DEPÓSITO LEGAL ZU2020000153
ISSN 0041-8811
E-ISSN 2665-0428

Revista de la Universidad del Zulia

Fundada en 1947
por el Dr. Jesús Enrique Lossada

Ciencias del Agro, Ingeniería y Tecnología

Año 14 N° 39
Enero - Abril 2023
Tercera Época
Maracaibo-Venezuela

REVISTA DE LA UNIVERSIDAD DEL ZULIA. 3ª época. Año 14, N° 39, 2023
Y. Kostikov & A. Romanenkov /// On solution of pseudohyperbolic equation... 225-232
DOI: http://dx.doi.org/10.46925//rdluz.39.12

On solution of pseudohyperbolic equation with constant coefficients

Yury Kostikov*
Aleksandr Romanenkov**

ABSTRACT

The paper proposes the method of forming the exact solution of the first initial boundary value problem for one-dimensional linear pseudohyperbolic equation with constant coefficients. To obtain the solution type, the modification of partition method (Fourier method) is used, when the type of one of the solution functional factors is considered to be known. At the same time, the initial problem is reduced to parameterized family of Cauchy problems for ordinary differential equations. The paper presents explicitly calculated formulas, which specify the solution. The qualitative research of the solution properties has been conducted. The conditions for coefficients in the form of inequalities have been obtained that is indicative of boundedness and variability of the solutions. Several examples confirming the results obtained have been considered.

KEYWORDS: Pseudohyperbolic equation, exact problem-solving solutions, solution properties.

*Candidate of Science in Physics and Mathematics, Head of Department 916, Moscow Aviation Institute (National Research University) Volokolamskoe shosse, 4, Moscow, 125993. E-mail: jkostikov@mail.ru

** Candidate of Technical Science, Associate Professor of Department 916, Moscow Aviation Institute (National Research University) Volokolamskoe shosse, 4, Moscow, 125993 Federal Research Center "Informatics and Control", Russian Academy of Sciences Vavilova St., 44, bld. 2, Moscow, 119333. ORCID: https://orcid.org/0000-0002-0700-8465. E-mail: romanaleks@gmail.com

Recibido: 13/09/2022 Aceptado: 16/11/2022

DOI: http://dx.doi.org/10.46925//rdluz.39.12

Sobre la solución de la ecuación pseudohiperbólica con coeficientes constantes

RESUMEN

El artículo propone el método para formar la solución exacta del primer problema de valor límite inicial para una ecuación pseudohiperbólica lineal unidimensional con coeficientes constantes. Para obtener el tipo de solución se utiliza el método de modificación de la partición (método de Fourier), cuando se considera conocido el tipo de uno de los factores funcionales de la solución. Al mismo tiempo, el problema inicial se reduce a la familia parametrizada de problemas de Cauchy para ecuaciones diferenciales ordinarias. El documento presenta fórmulas calculadas explícitamente, que especifican la solución. Se ha llevado a cabo la investigación cualitativa de las propiedades de la solución. Se han obtenido las condiciones para los coeficientes en forma de desigualdades que son indicativas de acotación y variabilidad de las soluciones. Se han considerado varios ejemplos que confirman los resultados obtenidos.

PALABRAS CLAVE: Ecuación pseudohiperbólica, soluciones exactas de resolución de problemas, propiedades de la solución.

Introduction

Let us consider the first initial boundary value problem for pseudohyperbolic equation. It is necessary to find function $u = u(x,t), u \in C^3[0;l] \times [0;T], T > 0$, which satisfies the linear homogeneous equation (1):

$$u_{tt} - A^2 u_{txx} + B^2 u_t - \alpha^2 u_{xx} + \beta^2 u = 0, \qquad (1)$$

where $A, B, \alpha, \beta \in \mathbb{R}_+$ to boundary conditions (2):

$$u|_{x=0} = u|_{x=l} = 0, \qquad (2)$$

and initial conditions (3):

$$u|_{t=0} = u_0(x), \qquad u_t|_{t=0} = u_1(x). \quad (3)$$

Let us point out that the equation considered is found in the liquid filtration theory. In (Chudnovsky, 1976). the equation (1) emerges as the mathematical model of evaporation and infiltration. When $\beta = 0$, the equation (1) is called Aller-Lykov moisture transfer equation. Steklov-type problem for this equation is considered in (Lafisheva, Kerefov, Dyshekova 2017), apriori estimates and numerical algorithm for forming the approximate

DOI: http://dx.doi.org/10.46925//rdluz.39.12

solution are obtained in the assumption of solution existence. In modern publications the significant attention is also paid to the research of inverse problems. Thus, for example, the renewal of the right part of one-dimensional equation is discussed in (Kenzhebai, 2021), and inverse problems for pseudohyperbolic equation are studied in (Kurmanbaeva, 2016), and the existence and uniqueness conditions for the solution of a special boundary value problem are obtained for constant coefficients of linear one-dimensional equation.

Generalized Aller equation of fractional order is considered in (Gekkieva, Karmokov, Kerefov, 2020; Kerefov, Gekkieva, 2019), for which the exact solution is written out in the form of finite integral formula. The integral operator kernel is explicitly written out in (Kerefov, Gekkieva, 2019), with the help of which the exact solution of the second boundary value problem is specified.

The aim of the work is to form the exact solution in the form of Fourier series of the first initial boundary value problem for pseudohyperbolic equation. The application of Fourier method (method of variable separation, for example, in (Ewans, (2003) is usually successful; due to the mixed derivative availability the variables cannot be separated directly. Therefore, referring to the type of boundary conditions and type of the differential operator in the equation (1), we will search for the solution in the following form:

$$u(x,t) = T(t)\sin\lambda x\,, \qquad (4)$$

where $\lambda -$ parameter, which can be a complex number.

The aim of the work is to obtain an exact solution of the initial boundary value problem in the form of a Fourier series using a special representation of the solution in the form of formula (4).

Obtaining of exact solution. Then we apply the expression for the required function from (4) to the equation (1):

$$T''(t)\sin\lambda x + A^2\lambda^2 T'(t)\sin\lambda x + B^2 T'(t)\sin\lambda x + \alpha^2\lambda^2 T(t)\sin\lambda x + \beta^2 T(t)\sin\lambda x = 0. \qquad (5)$$

After grouping the summands in (5) and simplifying by $\sin\lambda x$, we have

$$T''(t) + (A^2\lambda^2 + B^2)T'(t) + (\alpha^2\lambda^2 + \beta^2)T(t) = 0. \qquad (6)$$

The equation (6) is a linear equation with constant coefficients, the solution of which can be easily written out:

$$T(t) = C_1 e^{\mu_1 t} + C_2 e^{\mu_2 t}, \qquad (7)$$

DOI: http://dx.doi.org/10.46925//rdluz.39.12

where

$$\mu_{1,2} = -\frac{A^2\lambda^2 + B^2}{2} \pm \sqrt{\left(\frac{A^2\lambda^2 + B^2}{2} - \left(\frac{\alpha}{A}\right)^2\right)^2 + B^2\left(\frac{\alpha}{A}\right)^2 - \left(\left(\frac{\alpha}{A}\right)^4 + \beta^2\right)},$$

and

$$u(x,t) = \left(C_1 e^{\mu_1 t} + C_2 e^{\mu_2 t}\right)\sin\lambda x. \qquad (8)$$

The boundary conditions (2) allow specifying the set of values of parameter λ:

$$\lambda_n = \frac{\pi n}{l}, \qquad n \in \mathbb{N}. \qquad (9)$$

Further we introduce the designations of magnitude P_n, R_n

$$P_n = \frac{1}{2}\left(A^2\left(\frac{\pi n}{l}\right)^2 + B^2\right), \qquad (10)$$

$$R_n = \sqrt{\left(P_n - \left(\frac{\alpha}{A}\right)^2\right)^2 + B^2\left(\frac{\alpha}{A}\right)^2 - \left(\left(\frac{\alpha}{A}\right)^4 + \beta^2\right)}. \qquad (11)$$

$$= P_n\sqrt{1 - \left(\frac{\alpha}{A}\right)^2\left(\frac{2}{P_n} - \frac{B^2}{P_n^2}\right) - \frac{\beta^2}{P_n^2}}$$

Due to the linearity of the problem (1)-(3), the formula (8) specifies the solution for each λ_n from (9) and then we have Fourier series:

$$u(x,t) = \sum_{n=1}^{\infty}\left(C_{1n}e^{(R_n - P_n)t} + C_{2n}e^{-(R_n + P_n)t}\right)\sin\frac{\pi n x}{l}. \qquad (12)$$

Let us specify coefficients C_{1n} and C_{2n}. For this we use the initial condition (3):

$$u|_{t=0} = u_0(x) = \sum_{n=1}^{\infty}(C_{1n} + C_{2n})\sin\frac{\pi n x}{l},$$

$$u_t|_{t=0} = u_1(x) = \sum_{n=1}^{\infty}\left(C_{1n}(R_n - P_n) - C_{2n}(R_n + P_n)\right)\sin\frac{\pi n x}{l}.$$

Further we have the system to find the required coefficients:

$$\begin{cases} C_{1k} + C_{2k} = u_k^0 \\ C_{1k}(R_k - P_k) - C_{2k}(R_k + P_k) = u_k^1, \end{cases} \qquad (13)$$

where

DOI: http://dx.doi.org/10.46925//rdluz.39.12

$$u_k^0 = \frac{2}{l}\int_0^l u_0(x)\sin\frac{\pi kx}{l}dx, \qquad u_k^1 = \frac{2}{l}\int_0^l u_1(x)\sin\frac{\pi kx}{l}dx.$$

From the system (13) we have the expression for the required constants:

$$\begin{cases} C_{1k} = \dfrac{u_k^0(R_k + P_k) + u_k^1}{2R_k} \\ C_{2k} = \dfrac{u_k^0(R_k - P_k) - u_k^1}{2R_k}, \end{cases} \qquad (13)$$

Then we obtain Fourier series for the required function:

$$u(x,t) = \sum_{n=1}^{\infty} \frac{e^{-P_n t}}{2R_n}\left(\left(u_n^0(R_n + P_n) + u_n^1\right)e^{R_n t} + \left(u_n^0(R_n - P_n) - u_n^1\right)e^{-R_n t}\right)\sin\frac{\pi nx}{l}, \qquad (14)$$

or with the use of hyperbolic functions the formula (14) is as follows:

$$u(x,t) = \sum_{n=1}^{\infty} e^{-P_n t}\left(u_n^0 \operatorname{ch} R_n t + \frac{u_n^0 P_n + u_n^1}{R_n}\operatorname{sh} R_n t\right)\sin\frac{\pi nx}{l}. \qquad (15)$$

Practical results. On properties of solutions. The monotonous tend of magnitude P_n to infinity with the increase in index n is an obvious fact. It should be also pointed out that $R_n \sim P_n$, at $n \to \infty$. This fact follows from (11). Indeed, after elementary transformations we have

$$R_n = P_n\sqrt{1 - \frac{2}{P_n}\left(\frac{\alpha}{A}\right)^2 + \frac{1}{P_n^2}\left(\left(\frac{B\alpha}{A}\right)^2 - \beta^2\right)}. \qquad (16)$$

From (16) we have that if $R_n \in \mathbb{R}, \exists N: \forall n > N$ we have the inequality:

$$R_n < P_n, \qquad (17)$$

i.e. in (12) only a finite number of summands will not be limited with the increased t. At the same time, not all R_n can be real numbers but, again, only their finite number can be complex. Let us formulate these remarks as the lemma.

Lemma. i) Summands unlimited by t in (12) will correspond to indices n, which satisfy the following inequality:

$$2P_n < B^2 - \left(\frac{A}{\alpha}\beta\right)^2.$$

DOI: http://dx.doi.org/10.46925//rdluz.39.12

ii) Summands varying by t in (12) will correspond to indices n, which satisfy the following inequality:

$$\left(P_n - \left(\frac{\alpha}{A}\right)^2\right)^2 < \beta^2 + \left(\frac{\alpha}{A}\right)^4 \left(1 - \left(\frac{AB}{\alpha}\right)^2\right)$$

Proof: i) Let us require the fulfillment of $R_n > P_n$ and with the help of (16) we obtain the necessary result. ii) In (11) we will require the negativeness of the radical expression. After that we obtain the required inequality.

Examples. Practical results and examples. Let $A = \alpha = 1, l = \pi, B = 1$.

Example 1. Let us consider the initial boundary value problem (1)-(3) at

$$\beta = 5, \qquad u_0(x) = \sin x\,, \qquad u_1(x) = \sin x.$$

Then the equation (1) is as follows:

$$u_{tt} - u_{txx} + u_t - u_{xx} + 25u = 0,$$

Boundary conditions (2):

$$u|_{x=0} = u|_{x=\pi} = 0,$$

And initial conditions (3):

$$u|_{t=0} = \sin x\,, \qquad u_t|_{t=0} = \sin x.$$

For this case by the formula (9) we have that $\lambda_n = n$, where $n \in \mathbb{N}$ and by the formula (10) we have $P_n = \frac{n^2+1}{2}$. The initial conditions correspond to $n = 1$. According to point ii) of the lemma we should obtain the summands varying by t in the solution. It is not difficult to write out the solution for these parameters by the formula (14):

$$u(x,t) = \frac{e^{-t}}{5}(5\cos 5t + 2\sin 5t)\sin x$$

Example 2. Let us consider the initial boundary value problem (1)-(3) at

$$\beta = 1, \qquad u_0(x) = \sin 3x\,, \qquad u_1(x) = \sin 3x.$$

Then the equation (1) is as follows:

$$u_{tt} - u_{txx} + u_t - u_{xx} + u = 0,$$

Boundary conditions (2):

$$u|_{x=0} = u|_{x=\pi} = 0,$$

And initial conditions (3):

$$u|_{t=0} = \sin 3x\,, \qquad u_t|_{t=0} = \sin 3x.$$

DOI: http://dx.doi.org/10.46925//rdluz.39.12

As in the example 1, $\lambda_n = n$, where $n \in \mathbb{N}$ and by the formula (10) we have $P_n = \frac{n^2+1}{2}$. The initial conditions correspond to $n = 3$ and $P_3 = 5$, and condition ii) of the lemma is not fulfilled. Let us write out the problem solution by the formula (14):

$$u(x,t) = \frac{e^{-5t}}{2\sqrt{15}}\left(\left(\sqrt{15}+6\right)e^{\sqrt{15}t} + \left(\sqrt{15}-6\right)e^{-\sqrt{15}t}\right)\sin 3x.$$

We note that there are no coefficients varying by time here. At the same time, there are no values unlimited by time as well that is an expected factor since the inequality i) from the lemma is not fulfilled.

Example 3. Let $A = \alpha = 1, l = \pi, \beta = 1$. Let us consider the initial boundary value problem (1)-(3) at

$$B = 5, \qquad u_0(x) = \sin x\,, \qquad u_1(x) = \sin x.$$

Then the equation (1) is as follows:

$$u_{tt} - u_{txx} + 25u_t - u_{xx} + u = 0,$$

Boundary conditions (2):

$$u|_{x=0} = u|_{x=\pi} = 0,$$

And initial conditions (3):

$$u|_{t=0} = \sin x\,, \qquad u_t|_{t=0} = \sin x.$$

As in the example 1 we have that $\lambda_n = n$, where $n \in \mathbb{N}$ and by the formula (10) we have $P_n = \frac{n^2+1}{2}$. The initial conditions correspond to $n = 1$ and $P_1 = 1$. The inequality from point i) of the lemma is fulfilled. Let us write out the exact solution of the problem

$$u(x,t) = \frac{e^{-t}}{2\sqrt{23}}\left(\left(\sqrt{23}+2\right)e^{\sqrt{23}t} + \left(\sqrt{23}-2\right)e^{-\sqrt{23}t}\right)\sin x,$$

and we note that $e^{(-1+\sqrt{23})t}$ is unlimited with the increasing t.

Conclusion

The paper considers the linear equation with constant coefficients of pseudohyperbolic type. The first initial boundary value problem for this equation is set. The method of forming the exact solution in the form of Fourier series by countable trigonometric system consisting only of sinuses is proposed. The exact formulas for defining the solution coefficients, for the case of initial conditions from the corresponding functional space are

DOI: http://dx.doi.org/10.46925//rdluz.39.12

obtained. The quantitative behavior of solutions at some correlations on the coefficients of the initial differential equation is analyzed.

References

Chudnovsky, A.F. (1976). Thermal physics of soil. Moscow: Nauka, 352 p.

Ewans, L.K. (2003). Equations with partial derivatives. Translated from English. – Novosibirsk: Tamara Rozhkovskaya, 562 p.

Gekkieva, S.Kh.; Karmokov, M.M.; Kerefov, M.A. (2020). On one boundary value problem for generalized Aller equation // Bulletin of Samara State Technical University. Natural science series. V. 26, № 2. P. 7–14. DOI: http://doi.org/10.18287/2541-7525-2020-26-2-7-14.

Kenzhebai, Kh. (2021). Inverse problem of the renewal of the right part of one-dimensional pseudoparabolic equation. Bulletin of Kazan National University. Series: Mathematics, mechanics. informatics, [S.l.], v. 111, n. 3, p. 28-37, oct. ISSN 2617-4871.

Kerefov, M.A.; Gekkieva, S.Kh. (2019). Second boundary value problem for generalized Aller-Lykov moisture transfer equation. Bulletin of Samara State Technical University. Series: Physical and mathematical sciences. **23**:4, 607–621.

Kurmanbaeva, A.K. (2016). Linear inverse problems for pseudohyperbolic equations. Educational resources and technologies. №2 (14). URL: https://cyberleninka.ru/article/n/lineynye-obratnye-zadachi-dlya-psevdogiperbolicheskih-uravneniy (reference date: 22.08.2022).

Lafisheva, M.M.; Kerefov, M.A.; Dyshekova, R.V. (2017). Difference schemes for Aller-Lykov moisture transfer equation with nonlocal condition. Vladikavkaz mathematical journal. V. 19, issue 1. P. 41-49. DOI 10.23671/VNC.2017.1.5821