \in \mathbb{Z}$ such that

$$x - n_1 - 1 - av - au + (1-m)\,bu/4 = d\,|N(\mu)|\,q_2 + n_2,\ \ 0 \le n_2 < d\,|N(\mu)|\,.$$

Then

$$\begin{aligned}\alpha &= x + y\sigma_m \\ &= d\,|N(\mu)|\,q_2 + n_2 + n_1 + 1 + av + au - \left(\frac{1-m}{4}\right) bu + (au + bv + r_1)\left(\frac{-1}{2} + \frac{\sqrt{m}}{2}\right) \\ &= d\,|N(\mu)|\,q_2 + (v + u(1+\sigma_m))\gamma + n_2 + 1/2 + (n_1 + 1/2)\sqrt{m} \\ &\equiv (n_2 + 1/2) + (n_1 + 1/2)\sqrt{m} \pmod{\gamma}.\end{aligned}$$

Since $0 \le n_2 < d\,|N(\mu)|\,,\ 0 \le n_1 = (r_1 - 1)/2 \le (d-3)/2$ (because d is odd), we have $(n_2 + 1/2) + (n_1 + 1/2)\sqrt{m} \in T_2$.

II. Let $m \not\equiv 1 \pmod 4$. To show that the elements in T are distinct mod γ, let

$$\alpha_i = x_i + y_i\sqrt{m} \in T \qquad (i = 1, 2), \tag{2.9}$$

where $x_i, y_i \in \mathbb{Z}$, $0 \le x_i \le d\,|N(\mu)| - 1$ and $0 \le y_i \le d - 1$, be such that $\alpha_1 \equiv \alpha_2 \pmod{\gamma}$. Then there exists $\delta = a_2 + b_2\sqrt{m} \in \mathbb{Z}[\sqrt{m}]$ such that $\alpha_1 - \alpha_2 = \gamma\delta$, and so

$$(x_1 - x_2) + (y_1 - y_2)\sqrt{m} = d\,(a_1a_2 + b_1b_2m) + d\,(a_1b_2 + a_2b_1)\sqrt{m}. \tag{2.10}$$

Substituting into (2.10) and equating the irrational parts, we get $y_1 - y_2 = d(a_1b_2 + a_2b_1)$, showing that $d \mid (y_1 - y_2)$. Since $0 \le y_i \le d-1$, we have $0 \le |y_1 - y_2| \le d - 1 < d$, which together with the last divisibility imply that $y_1 = y_2$. Thus, (2.10) yields $\gamma|(x_1 - x_2)$. Since $0 \le x_i \le d\,|N(\mu)| - 1$, we have $0 \le |x_1 - x_2| \le d\,|N(\mu)| - 1 < d\,|N(\mu)|$. Invoking upon Lemma 1.1, we deduce that $x_1 = x_2$, and so $\alpha_1 = \alpha_2$.

Next, we show that each element $\alpha = x + y\sqrt{m} \in \mathbb{Z}[\sqrt{m}]$ is congruent mod γ to an element of T. By the Euclidean algorithm, there exist $q_1, r_1 \in \mathbb{Z}$ such that

$$y = dq_1 + r_1 \qquad (0 \le r_1 < d).$$

Since $d = \gcd(a, b)$, there exist $u, v \in \mathbb{Z}$ such that $au + bv = dq_1$. These last two relations give

$$y = au + bv + r_1. \tag{2.11}$$

By the Euclidean algorithm, there exist $q_2, r_2 \in \mathbb{Z}$ such that

$$x - av - ubm = d\,|N(\mu)|\,q_2 + r_2, \quad 0 \le r_2 < d\,|N(\mu)|\,,$$

and so

$$x = d\,|N(\mu)|\, q_2 + r_2 + av + ubm. \tag{2.12}$$

Using (2.11)-(2.12), we have

$$\begin{aligned}\alpha = x + y\sqrt{m} &= d\,|N(\mu)|\, q_2 + r_2 + av + ubm + (au + bv + r_1)\sqrt{m}\\ &= d\,|N(\mu)|\, q_2 + av + ubm + au\sqrt{m} + bv\sqrt{m} + r_2 + r_1\sqrt{m}\\ &= d\,|N(\mu)|\, q_2 + (v + u\sqrt{m})(a + b\sqrt{m}) + r_2 + r_1\sqrt{m}\\ &= d\,|N(\mu)|\, q_2 + (v + u\sqrt{m})\gamma + r_2 + r_1\sqrt{m}.\end{aligned}$$

From Lemma 1.1, we have

$$\alpha \equiv r_2 + r_1\sqrt{m} \pmod{\gamma}. \tag{2.13}$$

Since $0 \le r_2 < d\,|N(\mu)|$ and $0 \le r_1 < d$, we have $0 \le r_2 \le d\,|N(\mu)| - 1,\ 0 \le r_1 \le d-1$. Thus, modulo γ, we have $\alpha \equiv r_2 + r_1\sqrt{m} \in\ T$. □

3 Representation II

Our second representation makes use of lattice points in a parallelogram. We begin with a simple lemma.

Lemma 3.1. *For any $\alpha_1 = a_1 + b_1\sigma_m \in \mathbb{Z}[\sigma_m]$, we have*

$$\frac{\alpha_1}{\gamma} = (r_1 + s_1\sigma_m) + (R_1 + S_1\sigma_m), \tag{3.1}$$

where $r_1, s_1 \in \mathbb{Z}$, and $R_1, S_1 \in \mathbb{Q} \cap [-1/2, 1/2)$.

Proof. Multiplying $\alpha_1/\gamma = (a_1 + b_1\sigma_m)/(a + b\sigma_m)$ by the conjugate of the denominator, we get

$$\frac{\alpha_1}{\gamma} = \frac{a_1 + b_1\sigma_m}{a + b\sigma_m} = C_1 + D_1\sigma_m, \tag{3.2}$$

where

$$C_1 := \begin{cases} \left(a_1 a - a_1 b + \frac{1-m}{4} b_1 b\right)/N(\gamma) & \text{if } m \equiv 1 \pmod 4\\ \left(a_1 a - b_1 bm\right)/N(\gamma) & \text{if } m \not\equiv 1 \pmod 4\end{cases}$$

and $D_1 := (b_1 a - a_1 b)/N(\gamma)$. The desired shape follows by taking

$$r_1 = \left\lfloor C_1 + \frac{1}{2}\right\rfloor,\ s_1 = \left\lfloor D_1 + \frac{1}{2}\right\rfloor,\ R_1 = C_1 - r_1,\ S_1 = D_1 - s_1.$$

□

Our second representation is given in

Theorem 3.2. *Let V_1 be the collection of lattice points inside the parallelogram $ABCD$ whose vertices are, respectively,*

$$A = \frac{\gamma}{2}\left(1+\sigma_m\right),\ B = \frac{\gamma}{2}(1-\sigma_m),\ C = \frac{\gamma}{2}(-1-\sigma_m),\ D = \frac{\gamma}{2}(-1+\sigma_m),$$

and let V_2 be the collection of the lattice points on the half-open line segments BC and CD excluding the points B and D, but possibly including the points C (if $C \in \mathbb{Z}[\sigma_m]$). Then $V = V_1 \cup V_2$ is a $CRS(\gamma)$.

Proof. From Lemma 3.1, we have $\alpha_1 \equiv (R_1 + S_1\sigma_m)\gamma \pmod{\gamma}$. The equations of the line segments AB, BC, CD and DA are, respectively,

$$\gamma\left(\frac{1}{2}+\frac{2t-1}{2}\,\sigma_m\right),\ \gamma\left(\frac{2t-1}{2}-\frac{\sigma_m}{2}\right),\ \gamma\left(-\frac{1}{2}-\frac{2t-1}{2}\,\sigma_m\right),\ \gamma\left(-\frac{2t-1}{2}+\frac{\sigma_m}{2}\right),$$

where $t \in \mathbb{R} \cap [0,1]$.

- If $-1/2 < R_1 < 1/2$ and $-1/2 < S_1 < 1/2$, then $(R_1 + S_1\sigma_m)\gamma$ lies inside the parallelogram $ABCD$, yielding $(R_1 + S_1\sigma_m)\gamma \in V_1$.
- If $R_1 = -1/2$, then $(R_1 + S_1\sigma_m)\gamma$ lies on $\overline{CD}$ (excluding the point D), yielding $(R_1 + S_1\sigma_m)\gamma \in V_2$.
- If $S_1 = -1/2$, then $(R_1 + S_1\sigma_m)\gamma$ lies on $\overline{BC}$ (excluding the point B), yielding $(R_1 + S_1\sigma_m)\gamma \in V_2$.

These three possibilities show that each element of $\mathbb{Z}[\sigma_m]$ is congruent to some element of V. There remains to show that the elements in V are incongruent mod γ. Note first that each element $\alpha_1 \in V = V_1 \cup V_2$ when represented under the form (3.1) of Lemma 3.1 always has $r_1 = s_1 = 0$ and so (3.1) reduces to $\alpha_1 = (R_1 + S_1\sigma_m)\gamma$. Thus, for any $\alpha_1, \alpha_2 \in V$ with $\alpha_1 \equiv \alpha_2 \pmod{\gamma}$, we have $\alpha_1 = \alpha_2 + \delta\gamma$, where $\delta \in \mathbb{Z}[\sigma_m]$ satisfies

$$\delta = (R_1 - R_2) + (S_1 - S_2)\sigma_m.$$

Since $-1/2 \le R_1, R_2, S_1, S_2 < 1/2$, and $\delta \in \mathbb{Z}[\sigma_m]$, we deduce that $\delta = 0$, yielding $\alpha_1 = \alpha_2$. □

As pointed out in [1], it is of interest to find out when the set V_2 in Theorem 3.2 is empty, which we solve in the next proposition.

Proposition 3.3. *Keeping the notation of Theorem 3.2, let $m \equiv 1 \pmod{4}$.*
I. If $(1-m)/4$ is even, then the set V_2 is empty if and only if $N(\gamma)$ is not divisible by 2.
II. If $(1-m)/4$ is odd, then the set V_2 is empty if and only if γ is not divisible by 2.

Proof. I. Let $(1-m)/4$ be even. If V_2 is empty, assuming $N(\gamma)$ is divisible by 2, we see that

$$N(\gamma) = a^2 - ab + \left(\frac{1-m}{4}\right)b^2 = a(a-b) + \left(\frac{1-m}{4}\right)b^2$$

is even, showing that either a is even, or a and b are both odd. If a is even, since

$$C = -\frac{a}{2} - \frac{b}{2}\left(\frac{m-1}{4}\right) - \frac{a}{2}\sigma_m \in \mathbb{Z}[\sigma_m],$$

the vertex C is a point of V_2. If a and b are both odd, choosing $t = 1/2$ in the parametric representation of the line BC given in Theorem 3.2, we see that there is a vertex in V_2, viz.,

$$\gamma\left(-\frac{\sigma_m}{2}\right) = -\frac{b}{2}\left(\frac{m-1}{4}\right) + \left(\frac{-a+b}{2}\right)\sigma_m \in \mathbb{Z}[\sigma_m].$$

In either case, the set V_2 is non-empty, which is a contradiction.

On the other hand, if $N(\gamma)$ is not divisible by 2, assume that $V_2 \neq \phi$. For $\alpha_1 = a_1 + b_1\sigma_m \in V_2$, we see that α_1 lies either on $\overline{BC}$ or on $\overline{CD}$. If α_1 lies on $\overline{BC}$, then from (3.2), we have $\frac{b_1a - a_1b}{N(\gamma)} = -\frac{1}{2}$, and so $N(\gamma)$ is divisible by 2, a contradiction. If α_1 lies on $\overline{CD}$, then from (3.2), we have $\frac{1}{N(\gamma)}\left(a_1a - a_1b + \frac{1-m}{4}b_1b\right) = -\frac{1}{2}$, showing that $N(\gamma)$ is divisible by 2, again a contradiction.

II. Let $(1-m)/4$ be odd. If V_2 is empty, assuming $2|\gamma$, we see that the point C is

$$\frac{\gamma}{2}(-1-\sigma_m) = -\frac{a}{2} - \frac{b}{2}\left(\frac{m-1}{4}\right) - \frac{a}{2}\sigma_m \in \mathbb{Z}\left[\sigma_m\right],$$

and so $C \in V_2$, contradicting the emptiness of V_2.

On the other hand, assume now that $2 \nmid \gamma$. If V_2 is non-empty, then let $\alpha_1 = a_1 + b_1\sigma_m \in V_2$, so that α_1 lies either on $\overline{BC}$ or on $\overline{CD}$. We pause to prove an auxiliary result.

Claim. The number $N(\gamma)$ is divisible by 2 if and only if $2|\gamma$.

Proof of Claim. We have

$$N(\gamma) = a^2 - ab + \frac{1-m}{4}b^2 = (a-b)^2 + ab + \left(\frac{1-m}{4} - 1\right)b^2.$$

If $N(\gamma)$ is divisible by 2, since $(1-m)/4$ is odd, then $a-b$ and ab are of the same parity. If $a-b$ is odd, then a and b have opposite parity, yielding ab even, a contradiction. If $a-b$ is even, then a and b have the same parity. Since ab is even, both a and b are even, implying that γ is divisible by 2. The other implication is trivial, and the claim is proved.

Returning now to the proof of part II, if α_1 lies on $\overline{BC}$, from (3.2), we have $2\left(b_1a - a_1b\right) = -N(\gamma)$, while if α_1 lies on $\overline{CD}$, from (3.2), we have

$$2\left(a_1a - a_1b + \frac{1-m}{4}b_1b\right) = -N(\gamma).$$

In either case $N(\gamma)$ is divisible by 2. Using the claim, we deduce that γ is divisible by 2, which is a contradiction. □

Proposition 3.3 gives the following generalization of Bergum's result [1].

Theorem 3.4. *Let the notation be as in Theorem 3.2. Then $V_2 = \phi$ if and only if $N(\gamma)$ is not divisible by 2 .*

Proof. The case $m \equiv 1 \pmod 4$ has already been proved in Proposition 3.3. Consider now $m \not\equiv 1$ (mod 4).

If V_2 is empty, assuming $N(\gamma)$ is divisible by 2, we see that

$$N(\gamma) = a^2 - mb^2 \tag{3.3}$$

is even. We treat two possibilities cases depending on the parity of m.

Possibility 1: m is even. From (3.3), a is also even. Choosing $t = 1/2$ in the parametric representation of the line BC given in Theorem 3.2, we see that there is a vertex in V_2, viz.,

$$\gamma\left(-\frac{\sqrt{m}}{2}\right) = -\frac{bm}{2} - \frac{a\sqrt{m}}{2} \in \mathbb{Z}[\sqrt{m}], \tag{3.4}$$

showing that the set V_2 is non-empty, which is a contradiction.

Possibility 2: m is odd, say $m = 2k+1$ ($k \in \mathbb{Z}$). Substituting into (3.3), we get

$$N(\gamma) = (a-b)(a+b) - 2kb^2. \tag{3.5}$$

Since $N(\gamma)$ is even, either a and b are both even, or a and b are both odd. If a and b are both even, the relation (3.4) yields $\gamma\left(-\sqrt{m}/2\right) \in V_2$. If a and b are both odd, since

$$C = \frac{\gamma}{2}(-1-\sqrt{m}) = -\frac{a+bm}{2} - \frac{a+b}{2}\sqrt{m} \in \mathbb{Z}[\sqrt{m}],$$

the vertex C is a point of V_2. In either case, the set V_2 is non-empty, which is a contradiction.

To establish the other implication, assume that $N(\gamma)$ is not divisible by 2. If $V_2 \neq \phi$, then for $\alpha_1 = a_1 + b_1\sqrt{m} \in V_2$, we see that α_1 lies either on $\overline{BC}$ or on $\overline{CD}$. If α_1 lies on $\overline{BC}$, then from (3.2), we have

$$\frac{ab_1 - a_1 b}{N(\gamma)} = -\frac{1}{2},$$

and so $N(\gamma)$ is divisible by 2, a contradiction. If α_1 lies on $\overline{CD}$, then from (3.2), we have

$$\frac{a_1 a - b_1 bm}{N(\gamma)} = -\frac{1}{2},$$

showing that $N(\gamma)$ is divisible by 2, again a contradiction. □

4 Representation III

Our last representation makes use of lattice points in a hexagon. Since this representation is so constructed to be minimal (in the sense that the sum of their absolute values is minimal), we need to adjust the parameters in Lemma 3.1 appropriately using the following claim.

Lemma 4.1. *For any $\alpha_1 = a_1 + b_1\sigma_m \in \mathbb{Z}[\sigma_m]$, there are rational integers r, s and rational numbers R, S such that*

$$\frac{\alpha_1}{\gamma} = (r + s\sigma_m) + (R + S\sigma_m), \tag{4.1}$$

where

$$-1 \le 2R - S < 1 \tag{4.2}$$

$$-\frac{|m|+1}{4} \le R + \left(\frac{|m|-1}{2}\right)S < \frac{|m|+1}{4} \tag{4.3}$$

$$-\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right)S - R < \frac{|m|+1}{4}. \tag{4.4}$$

(For convenience, a number written under the form (4.1) subject to (4.2)–(4.4) is said to be in a *standard form*).

Proof. By Lemma 3.1, we have $\alpha_1/\gamma = (r_1 + s_1\sigma_m) + (R_1 + S_1\sigma_m)$, where $r_1, s_1 \in \mathbb{Z}$, and $R_1, S_1 \in \mathbb{Q} \cap [-1/2, 1/2)$. We treat four possible cases depending on the subdivision of the ranges of R_1 and S_1, namely,

i) $-1/2 \le R_1 \le 0,\ -1/2 \le S_1 \le 0,$

ii) $0 < R_1 < 1/2,\ 0 < S_1 < 1/2,$

iii) $-1/2 \le R_1 \le 0,\ 0 < S_1 < 1/2,$

iv) $0 < R_1 < 1/2,\ -1/2 \le S_1 \le 0.$

For the cases i) and ii), the lemma follows by taking $r = r_1,\ s = s_1,\ R = R_1$ and $S = S_1$.
As for case iii), since

$$-\tfrac{1}{2} < R_1 + \left(\tfrac{|m|-1}{2}\right) S_1 < \tfrac{|m|-1}{4}\ ,\ -\tfrac{3}{2} < 2R_1 - S_1 < 0\ ,\ 0 < \left(\tfrac{|m|+1}{2}\right) S_1 - R_1 < \tfrac{|m|+3}{4},$$

we split our consideration into eight possibilities.

iii.1) $-\frac{1}{2} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-3}{4},\ -\frac{3}{2} < 2R_1 - S_1 < -1$ and $0 < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+1}{4}$.
The result follows by taking $r = r_1 - 1,\ s = s_1,\ R = R_1 + 1,\ S = S_1$.

iii.2) $-\frac{1}{2} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-3}{4},\ -\frac{3}{2} < 2R_1 - S_1 < -1$ and $\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+3}{4}$.
The result follows by taking $r = r_1 - 1,\ s = s_1,\ R = R_1 + 1,\ S = S_1$.

iii.3) $-\frac{1}{2} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-3}{4},\ -1 \le 2R_1 - S_1 < 0$ and $0 < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+1}{4}$.
The result follows by taking $r = r_1,\ s = s_1,\ R = R_1,\ S = S_1$.

iii.4) $-\frac{1}{2} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-3}{4},\ -1 \le 2R_1 - S_1 < 0$ and $\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+3}{4}$.
These three sets of inequalities are self-contradictory, so this possibility is ruled out.

iii.5) $\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-1}{4},\ \ -3/2 < 2R_1 - S_1 < -1$ and $0 < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+1}{4}$.
The inequalities are self-contradictory.

iii.6) $\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-1}{4},\ -3/2 < 2R_1 - S_1 < -1$ and $\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+3}{4}$.
The result follows by taking $r = r_1,\ s = s_1 + 1,\ R = R_1,\ S = S_1 - 1$.

iii.7) $\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-1}{4},\ -1 \le 2R_1 - S_1 < 0$ and
$0 < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+1}{4}$.
The result follows by taking $r = r_1,\ s = s_1,\ R = R_1,\ S = S_1$.

iii.8) $\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{|m|-1}{4},\ -1 \le 2R_1 - S_1 < 0$ and
$\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < \frac{|m|+3}{4}$.
The result follows by taking $r = r_1,\ s = s_1 + 1,\ R = R_1,\ S = S_1 - 1$.

We next turn to case iv). Since

$$-\frac{|m|-1}{4} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{1}{2},\ 0 < 2R_1 - S_1 < \frac{3}{2},\ -\frac{|m|+3}{4} < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < 0,$$

we again split our consideration into eight possibilities.

iv.1) $-\frac{|m|-1}{4} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < -\frac{|m|-3}{4},\ 0 < 2R_1 - S_1 < 1$ and
$-\frac{|m|+3}{4} < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < -\frac{|m|+1}{4}$.
The result follows by taking $r = r_1,\ s = s_1 - 1,\ R = R_1,\ S = S_1 + 1$.

iv.2) $-\frac{|m|-1}{4} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < -\frac{|m|-3}{4},\ 0 < 2R_1 - S_1 < 1$ and
$-\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < 0$.
The result follows by taking $r = r_1,\ s = s_1,\ R = R_1,\ S = S_1$.

iv.3) $-\frac{|m|-1}{4} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < -\frac{|m|-3}{4},\ 1 \le 2R_1 - S_1 < \frac{3}{2}$ and
$-\frac{|m|+3}{4} < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < -\frac{|m|+1}{4}$.
The result follows by taking $r = r_1,\ s = s_1 - 1,\ R = R_1,\ S = S_1 + 1$.

iv.4) $-\frac{|m|-1}{4} < R_1 + \left(\frac{|m|-1}{2}\right) S_1 < -\frac{|m|-3}{4},\ 1 \le 2R_1 - S_1 < \frac{3}{2}$ and
$-\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < 0$.
The inequalities are self-contradictory.

iv.5) $-\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{1}{2},\ 0 < 2R_1 - S_1 < 1$ and
$-\frac{|m|+3}{4} < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < -\frac{|m|+1}{4}$.
The inequalities are self-contradictory.

iv.6) $-\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{1}{2},\ 0 < 2R_1 - S_1 < 1$ and
$-\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < 0$.
The result follows by taking $r = r_1,\ s = s_1,\ R = R_1,\ S = S_1$.

iv.7) $-\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{1}{2},\ 1 \le 2R_1 - S_1 < \frac{3}{2}$ and
$-\frac{|m|+3}{4} < \left(\frac{|m|+1}{2}\right) S_1 - R_1 < -\frac{|m|+1}{4}$.
The result follows by taking $r = r_1 + 1,\ s = s_1,\ R = R_1 - 1,\ S = S_1$.

iv.8) $-\frac{|m|-3}{4} \le R_1 + \left(\frac{|m|-1}{2}\right) S_1 < \frac{1}{2},\ 1 \le 2R_1 - S_1 < \frac{3}{2}$ and
$-\frac{|m|+1}{4} \le \left(\frac{|m|+1}{2}\right) S_1 - R_1 < 0.$
The result follows by taking $r = r_1 + 1,\ s = s_1,\ R = R_1 - 1,\ S = S_1.$

□

We now state our third representation.

Theorem 4.2. *Let $\gamma = a + b\sigma_m \in \mathbb{Z}[\sigma_m] \setminus \{0\}$. Let W_1 be the collection of lattice points inside the hexagon $ABCDEF$ whose vertices are, respectively,*

$$A = \frac{\gamma}{|m|}\left(\frac{3|m|-1}{4} + \frac{|m|-1}{2}\sigma_m\right),\ B = \frac{\gamma}{|m|}\left(\frac{|m|+1}{4} + \frac{|m|+1}{2}\sigma_m\right),$$
$$C = \frac{\gamma}{|m|}\left(-\frac{|m|+1}{4} + \frac{|m|-1}{2}\sigma_m\right),\ D = \frac{\gamma}{|m|}\left(-\frac{3|m|-1}{4} - \frac{|m|-1}{2}\sigma_m\right),$$
$$E = \frac{\gamma}{|m|}\left(-\frac{|m|+1}{4} - \frac{|m|+1}{2}\sigma_m\right),\ F = \frac{\gamma}{|m|}\left(\frac{|m|+1}{4} - \frac{|m|-1}{2}\sigma_m\right),$$

and let W_2 be the collection of lattice points on the line segments CD, DE and EF excluding the vertices C, F, but possibly including the endpoints D (if $D \in \mathbb{Z}[\sigma_m]$) and E (if $E \in \mathbb{Z}[\sigma_m]$). Then $W = W_1 \cup W_2$ is a $CRS(\gamma)$.

Proof. We begin by showing that any $\alpha_1 = a_1 + b_1\sigma_m \in \mathbb{Z}[\sigma_m]$ is congruent mod γ to an element in W. From Lemma 4.1, we see that $\alpha_1 \equiv (R + S\sigma_m)\gamma \pmod{\gamma}$. We show next that the point $\mathcal{P} := (R + S\sigma_m)\gamma$ belongs to the set $W = W_1 \cup W_2$. Since the line segments AB, BC, CD, DE, EF and FA are given, respectively, by

$$\frac{\gamma}{|m|}\left\{\frac{|m|+1}{4} + \frac{|m|-1}{2}t + \left(\frac{|m|+1}{2} - t\right)\sigma_m\right\},$$
$$\frac{\gamma}{|m|}\left\{-\frac{|m|+1}{4} + \frac{|m|+1}{2}t + \left(\frac{|m|-1}{2} + t\right)\sigma_m\right\},$$
$$\frac{\gamma}{|m|}\left\{-\frac{3|m|-1}{4} + \frac{|m|-1}{2}t + \left(-\frac{|m|-1}{2} + (|m|-1)t\right)\sigma_m\right\},$$
$$\frac{\gamma}{|m|}\left\{-\frac{|m|+1}{4} + \frac{-|m|+1}{2}t + \left(-\frac{|m|+1}{2} + t\right)\sigma_m\right\},$$
$$\frac{\gamma}{|m|}\left\{\frac{|m|+1}{4} - \frac{|m|+1}{2}t + \left(-\frac{|m|-1}{2} - t\right)\sigma_m\right\},$$
$$\frac{\gamma}{|m|}\left\{\frac{3|m|-1}{4} + \frac{-|m|+1}{2}t + \left(\frac{|m|-1}{2} + (-|m|+1)t\right)\sigma_m\right\},$$

where $t \in \mathbb{R} \cap [0, 1]$, the location of the point $\mathcal{P}$ is easily checked as follows:

- if $-\frac{|m|+1}{4} < R + \frac{|m|-1}{2}S < \frac{|m|+1}{4}$, $-1 < 2R - S < 1$, $-\frac{|m|+1}{4} < \frac{|m|+1}{2}S - R < \frac{|m|+1}{4}$, then $\mathcal{P}$ lies inside the hexagon $ABCDEF$, i.e., $\mathcal{P} \in W_1$;
- if $2R - S = -1$, then $\mathcal{P}$ lies on $\overline{CD}$ (excluding the point C), i.e., $\mathcal{P} \in W_2$;

- if $R + \frac{|m|-1}{2}S = -\frac{|m|+1}{4}$, then $\mathcal{P}$ lies on $\overline{DE}$, i.e., $\mathcal{P} \in W_2$;
- if $\frac{|m|+1}{2}S - R = -\frac{|m|+1}{4}$, then $\mathcal{P}$ lies on $\overline{EF}$ (excluding the point F), i.e., $\mathcal{P} \in W_2$.

There remains to check that any two distinct elements of W are incongruent modulo γ. To this end, let $\alpha_1 \in W$, and assume without loss of generality that it is written in standard form as

$$\frac{\alpha_1}{\gamma} = (r + s\sigma_m) + (R + S\sigma_m) = (r + R) + (s + S)\sigma_m$$

with $r, s \in \mathbb{Z}$; $R, S \in \mathbb{Q}$ satisfying (4.2)–(4.4). Since $\alpha_1 \in W$, i.e., α lies inside the hexagon or on the line segments CD, DE, EF (excluding the vertices C, F, but possibly including the points D, E), its coordinates must satisfy

$$-1 \leq 2(R + r) - (S + s) < 1 \tag{4.5}$$

$$-\frac{|m|+1}{4} \leq (R + r) + \left(\frac{|m|-1}{2}\right)(S + s) < \frac{|m|+1}{4} \tag{4.6}$$

$$-\frac{|m|+1}{4} \leq \left(\frac{|m|+1}{2}\right)(S + s) - (R + r) < \frac{|m|+1}{4}. \tag{4.7}$$

Solving (4.2) and (4.5) and using the fact that $r, s \in \mathbb{Z}$, we get

$$2r - s = 0. \tag{4.8}$$

Solving (4.3) and (4.6), we get

$$-\frac{|m|+1}{2} < r + \left(\frac{|m|-1}{2}\right)s < \frac{|m|+1}{2}. \tag{4.9}$$

Solving (4.8) and (4.9), we get

$$-|m| < -\frac{|m|+1}{2} < |m|\, r < \frac{|m|+1}{2} < |m|\,.$$

Since $r \in \mathbb{Z}$, we must have $r = s = 0$, i.e., $\alpha_1 = (R + S\sigma_m)\gamma$. Thus, any element α_2 of W is of the form

$$\alpha_2 = (U + V\sigma_m)\gamma, \quad \text{where } U, V \text{are rational numbers satisfying (4.2)–(4.4)} \\ \text{with } U \text{ in place of } R \text{ and } V \text{ in place of } S. \tag{4.10}$$

If $\alpha_1 \equiv \alpha_2 \pmod{\gamma}$, then $\alpha_1 = \alpha_2 + \gamma\delta$ for some $\delta \in \mathbb{Z}[\sigma_m]$. If $\delta \neq 0$, then $\gamma\delta \in \mathbb{Z}[\sigma_m] \setminus \{0\}$, which is a contradiction because α_2 is of the form (4.10) but α_1 is not. Thus, $\delta = 0$ yielding $\alpha_1 = \alpha_2$. □

Our final discussion deals with the concept of minimal representation, which is defined ([1]) as follows: a representation S of a complete residue system modulo γ is said to be an *absolute minimal representation* if and only if for any representation R of a complete residue system modulo γ, we have

$$\sum_{\alpha \in S} |N(\alpha)| \leq \sum_{\beta \in R} |N(\beta)|\,.$$

Bergum in [1] discovered an absolute minimal representation modulo γ for $\mathbb{Z}[\sigma_{-3}]$. Using our third representation, this result of Bergum is now generalized but only for the case of negative integer m.

Theorem 4.3. *Let W be as defined as in Theorem 4.2. Assume that $m < 0$. If $\alpha \in W$ and if $\beta \in \mathbb{Z}[\sigma_m]$ is such that $\beta \equiv \alpha \pmod{\gamma}$, then $|N(\beta)| \geq |N(\alpha)|$.*

Proof. From the latter half of the proof of Theorem 4.2, we can write α in its standard form as $\alpha = (R + S\sigma_m)\gamma$, with the three sets of governing inequalities (4.2)–(4.4).

Consider first the case $m \equiv 1 \pmod 4$. Since $\beta \equiv \alpha \pmod{\gamma}$, we have $\beta - \alpha = \gamma(c + d\sigma_m)$ for some $c + d\sigma_m \in \mathbb{Z}[\sigma_m]$. Therefore,

$$N\left(\frac{\beta}{\gamma}\right) = E + N\left(\frac{\alpha}{\gamma}\right),$$

where $E = 2Rc + c^2 - Rd - cS - cd + \left(\frac{1-m}{2}\right) Sd + \left(\frac{1-m}{4}\right) d^2$. To prove the theorem, it suffices to check six possibilities.

1. If $c = 0$, from (4.4), we have $E = \left(\frac{1-m}{4}\right)\left\{d^2 + d\left(\frac{-4R+(2-2m)S}{1-m}\right)\right\} \geq 0$.

2. If $c = d$, from (4.3), we have $E = \left(\frac{1-m}{4}\right)\left\{d^2 + d\left(\frac{4R+(-2-2m)S}{1-m}\right)\right\} \geq 0$.

3. If $c < d$ and $c < 0$, from (4.2), we have $2R - S - d < -d + 1 \leq -c$. Thus, $c^2 + (2R - S - d)c > 0$ and (4.4) yields
$$E = c^2 + (2R - S - d)c + \left(\frac{1-m}{4}\right)\left\{d^2 + d\left(\frac{-4R+(2-2m)S}{1-m}\right)\right\} \geq 0.$$

4. If $c < d$ and $c > 0$, from (4.4), we have $c \leq d - 1 \leq \frac{-4R+(2-2m)S}{1-m} + d$, which after simplification gives $d\left\{-R + \left(\frac{1-m}{2}\right) S + \left(\frac{1-m}{4}\right) d\right\} - \left(\frac{1-m}{4}\right) cd \geq 0$. Using $\left(\frac{-3-m}{4}\right) cd \geq 0$ and (4.2), we get
$$\begin{aligned} E = {} & d\left\{-R + \left(\frac{1-m}{2}\right) S + \left(\frac{1-m}{4}\right) d\right\} - \left(\frac{1-m}{4}\right) cd + \left(\frac{-3-m}{4}\right) cd \\ & + (c^2 + c(2R - S)) \geq 0. \end{aligned}$$

5. If $c > d$ and $c < 0$, from (4.4), we get $\frac{-4R+(2-2m)S}{1-m} + d < d + 1 \leq c$, which after simplification gives
$$d\left\{-R + \left(\frac{1-m}{2}\right) S + \left(\frac{1-m}{4}\right) d\right\} - \left(\frac{1-m}{4}\right) cd > 0.$$
Using $d < c < 0$ and (4.2), we have
$$\begin{aligned} E = {} & d\left\{-R + \left(\frac{1-m}{2}\right) S + \left(\frac{1-m}{4}\right) d\right\} - \left(\frac{1-m}{4}\right) cd + \left(\frac{-3-m}{4}\right) cd \\ & + (c^2 + c(2R - S)) \geq 0. \end{aligned}$$

6. If $c > d$ and $c > 0$, from (4.2), we have $d \leq c - 1 \leq 2R - S + c$. Thus, $c(2R - S + c) - cd \geq 0$ and (4.4) yields
$$E = c(2R - S + c) - cd + \left(\frac{1-m}{4}\right)\left\{d^2 + d\left(\frac{-4R+(2-2m)S}{1-m}\right)\right\} \geq 0.$$

Next, consider the case $m \not\equiv 1 \pmod 4$. Since $\beta \equiv \alpha \pmod \gamma$, we have $\beta - \alpha = \gamma(c + d\sqrt{m})$ for some $c + d\sqrt{m} \in \mathbb{Z}[\sqrt{m}]$. From $\frac{\beta}{\gamma} = (R + c) + (S + d)\sqrt{m}$, we get

$$N\left(\frac{\beta}{\gamma}\right) = (R + c)^2 - m\,(S + d)^2 = N\left(\frac{\alpha}{\gamma}\right) + E, \tag{4.11}$$

where $E = 2Rc + c^2 - 2mSd - md^2$. Since $R, S \in [-1/2, 1/2)$, and c, d, m are rational integers with m being negative, we have $E = (c^2 + 2Rc) - m(d^2 + 2Sd) \geq 0$. Thus, (4.11) implies $|N(\beta)| \geq |N(\alpha)|$.

□

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Lema de Poincaré para un álgebra de Heisenberg semitrenzada

Poincaré Lemma for a quasibraided Heisenberg algebra

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Resumen

Un álgebra semitrenzada es un álgebra A sobre un anillo conmutativo Λ con unidad, equipada con un operador $R \in End(A \otimes A)$ que satisface la ecuación de Yang-Baxter, $R(1 \otimes a) = a \otimes 1$ y $R(a \otimes 1) = 1 \otimes a$. El cálculo diferencial semitrenzado $\Omega_R(A)$ se obtiene del cálculo diferencial universal módulo las relaciones $a\,db = \sum_i (db^i)a_i$, donde $R(a \otimes b) = \sum_i a_i \otimes b_i$. Demostramos una versión del Lema de Poincaré para el álgebra semitrenzada de Heisenberg sobre $\mathbb{R}[x]$.

Palabras y frases clave: formas diferenciales no conmutativas, álgebra semitrenzada, Lema de Poincaré.

Abstract

A quasi-braided algebra is an algebra A on a commutative ring Λ with unit, equipped with an operator $R \in End(A \otimes A)$ which satisfaces the Yang-Baxter equation, $R(1 \otimes a) = a \otimes 1$ and $R(a \otimes 1) = 1 \otimes a$. The quasi-braided differential calculus $\Omega_R(A)$ is obtained from the universal differential calculus modulo the relations $a\,db = \sum_i (db^i)a_i$, where $R(a \otimes b) = \sum_i a_i \otimes b_i$. We show a version of Poincaré's Lemma for a quasibraided Heisenberg algebra on $\mathbb{R}[x]$.

Key words and phrases: non-commutative differential forms, quasi-braided algebra, Poincaré's Lemma.

1 Introducción

El Lema de Poincaré es un resultado clásico en topología diferencial que establece que si M es una variedad suavemente contráctil a un punto $p \in M$, entonces que toda forma diferencial cerrada sobre M es exacta (ver Corolario 18 en Spivak [16]). En términos del complejo de De Rham $(\Omega^*(M), d)$, el resultado establece que para tal variedad la cohomología es trivial, por ejemplo, en el caso de coeficientes reales, tenemos que

$$H^i(M) = \begin{cases} \mathbb{R} & \text{si } i = 0, \\ 0 & \text{si } i > 0. \end{cases}$$

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El Lema de Poincaré constituye, junto con el Teorema de Stokes, las piezas fundamentales para demostrar el Teorema de De Rham que establece un isomorfismo entre la cohomología del complejo de formas diferenciales y la cohomolgía singular de la variedad (ver Spivak [16, p. 457]).

Algunos resultados se han obtenido al aplicar versiones generalizadas del complejo de De Rham. Por ejemplo, Cenkl y Porter [5] estudiaron el complejo de formas diferenciales moderadas. Similarmente, Karoubi [9] analizó el complejo de formas diferenciales no conmutativas con coeficientes racionales y Mejías [13] trabajó con el complejo de formas no conmutativas moderadas.

Un álgebra trenzada es un álgebra A sobre un anillo conmutativo Λ con unidad y equipada con un operador de Yang-Baxter R, es decir, un operador invertible $R \in End(A \otimes A)$ tal que, si $R_1(a_1 \otimes a_2 \otimes a_3) = R(a_1 \otimes a_2) \otimes a_3$ y $R_2(a_1 \otimes a_2 \otimes a_3) = a_1 \otimes R(a_2 \otimes a_3)$, se satisface la ecuación $R_1R_2R_1 = R_2R_1R_2$. Además, $R(1 \otimes a) = a \otimes 1$, $R(a \otimes 1) = 1 \otimes a$, $R(\mu \otimes \mathrm{id}) = (\mathrm{id} \otimes \mu)R_1R_2$ y $R(\mathrm{id} \otimes \mu) = (\mu \otimes \mathrm{id})R_2R_1$, siendo μ la multiplicación en A. Las dos últimas igualdades establecen cierta compatibilidad entre un álgebra trenzada y el grupo trenzado B_n (ver Baez [1] y Kassel [11]).

El cálculo diferencial trenzado $\Omega_R(A)$ se obtiene del cálculo diferencial universal $\Omega_u(A)$ (formas diferenciales no conmutativas introducidas por Connes [6]) módulo las relaciones $a\,db = \sum_i (db^i)a_i$, donde $R(a \otimes b) = \sum_i a_i \otimes b^i$.

En este artículo consideramos un concepto más débil que el de álgebra trenzada: Un "álgebra semitrenzada", un álgebra A sobre un anillo conmutativo Λ con unidad, equipada con un operador $R \in End(A \otimes A)$ que satisface la ecuación de Yang-Baxter, $R(1 \otimes a) = a \otimes 1$ y $R(a \otimes 1) = 1 \otimes a$; es decir, suprimimimos la condición de invertibilidad y la compatibilidad con el grupo trenzado. Seguidamente se establece el cálculo diferencial semitrenzado $\Omega_R(A)$ como el cociente del cálculo diferencial universal $\Omega_u(A)$ por el ideal generado por las relaciones $a\,db = \sum_i (db^i)a_i$, donde $R(a \otimes b) = \sum_i a_i \otimes b^i$.

Aplicando construcciones y técnicas similares a las de Baez [1] demostramos una versión del Lema de Poincaré para el álgebra semitrenzada de Heisenberg sobre $\mathbb{R}[x]$ (Teorema 6.2).

2 El cálculo diferencial universal

El cálculo diferencial universal sobre un álgebra fue introducido por Connes [6] y [7] como una generalización del complejo de formas diferenciales sobre una variedad y fue utilizado posteriormente por Karoubi [10] para construir el complejo no conmutativo de De Rham.

Sea A un álgebra sobre un anillo conmutativo Λ con unidad. Las *formas diferenciales de grado* n son los elementos del producto tensorial de Λ-álgebras

$$T^n(A) = \underbrace{A \otimes_\Lambda A \otimes_\Lambda \cdots \otimes_\Lambda A}_{n+1 \text{ factores}}.$$

Tenemos pues que $T^*(A) = \bigoplus_{n\geq 0} T^n(A)$ es una Λ-álgebra con multiplicación $\cdot : T^n(A) \otimes T^m(A) \to T^{n+m}(A)$ definida por

$$\alpha \cdot \beta = a_0 \otimes a_1 \otimes \cdots \otimes (a_n \cdot b_0) \otimes b_1 \otimes \cdots \otimes b_m.$$

para todos $\alpha = a_0 \otimes a_1 \otimes \cdots \otimes a_n \in T^n(A)$ y $\beta = b_0 \otimes b_1 \otimes \cdots \otimes b_m \in T^m(A)$.

El *operador diferencial* $D : T^n(A) \to T^{n+1}(A)$ está definido por

$$\begin{aligned} D(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= 1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_n \\ &\quad + \sum_{j=1}^{n} (-1)^j a_0 \otimes a_1 \otimes \cdots \otimes a_{j-1} \otimes 1 \otimes a_j \cdots \otimes a_n \\ &\quad + (-1)^{n+1} a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1. \end{aligned}$$

Teorema 2.1. *Si $\omega \in T^n(A)$ y $\theta \in T^m(A)$, entonces*

(1) $D^2(\omega) = 0$.

(2) $D(\omega \cdot \theta) = D(\omega) \cdot \theta + (-1)^n \omega \cdot D(\theta)$ *(la identidad de Leibniz).*

Es decir, $T^*(A)$ es un álgebra diferencial graduada. La cohomología del complejo $(T^*(A), D)$ es trivial.

Ahora consideremos $\Omega^0(A) = A$ y $\Omega^1(A) = \ker(\mu)$, entonces el módulo $\Omega^1(A)$ sobre Λ es un bimódulo sobre A. Las *formas diferenciales no conmutativas de grado* n son los elementos del producto tensorial de A-módulos

$$\Omega^n(A) = \underbrace{\Omega^1(A) \otimes_A \Omega^1(A) \otimes_A \cdots \otimes_A \Omega^1(A)}_{n \text{ factores}}.$$

La suma directa

$$\Omega_u(A) = \bigoplus_{n \geq 0} \Omega^n(A)$$

es un álgebra graduada cuyo producto está definido por yuxtaposición de productos tensoriales. La *diferencial* $d : \Omega^0(A) \to \Omega^1(A)$ está dada por

$$d(a) = 1 \otimes a - a \otimes 1.$$

Así tenemos el isomorfismo de Λ-módulos $A \otimes A/\Lambda \to \Omega^1(A)$ tal que $a \otimes \bar{b} \mapsto a\, db$, entonces $\Omega^n(A)$ puede ser identificado con el producto tensorial de Λ-módulos

$$A \otimes \underbrace{A/\Lambda \otimes A/\Lambda \otimes \cdots \otimes A/\Lambda}_{n \text{ factores}}.$$

Una forma $\omega \in \Omega^1(A)$ puede escribirse como una combinación lineal de términos del tipo $a_0\, da_1 da_2 \ldots da_n$ y el morfismo d se extiende a las formas de grado n de $\Omega^n(A)$ por

$$d(a_0\, da_1 \ldots da_n) = da_0 da_1 \ldots da_n = 1\, da_0 da_1 \ldots da_n.$$

Teorema 2.2. *Si $\omega \in \Omega^n(A)$ y $\theta \in \Omega^m(A)$, entonces*

(1) $d^2(\omega) = 0$.

(2) $d(\omega \cdot \theta) = d(\omega) \cdot \theta + (-1)^n \omega \cdot d(\theta)$ *(la identidad de Leibniz).*

El álgebra $\Omega_u(A)$ es *el cálculo diferencial universal* para A debido a que constituye la solución a un problema universal: Para un álgebra diferencial graduada B^* y un morfismo de álgebras $f : A \to B^0$ existe un único morfismo de álgebras diferenciales graduadas $f^* : \Omega_u(A) \to B^*$ el cual coincide con f en grado 0.

Existe una inclusión que envía $\Omega_u(A) \to T^*(A)$. Por otra parte, para cualquier $n \geq 0$ existe un *operador proyección* $J : T^n(A) \to \Omega^n(A)$ dado por $J(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0\, da_1 \ldots da_n$.

3 Álgebras trenzadas

Considerables esfuerzos han sido realizados, particularmente por Karoubi [9, 10], Baez [1, 2] y Cenkl [4] para el estudio de la cohomología del cálculo diferencial de un álgebra trenzada, la cual es un álgebra A sobre un anillo Λ con un operador $R \in End(A \otimes A)$ que generaliza la función $a \otimes b \mapsto b \otimes a$ y está relacionado de una forma especial con el grupo trenzado.

Si $R \in End(A \otimes A)$, entonces para cada i, con $1 \leq i \leq n-1$, definimos $R_i \in End(A^{\otimes n})$ por

$$R_i(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_{i-1} \otimes R(a_i \otimes a_{i+1}) \otimes a_{i+2} \otimes \cdots \otimes a_n.$$

La siguiente igualdad sobre $A^{\otimes 3}$

$$R_1 R_2 R_1 = R_2 R_1 R_2 \tag{3.1}$$

es conocida como la *ecuación de Yang-Baxter*. Si $R \in End(A \otimes A)$ es invertible y satisface la ecuación (3.1) decimos que R es un *operador de Yang-Baxter*. Decimos que R es fuerte si $R^2 = \mathrm{id}$.

Ejemplo 3.1. Para cualquier álgebra A, la función $R : A \otimes A \to A \otimes A$ dada por

$$R(a \otimes b) = b \otimes a$$

es un operador de Yang-Baxter fuerte.

Ejemplo 3.2. Si A es un álgebra graduada, la función $R : A \otimes A \to A \otimes A$ dada por

$$R(a \otimes b) = (-1)^{\deg\, a \deg\, b}\, b \otimes a$$

es un operador de Yang-Baxter (deg denota el grado).

Ejemplo 3.3. La función $R : \mathbb{R}[x] \otimes \mathbb{R}[x] \to \mathbb{R}[x] \otimes \mathbb{R}[x]$ dada por

$$R(x^m \otimes x^n) = 2^{mn}\, x^n \otimes x^n$$

satisface la igualdad (3.1), pero no es un operador de Yang-Baxter porque no es invertible.

El *grupo trenzado* B_n generado por $s_1, \ldots, s_n$ es aquél determinado por las relaciones

$$s_i s_j = s_j s_i, \quad \text{si } |i-j| \geq 2,$$

y

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

para $i = 1, 2, \ldots, n$. Entonces $R \in End(A \otimes A)$ es un operador de Yang-Baxter si solo si las n funciones $s_i \mapsto R_i$ se extienden a una representación ρ de B_n a $A^{\otimes(n+1)}$.

Un *álgebra trenzada* o una *r-estructura* es un par (A, R), donde A es un álgebra con multiplicación μ y R es un operador de Yang-Baxter sobre A, tal que se satisfacen las siguientes ecuaciones

$$R(1 \otimes a) = a \otimes 1 \quad \text{y} \quad R(a \otimes 1) = 1 \otimes a. \tag{3.2}$$

para todo $a \in A$, y

$$R(\mu \otimes \mathrm{id}) = (\mathrm{id} \otimes \mu) R_1 R_2 \quad \text{y} \quad R(\mathrm{id} \otimes \mu) = (\mu \otimes \mathrm{id}) R_2 R_1, \tag{3.3}$$

como funciones de $A^{\otimes 3}$ en $A^{\otimes 2}$. Cuando en el contexto está claro el operador R nos referimos a A como un álgebra trenzada. Las ecuaciones (3.3) describen la relación entre el operador R y el grupo trenzado B_n (ver [1]).

Así, en particular, las estructuras indicadas en los ejemplos 3.1 y 3.1 son álgebras trenzadas. El operador indicado en el Ejemplo 3.3, además de no ser invertible, no cumple la segunda ecuación de (3.3).

Un *ideal trenzado* I de un álgebra trenzada A es un ideal $I \subset A$ tal que R preserva a $I \otimes A + A \otimes I$. Dadas dos álgebras trenzadas (A, R_A) y (B, R_B) decimos que $f : A \to B$ es un *morfismo de álgebras trenzadas* si es un morfismo y $(f \otimes f)R_A = R_B(f \otimes f)$.

Para algunos casos especiales Baez [1] estudia el cálculo diferencial $\Omega_R(A)$ de un álgebra trenzada (A, R), considerado como el cociente del cálculo diferencial universal $\Omega_u(A)$ por las relaciones

$$a\, db = \sum_i (db^i) a_i, \tag{3.4}$$

donde

$$R(a \otimes b) = \sum_i b^i \otimes a_i. \tag{3.5}$$

4 Álgebras semitrenzadas

Como se indicó en el Ejemplo 3.3, el hecho de que un operador $R \in End(A \otimes A)$ sea solución de la ecuación (3.1) no dota al álgebra A con una estructura trenzada porque no es invertible ni es compatible con el grupo trenzado B_n. Sin embargo, para este tipo de operadores, se obtienen algunos resultados parciales en el espíritu de aquéllos que Baez [1] probó para álgebras trenzadas.

Un *álgebra semitrenzada* es un par (A, R), donde A es un álgebra y $R \in End(A \otimes A)$ satisface las ecuaciones (3.1) y (3.2). Un *ideal semitrenzado* I de un álgebra semitrenzada A es un ideal $I \subset A$, tal que R preserva a $I \otimes A + A \otimes I$. Dadas dos álgebras trenzadas (A, R_A) y (B, R_B) decimos que $f : A \to B$ es un *morfismo de álgebras semitrenzadas* si es un morfismo y $(f \otimes f)R_A = R_B(f \otimes f)$.

Análogamente, el cálculo diferencial $\Omega_R(A)$ para un álgebra semitrenzada (A, R) se define por ecuaciones (3.4) y (3.5) puestas en contexto.

5 La fórmula de Künneth y el Lema del Diamante

En esta sección presentamos dos teoremas clásicos que cumplen una función de resultados auxiliares en la prueba del “Lema de Poincaré”. En primer lugar la fórmula de Künneth para cohomología, que establece que si Λ es un campo, y X e Y son dos espacios topológicos con $H_i(X)$ de tipo finito, entonces existe un isomorfismo de álgebras

$$H^*(X) \otimes H^*(Y) \to H^*(X \times Y). \tag{5.1}$$

Esta ecuación es una consecuencia del siguiente resultado.

Teorema 5.1. *Sean C y C' complejos de cadenas que se anulan por debajo de cierta dimensión. Supongamos que C es libre y finitamente generado en cada dimensión, entonces existe una sucesión exacta natural*

$$0 \longrightarrow \oplus_{p+q=m} H^p(C) \otimes H^q(C) \xrightarrow{\Theta} H^m(C \otimes C') \longrightarrow \oplus_{p+q=m} H^{p+1}(C) * H^q(C') \longrightarrow 0.$$

Esta sucesión se escinde si C' es libre y finitamente generada en cada dimensión.

Para la demostración de este resultado ver Munkres [14, pp. 357-358]. También puede considerarse la presentación análoga en Spanier [15, pp. 246-247].

El otro resultado auxiliar es un resultado que algunas veces es conocido como el “Lema del Diamante”. Sean Λ un anillo conmutativo, $\mathcal{L}[\cdot,\cdot]$ un álgebra de Lie sobre Λ y $\Lambda\langle X\rangle$ el álgebra libre sobre Λ. Denotemos por $\Lambda[\mathcal{L}]$ al cociente $\Lambda\langle X\rangle/\mathcal{I}$, donde $\mathcal{I}$ es el ideal generado por todos los elementos $ab-ba-[a,b]$ con $a,b\in\mathcal{L}$. Identificamos a $\mathcal{L}$ con el submódulo de $\Lambda\langle X\rangle$ generado por X y para $a\in\mathcal{L}$ denotamos por a' la imagen de a en $\Lambda[\mathcal{L}]$.

Teorema 5.2 (Poincaré-Birkhoff-Witt)**.** *Si $\preceq$ es un orden total en X, entonces $k[\mathcal{L}]$ es un k-módulo libre con una base formada por todos los productos $x'y'\cdots z'$ tal que $x,y,...,z\in X$, y $x\preceq y\preceq\cdots\preceq z$.*

Este resultado es una consecuencia de un resultado de gran generalidad conocido más frecuentemente como el Lema del Diamante. Para la demostración del Teorema 5.2 ver Bergman [3, pp. 186-187].

6 Lema de Poincaré para álgebras semitrenzadas

En esta sección demostramos el “Lema de Poincaré” para un álgebra semitrenzada construida a partir del Ejemplo 3.3, utilizando técnicas similares a las de Baez [1]. Recordemos que para un álgebra de Lie $(\mathcal{L},[\cdot,\cdot])$, el *cubrimiento universal* $\mathcal{U}(L)=T(L)/\mathcal{I}$, donde $\mathcal{I}$ es el ideal generado por los elementos de la forma $xy-yx-[x,y]$. Dados un espacio vectorial V sobre un cuerpo $\mathbb{F}$ y $\phi: V\times V\to\mathbb{F}$ una forma bilineal antisimétrica, entonces el espacio $V\oplus\mathbb{F}$ es un álgebra de Lie con:

$$[u+\alpha \mathrm{e}, v+\beta \mathrm{e}]=\phi(u,v)\mathrm{e},$$

para todos $u,v\in V$, $\alpha,\beta\in\mathbb{F}$, $\mathrm{e}=(0,1)\in V\oplus\mathbb{F}$. El *álgebra de Heisenberg* $\mathfrak{H}$ sobre V es el cubrimiento universal $U(V\oplus\mathbb{F})$. Dado $\hbar\in\mathbb{F}$, el *álgebra de Weyl* $\mathfrak{H}_\hbar$ es el cociente de $\mathfrak{H}$ por ideal generado por $\mathrm{e}-\hbar$ y denotemos por j la proyección $j:\mathfrak{H}\to\mathfrak{H}_\hbar$. Por el Teorema 1 de Baez [1] existe una única estructura de álgebra trenzada sobre $\mathfrak{H}_\hbar$ dada por

$$\widehat{R}(u\otimes v)=v\otimes u+\hbar\phi(u,v)(1\otimes 1),$$

para todos $u,v\in V$.

Tomando el caso $\mathbb{F}=\mathbb{R}$ y $V=\mathbb{R}[x]$ podemos utilizar $\widehat{R}$ y el Ejemplo 3.3 para definir una estructura semitrenzada sobre $\mathfrak{H}_\hbar$ así:

$$R(x^m\otimes x^n)=2^{mn}x^n\otimes x^m+\hbar\phi(x^m,x^n)(1\otimes 1),$$

Ahora consideramos $\Omega(\mathfrak{H}_\hbar)=\Omega_R(\mathfrak{H}_\hbar)$ el cociente $\Omega_u(\mathfrak{H}_\hbar)$ por las relaciones

$$a\,db=\sum_i (db^i)a_i,$$

donde

$$R(a\otimes b)=\sum_i b^i\otimes a_i.$$

Teorema 6.1. *Sean $\hbar \in \mathbb{R}$, $j_* : \Omega(\mathfrak{H}) \to \Omega(\mathbb{R}[x]) \otimes \Omega(\mathfrak{H}_\hbar)$ inducida por la proyección $j : \mathfrak{H} \to \mathfrak{H}_\hbar$ y $p : \Omega(\mathbb{R}[x]) \to \mathbb{R}$ el homomorfismo determinado por $p(x) = \hbar$ y $p(dx) = 0$, entonces existe un isomorfismo de complejos diferenciales y de $\Omega(\mathbb{R}[x])$-módulos $\varphi : \Omega(\mathfrak{H}) \to \Omega(\mathbb{R}[x]) \otimes \Omega(\mathfrak{H}_\hbar)$ tal que*

$$(p \otimes \mathrm{id}) \circ \varphi = j_*.$$

Demostración. Por el Teorema 5.2 tenemos que los elementos del tipo

$$\omega = x^{i_0}(dx)^{k_0} x^{i_1}(dx)^{k_1} \cdots x^{i_n}(dx)^{k_n}$$

constituyen una base para $\Omega(\mathfrak{H})$. Definimos $\varphi : \Omega(\mathfrak{H}) \to \Omega(\mathbb{R}[x]) \otimes \Omega(\mathfrak{H}_\hbar)$ por

$$\varphi(\omega) = x^{i_0}(dx)^{k_0} \otimes x^{i_1}(dx)^{k_1} \otimes \cdots \otimes x^{i_n}(dx)^{k_n}.$$

Es fácil comprobar que φ es un morfismo de complejos diferenciales y de $\Omega(\mathbb{R}[x])$-módulos y que $(p \otimes \mathrm{id}) \circ \varphi = j_*$. Para demostrar que φ es inyectiva basta aplicar nuevamente el "Lema del Diamante" para concluir que los elementos de la forma

$$x^{i_0}(dx)^{k_0} \otimes x^{i_1}(dx)^{k_1} \otimes \cdots \otimes x^{i_n}(dx)^{k_n}$$

Constituyen una base para $\Omega(\mathbb{R}[x]) \otimes \Omega(\mathfrak{H}_\hbar)$. □

Finalmente tenemos la siguiente versión del Lema de Poincaré para el álgebra de Heisenberg:

Teorema 6.2. *Tenemos que la cohomología de De Rham $H^p(\Omega(\mathfrak{H})) = 0$ si $p > 0$ y $H^0(\Omega(\mathfrak{H})) = \mathbb{R}$.*

Demostración. Por el Teorema 6.1 y la fórmula de Künneth (ecuación (5.1)) tenemos que

$$H(\Omega(\mathfrak{H})) = H(\Omega(\mathbb{R}[x])) \otimes H(\Omega(\mathfrak{H}_\hbar)), \quad \text{para todo } \hbar \in \mathbb{R},$$

Pero tenemos que $H^p(\Omega(\mathbb{R}[x])) = 0$ si $p > 0$ y $H^0(\Omega(\mathbb{R}[x])) = \mathbb{R}$, por lo tanto $H^p(\Omega(\mathfrak{H}_\hbar))$ tambio cumple. Para concluir la prueba basta tomar $\hbar = 0$. □

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Schlict solution of Briot-Bouquet differential subordinations involving linear sums

Solución Schlict de subordinaciones diferenciales Briot-Bouquet que envuelven sumas lineales

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Abstract

It is well know that many important classes of univalent functions, for example the convex and starlike functions, are related through their derivatives by functions of positive real parts. These functions plays an important part in problem solving from signal theory, moment problems and in constructing quadrature formulas among other applications. This paper focus on an important classes of an analytic function with positive real part defined by linear sums, of particular interest is its order of schlictness in the unit disc E.

Key words and phrases: Analytic functions, subordination, Hadamard product, linear combination.

Resumen

Es bien sabido que muchas clases importantes de funciones univalentes, por ejemplo las funciones convexas y estrelladas, están relacionadas a través de sus derivadas por funciones de partes reales positivas. Estas funciones desempeñan un papel importante en la resolución de problemas desde la teoría de señales, problemas de momento y en la construcción de fórmulas de cuadratura entre otras aplicaciones. Este trabajo se centra en una clase importante de una función analítica con una parte real positiva definida por las sumas lineales, de particular interés es su orden de escisi el disco unitario E.

Palabras y frases clave: Funciones analíticas, subordinación, producto de Hadamard, combinación lineal.

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1 Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $E = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Let S be the subclass of $\mathcal{A}$ consisting of analytic univalent functions of the form (1.1).

Let $f(z)$ and $g(z)$ be analytic functions in E, $f(z)$ is said to be subordinate to $g(z)$ in E, written $f \prec g$ or $f(z) \prec g(z)$ $(z \in E)$, if there exists a Schwarz function $w(z)$, analytic in E with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, $z \in E$. It is well know that if the function g is univalent in E, then the above subordination is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$. For some works on application of subordination see [6, 11, 13, 18]. Also, let S denote the class of all functions in $\mathcal{A}$ which are univalent in U. Well know know subclasses of S include, for example, the class $S^*(\beta)$ of starlike functions of order β in E and the class $K(\beta)$ of convex functions of order β in E. There are defined as follow:

$$S^*(\beta) = \left\{ f \in S : Re\left(\frac{zf'(z)}{f(z)}\right) > \beta, 0 \leq \beta < 1, z \in U \right\} \tag{1.2}$$

$$K(\beta) = \left\{ f \in S : Re\left(1 + \frac{zf'(z)}{f'(z)}\right) > \beta, 0 \leq \beta < 1, z \in E \right\}$$

The concept of linear combination is dated as far back as 1969 with the work of [10], where he introduced the idea of α-convex functions. So many other authors like [1, 2, 16, 17] build on the celebrated idea with a lot of works in that regard scattered in this area of study. Babalola in [3] considered the linear combination of some geometric expressions.

$$Re\left[(1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)}\right] > \beta, z \in U \tag{1.3}$$

where

$$\mathcal{D}^n f(z) = \frac{z}{(1-z)^{n+1}} * f(z) = \frac{z(z^{n-1} f(z))^{(n)}}{n!},$$

as defined in [14] and $*$ stands for the familiar Hadamard product. He show that if (1.3) holds then

$$Re\left\{\frac{\mathcal{D}^n f(z)}{z}\right\} > 0.$$

Definition 1.1. Let λ, β be real numbers such that $0 \leq \beta < 1$ we define the class

$$\mathcal{B}_\lambda^n(\beta) = \left\{ f \in A : Re\left[(1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)}\right] > \beta, z \in E \right\} \tag{1.4}$$

Thus $f \in \mathcal{B}_\lambda^n(\beta)$ iff

$$(1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda \frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \prec \frac{1 + (1-2\beta)z}{1-z}$$

In recent years, the method of Briot-Bouquet differential subordination have been implored by many researchers in this noble field of univalent function theory, to sharpen and improve on many well know results [4, 6, 7, 12, 13, 15].

Definition 1.2. A function of the form $p(z)$ is said to satisfy Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad z \in E$$

for β and γ are complex constants and $h(z)$ a complex function with $h(0) = 1$ and $Re[\beta h(z)+\gamma] > 0$ in E. If

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} \prec h(z), \qquad q(0) = 1 \tag{1.5}$$

has univalent solution $q(z)$ in E, then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant.

Suppose $\Psi : \mathbb{C}^3 \times E \to E$ and let h be univalent in E and satisfies the second-order differential subordinations

$$\Psi\left(p(z), zp'(z), z^2p''(z) : z\right) \prec h(z), z \in E.$$

Then $p(z)$ is called the solution of the differential subordination. The univalent function q is called dominant if $p \prec q$ for all p satisfying (1.2). A dominant $\breve{q}$ that satisfy $\breve{q} \prec q$ for all dominant q of (1.2) is said to be the best dominant of (1.2). The best dominant is unique up to rotation of E.

We shall need the following lemmas in the sequel to prove of theorems.

Lemma 1.1. *(cf. [8]) Let η and μ be complex constant and $h(z)$ a convex univalent function in E satisfying $h(0) = 1$ and $Re[\eta h(z) + \mu] > 0$. Suppose $p \in P$ satisfies the differential subordination*

$$p(z) + \frac{zp'(z)}{\eta p(z) + \mu} \prec h(z), \quad z \in E. \tag{1.6}$$

If the differential equation

$$q(z) + \frac{zq'(z)}{\eta q(z) + \mu} = h(z), \quad q(0) = 1$$

have univalent solution $q(z)$ in E then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant of (1.5). The formal solution of (1.6) is given as

$$q(z) = \frac{zf'(z)}{f(z)} = \frac{\eta + \mu}{\eta}\left(\frac{H(z)}{F(z)}\right)^{\eta} - \frac{\mu}{\eta}$$

where

$$F(z)^n = \frac{\eta + \mu}{z^{\mu}} \int_{\circ}^{z} t^{\mu - 1} f(z)^{\eta} dt$$

and

$$H(z) = z\ exp\left(\int_{\circ}^{z} \frac{h(z) - 1}{t} dt\right)$$

Lemma 1.2. *(cf. [20]) Let V be a positive measure on $[0, 1]$.Lets h be a complex value function defined on $E \times [0, 1]$ such that $h(z, t)$ is analytic in E for each $t \in [0, 1]$ for all $z \in E$. In addition, suppose the $Re[h(z, t)] > 0$, $h(-\gamma, t)$ is real and $Re\left[\frac{1}{h(s,t)}\right] \geq \frac{1}{h(-\gamma,t)}$ for $|z| \leq \gamma < 1$ and $t \in [0, 1]$, if $h(z) = \int_{\circ}^{1} h(z, t)dv(t)$, then $Re\left[\frac{1}{h(z)}\right] \geq \frac{1}{h(-\gamma)}$.*

Lemma 1.3. *(cf. [19]) For a real numbers a, b, c $(c \neq 0, -1, -2, \cdots)$, we have*

$$\int_{\circ}^{1} t^{b-1}(1-t)^{-b-1}(1-z)^{-a}dt = \frac{\Gamma(\beta)\Gamma(c-b)}{\Gamma(c)} 2F_1(a, b, c; z) \tag{1.7}$$

$$2F_1(a, b, c; z) = 2F_1(b, a, c; z)$$

$$2F_1(a, b, c; z) = (1-z)^{-a} 2F_1(-b, a, c; \frac{z}{z-1})$$

$$2F_1(a, b, c; z) = 1 + \frac{a.b}{1.c}z + \frac{a(a+1)b(b+1)}{1.2.c(c+2)}z^2 + \cdots \tag{1.8}$$

2 Main Results

Theorem 2.1. *Let the function $f(z)$ as defined in (1.1) be in the class $\mathcal{B}_{\lambda}^{n}(\beta)$, if $\beta \geq \lambda \geq 0$ then*

$$\frac{\mathcal{D}^n f(z)}{z} \prec \frac{\lambda}{(n+1)(1-\lambda)Q(z)} = q(z) \prec \frac{1+(1-2\beta)z}{1-z}$$

Where

$$Q(z) = \left\{ \int_0^1 s^{\frac{(n+1)(1-\lambda)}{\lambda}-1} \left(\frac{1-z}{1+sz}\right)^{\frac{2(1-\beta)(n+1)}{\lambda}} ds \right.$$

and $q(z)$ is the best dominant. Furthermore,

$$Re\left(\frac{(\mathcal{D}^n f(z))}{z}\right) > \rho$$

Where

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{(n+1)(1-\lambda)}{\lambda} + 1; \frac{1}{2}\right)\right]^{-1}$$

Proof. If $f \in \mathcal{B}_{\lambda}^{n}(\beta)$ then

$$(1-\lambda)\frac{\mathcal{D}^n f(z)}{z} + \lambda\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} \prec \frac{1+(1-2\beta)z}{1-z} \tag{2.1}$$

let

$$\frac{\mathcal{D}^n f(z)}{z} = p(z) \tag{2.2}$$

Then $p(z)$ is analytic in E with $p(0) = 1$ taking the logarithm differentiation on both sides of (2.2), we have

$$\frac{\mathcal{D}^{n+1} f(z)}{\mathcal{D}^n f(z)} = 1 + \frac{zp'(z)}{(n+1)p(z)}$$

With reference to (1.4) we then have

$$\lambda + (1-\lambda)\left[p(z) + \frac{zp'(z)\lambda}{(n+1)(1-\lambda)p(z)}\right] = B_{n,\lambda}(f) \prec \frac{1+(1-2\beta)z}{1-z}$$

If

$$\lambda + (1-\lambda)\left[q(z) + \frac{zq'(z)\lambda}{\frac{(n+1)(1-\lambda)q(z)}{\lambda}}\right] = \frac{1+(1-2\beta)z}{1-z}$$

Then we have

$$q(z) + \frac{zq'(z)}{\frac{(n+1)(1-\lambda)q(z)}{\lambda}} = \frac{1-\lambda+z(1-2\beta+\lambda)}{(1-\lambda)(1-z)} = h(z) \tag{2.3}$$

It can be verified that $[\eta h(z)+\mu]$ have positive real parts given that $0 \leq \lambda \leq \beta$ Therefore by Lemma 1.1 satisfied the differential subordination $p(z) + \frac{zp'(z)}{\eta h(z)+\mu} \prec h(z)$ and hence we have that

$$\frac{\mathcal{D}^n f(z)}{z} \prec q(z) \prec h(z)$$

Where $q(z)$ is the solution solution of the differential equation (2.3) obtain through the following processes, with

$$\eta = \frac{(n+1)(1-\lambda)}{\lambda}, \quad \mu = 0$$

$$H(z) = z\ exp\left(\int_0^z \frac{h_1(t)-1}{t}dt\right) = z\ exp\left(\int_0^z \frac{2t(1-\beta)}{(1-t)(1-\lambda)t}dt\right).$$

After simplification, we have;

$$H(z) = z(1-z)^{\frac{2(1-\beta)}{1-\lambda}}$$

$$F^\eta(z) = \frac{\eta+\mu}{z^\mu}\int_0^z t^{\mu-1}H^\eta(z), \quad \text{and}\eta = \frac{(n+1)(1-\lambda)}{\lambda}, \mu = 0$$

$$F(z) = \left[\frac{(n+1)(1-\lambda)}{\lambda}\int_0^z t^{-1}\left(t^{-1}(1-t)^{\frac{2(1-\beta)}{1-\lambda}}\right)^{\frac{(n+1)(\lambda)}{\lambda}}\right]^{\frac{1}{\eta}}$$

But

$$q(z) = \frac{\eta+\mu}{\mu}\left(\frac{H(z)}{F(z)}\right)^\eta - \frac{\mu}{\eta}$$

Thus we have

$$q(z) = \frac{\lambda\left[z(1-z)^{\frac{2(1-\beta)}{1-\lambda}}\right]^{\frac{(n+1)(1-\lambda)}{\lambda}}}{(n+1)(1-\lambda)\int_0^z t^{\frac{(n+1)(1-\lambda)}{\lambda}-1}(1-t)^{\frac{2(1-\beta)(n+1)}{\lambda}}dt}$$

Let

$$s = \frac{t}{z}\text{and} \quad dt = zds.$$

After some simplification we have

$$q(z) = \frac{\lambda}{(n+1)(1-\lambda)\int_0^z s^{\frac{(n+1)(1-\lambda)}{\lambda}-1}\left(\frac{1-sz}{1-z}\right)^{\frac{2(1-\beta)(n+1)}{\lambda}}}ds$$

Therefore

$$q(z) = \frac{\lambda}{(n+1)(1-\lambda)Q'(z)}$$

where

$$Q(z) = \int_0^z s^{\frac{(n+1)(1-\lambda)}{\lambda}-1} \left(\frac{1-sz}{1-z}\right)^{\frac{2(1-\beta)(n+1)}{\lambda}} ds,$$

Next we show that

$$\inf_{|z|<1} \{Req(z)\} = q(-1), \quad z \in E \tag{2.4}$$

To prove (2.4) it suffices to show that

$$Re\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-1)}$$

$$Q(z) = \int_0^1 s^{\frac{(n+1)(1-\lambda)}{\lambda}-1} \left(\frac{1-sz}{1-z}\right)^{\frac{2(1-\beta)(n+1)}{\lambda}} ds.$$

For an appropriate choice for a ,b and c with some simplification we have

$$Q(z) = (1-z)^a \int_0^1 s^{b-1}(1-sz)^{-a} ds, \quad c > b > 0 \tag{2.5}$$

Applying (1.7) on (2.5) yields

$$Q(z) = \frac{\Gamma(b)\Gamma(1)}{\Gamma(c)} 2F_1\left(1, a; c; \frac{Bz}{1+Bz}\right)$$

Which further yields

$$Q(z) = \int_0^1 h(z,s)d\mu(s),$$

where

$$h(z,s) = \frac{1-z}{1-(1-s)z}$$

and

$$d\mu(s) = \frac{\Gamma(c)}{\Gamma(b)} s^{\frac{(n+1)(1-\lambda)}{\lambda}-1} ds.$$

Which is a positive measure on $[0,1]$. For $-1 \leq r < 0$ note that; $Re\{h(s,z)\} > 0, h(-\gamma, s)$ is real for $0 \leq \gamma < 1$

$$Re\left\{\frac{1}{h(z,s)}\right\} = Re\left\{\frac{1+(1-s)z}{1-z}\right\} \geq \frac{1-(1-s)r}{1+r} = \frac{1}{h(-r,s)}$$

For $|z| \leq r < 1$ and $s \in [0,1]$, therefore using $Re\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-r)}$ and letting $r \to 1^-$, we have

$$Re\left\{\frac{1}{Q(z)}\right\} \geq \frac{1}{Q(-1)},$$

which implies (2.4). Further simplification gives

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}$$

The bound ρ is best possible. □

Using (1.8) we can write ρ in series form as follow;

$$\frac{1}{\rho} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^{k} \frac{2(\beta-1)(n+1)+j\lambda}{n(1+\lambda)+1+j\lambda}$$

Corollary 2.1. *Let $f(z) \in B_\lambda^n(\beta)$ and $\beta \geq \lambda \geq 0$ then $\frac{\mathcal{D}^n f(z)}{z} \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}$ where $z \in E$ and $q(z)$ is the best dominant. Furthermore, Re $\left\{\frac{\mathcal{D}^n f(z)}{z}\right\} > \rho$, where*

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}.$$

Proof. If $\lambda = 0$ in (2.1), we have $\frac{\mathcal{D}^n f(z)}{z} \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}$, and q(z) is best dominant. Furthermore, *Re* $\left\{\frac{\mathcal{D}^n f(z)}{z}\right\} > \rho$, where

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}.$$

□

Putting $n = 0$ in Corollary 2.1, we obtain:

Corollary 2.2. *If $f(z) \in B_\lambda^n(\beta)$ then $\frac{f(z)}{z} \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}$ and $q(z)$ is the best dominant. Furthermore, Re $\left\{\frac{f(z)}{z}\right\} > \rho$, where*

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}$$

Putting $n = 1$ in Corollary 2.1, we obtain:

Corollary 2.3. *Let $f(z) \in B_\lambda^n(\beta)$ then $f'(z) \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}$, and $q(z)$ is the best dominant. Furthermore, Re $\{f'(z)\} > \rho$, where*

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}$$

Corollary 2.4. *Let $f(z) \in B_\lambda^n(\beta)$, $z \in E$ then $\frac{zf'(z)}{f(z)} \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}$, and $q(z)$ is the best dominant. Furthermore, Re $\left\{\frac{f(z)}{z}\right\} > \rho$, where*

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}$$

Proof. Take $\lambda = 1$ in (2.1), we have

$$\frac{f(z)}{z} \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}, \quad z \in E$$

putting $n = 0$ in the above we have $\frac{f(z)}{z} \prec q(z)$ and $q(z)$ is the best dominant. Furthermore $Re\left\{\frac{f(z)}{z}\right\} > \rho$, where

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}$$

□

Putting $n = 1$ in Corollary 2.4, we have:

Corollary 2.5. *Let* $f(z) \in B^n_\lambda(\beta)$, $z \in E$ *then* $1 + \frac{zf''(z)}{2f'(z)} \prec q(z) \prec \frac{1+(1-2\beta)z}{1-z}$ *where* $q(z)$ *is the best dominant. Furthermore* $Re\left\{\frac{f(z)}{z}\right\} > \rho$, *where*

$$\rho = \left[2F_1\left(1, \frac{2(\beta-1)(n+1)}{\lambda}; \frac{n(1+\lambda)+1}{\lambda}; \frac{1}{2}\right)\right]^{-1}$$

Remark 2.1. It is interesting to note that the results obtained herein, sharpened and improved previous results. The above corollaries speaks volumes in this regards.

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Problemas y Soluciones

Problems and Solutions

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Los problemas apropiados para esta sección son aquellos que puedan ser abordados por un estudiante de matemática no graduado sin conocimientos especializados. Problemas abiertos conocidos no son aceptables. Se prefieren problemas originales e interesantes. Las soluciones y los problemas propuestos deben dirigirse al editor por correo electrónico, en español o inglés, a la dirección arriba indicada (preferiblemente como un archivo fuente en LaTeX). Las propuestas deben acompañarse de la solución, o al menos de información suficiente que haga razonable pensar que una solución puede ser hallada.

Appropriate problems for this section are those which may be tackled by undergraduate math students without specialized knowledge. Known open problems are not suitable. Original and interesting problems are preferred. Problem proposals and solutions should be e-mailed to the editor, in Spanish or English, to the address given above (preferably as a LaTeX source file). Proposals should be accompanied by a solution or, at least, enough information on why a solution is likely.

1 Problemas propuestos

El problema propuesto a continuación se planteó en la 58^a Olimpiada Internacional de Matemáticas (IMO) celebrada en Río de Janeiro, Brasil, del 12 al 23 de Julio de este año, con la participación de 615 jóvenes provenientes de 111 países de los cinco continentes. La delegación venezolana estuvo integrada por cinco estudiantes, Wemp Pacheco Rodríguez del colegio Calicantina, Maracay, medalla de bronce por segundo año consecutivo, Amanda Vanegas Ledesma, colegio San Francisco de Asís, Maracaibo, medalla de bronce, Laura Queipo Morales, colegio San Vicente de Paul, Maracaibo, mención honorífica, Iván Rodríguez, colegio Santiago León de Caracas, mención honorífica y Onice Aguilar, colegio La Presentación, Mérida. El líder de la delegación fue el profesor Rafael Sánchez Lamoneda, y el vice-líder el editor de esta sección. Venezuela obtuvo en esta ocasión la mayor puntuación total de todas las 24 IMO a las que ha asistido.

142. (58^a IMO) Sea $N \geq 2$ un entero dado. Los $N(N+1)$ jugadores de un grupo de futbolistas, todos de distinta estatura, se colocan en fila. El técnico desea quitar $N(N-1)$ jugadores de esta fila, de modo que la fila resultante formada por los $2N$ jugadores restantes satisfaga las N condiciones siguientes:

   (1) Que no quede nadie ubicado entre los dos jugadores más altos.

   (2) Que no quede nadie ubicado entre el tercer jugador más alto y el cuarto jugador más alto.

   $\vdots$

   (N) Que no quede nadie ubicado entre los dos jugadores de menor estatura.

   Demostrar que esto siempre es posible.

2 Soluciones

Recordamos que no se han recibido soluciones a los problemas 24–28, 44, 51, 54, 59, 69, 72, 79–91, 94–106, 108–113, 116, 118–123, 125–130 y 132–141. Invitamos a los lectores a enviarnos sus soluciones a los problemas mencionados en la lista anterior.

74. [**11**(2) (2003) p. 162.] Sea S una circunferencia y AB un diámetro de ella. Sea t la recta tangente a S en B y considere dos puntos C, D en t tales que B esté entre C y D. Sean E y F las intersecciones de S con AC y AD y sean G y H las intersecciones de S con CF y DE. Demostrar que $AH = AG$.

    Solución del editor: Como $\angle AGB = \angle AHB = 90°$ y los triángulos AGB y AHB tienen el lado común AB, será suficiente demostrar que $\angle ABH = \angle ABG$, pues entonces los triángulos AGB y AHB son congruentes y por tanto $AG = AH$. Como $AEBH$ y $AGBF$ son cuadriláteros cíclicos, entonces $\angle AEH = \angle ABH$ y $\angle GBA = \angle GFA$.

    Ahora, como $\angle AEH + \angle CED = 180° = \angle GFA + \angle CFD$, si $\angle CED = \angle CFD$ entonces $\angle AEH = \angle GFA$.

    Para demostrar que $\angle CED = \angle CFD$ basta mostrar que $CEFD$ es cíclico. Esto se sigue del hecho de que el triángulo ABE es semejante al ABC y de que el triángulo AFB es semejante al ABD. Entonces, de la primera semejanza, $AB^2 = AE \cdot AC$. Y de la segunda semejanza, $AB^2 = AD \cdot AF$. En consecuencia, $AE \cdot AC = AD \cdot AF$ y $CEFD$ es cíclico.

75. [**11**(2) (2003) p. 162.] Sean a, b enteros positivos, con $a > 1$ y $b > 2$. Demostrar que $a^b + 1 \geq b(a+1)$ y determinar cuándo se tiene la igualdad.

    Solución del editor: Se procederá por inducción sobre b. Para $b = 3$, se tiene que $a^3 + 1 = (a+1)(a^2 - a + 1)$. Para mostrar que esta expresión es mayor que $3(a+1)$ es suficiente demostrar que $(a^2 - a + 1) \geq 3$, lo cual es cierto pues $a^2 - a + 1 > a(a-1) \geq 2$.

    Ahora supóngase que la expresión es cierta para algún valor de b, es decir, se cumple que $a^b + 1 \geq b(a+1)$. Se demostrará ahora para $b + 1$.

    Nótese que

    $$a^{b+1} + 1 = a(a^b + 1) - (a+1) + 2 \geq ab(a+1) - (a+1) + 2,$$

    donde la última desigualdad se tiene por la hipótesis de inducción. La última expresión se puede reescribir como

    $$ab(a+1) - (a+1) + 2 = (a+1)(ab-1) + 2 > (ab-1)(a+1).$$

    Finalmente, $ab - 1 \geq 2b - 1 = (b+1) + (b-2) > b + 1$, lo cual es cierto.

    Por tanto, la desigualdad se vuelve estricta después de $b = 3$. Retomando el caso $b = 3$, se observa que $a(a-1) = 2$ únicamente cuando $a = 2$. Por tanto, se ha demostrado por inducción que la desigualdad siempre se tiene, y que la igualdad se da únicamente en el caso $a = 2$, $b = 3$. Esto concluye la solución.

76. [**11**(2) (2003) p. 162.] Sean S_1 y S_2 dos circunferencias que se intersectan en dos puntos distintos P y Q. Sean ℓ_1 y ℓ_2 dos rectas paralelas, tales que:

    i. ℓ_1 pasa por el punto P e intersecta a S_1 en un punto A_1 distinto de P y a S_2 en un punto A_2 distinto de P.

ii. ℓ_2 pasa por el punto Q e intersecta a S_1 en un punto B_1 distinto de Q y a S_2 en un punto B_2 distinto de Q.

Demostrar que los triángulos A_1QA_2 y B_1PB_2 tienen igual perímetro.

Solución del editor: Se demostrará que los triángulos A_1QA_2 y B_1PB_2 son congruentes, de donde el resultado se sigue de forma inmediata.

Nótese inicialmente que como $A_1P \parallel B_1Q$, entonces A_1PQB_1 es un trapecio isósceles y sus diagonales son iguales, de donde $A_1Q = B_1P$. Ahora bien, como $\angle PA_1Q = \angle PB_1Q$ por estar inscritos en el mismo arco, y $\angle PA_2Q = PB_2Q$ por la misma razón, entonces $\triangle A_1QA_2$ y $\triangle B_1PB_2$ son semejantes. Como ya se demostró la igualdad entre un par de lados adyacentes, se sigue la congruencia de los triángulos.

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Divulgaciones Matemáticas es una revista arbitrada que considera para su publicación trabajos inéditos de investigación, en todas las áreas de la Matemática y sus aplicaciones, historia o enseñanza. Contribuciones adecuadas trabajos de investigación, de divulgación e históricos y de enseñanza matemática. Se presta especial atención a los temas tratados en la reunión anual e itinerante de las **Jornadas Venezolanas de Matemáticas** celebradas en Venezuela. Además, contempla una sección de problemas y soluciones, la cual presenta problemas que puedan ser abordados por un estudiante de matemática no graduado, sin conocimientos especializados.

El primer requisito para que un artículo sea publicable es su corrección matemática. En segundo lugar, el estilo expositivo debe ser atrayente y lo más fluido y organizado que sea posible. En los trabajos de investigación se tomarán en cuenta la relevancia y originalidad de los resultados obtenidos. El tercer requisito para que el cuerpo editorial de la revista acepte un artículo, para someterlo a evaluación y posible publicación, es que el mismo debe estar elaborado en LaTeX, utilizando una plantilla predefinida por la revista, se le pide a los autores respetar las instrucciones internas indicadas en la plantilla mencionada. El archivo fuente (.tex) y una versión en formato .dvi, .pdf o .ps (imprimible) debe enviarse por correo electrónico a `divulgaciones@demat-fecluz.org`. Si el artículo contiene figuras, éstas deben adjuntarse como archivos separados en formatos .png o .jpg.

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