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The Relationships Between Discrete Dynamical Systems in Topological Spaces and their respective Hyperextensions to sets of Compact Spaces

Héctor Méndez-Gómez *
Jorge Luís Yaulema-Castañeda **
Paulina Fernanda Bolaños-Logroño ***
Fernando Ricardo Márquez-Sañay ****

ABSTRACT

Several studies have been carried out related to the analysis of the relationship with respect to the dynamic properties of f and its hyperextension ¯f. However, the literature regarding the analysis of the effects of individual and collective chaos on their behaviour is scarce. Therefore, in this article several conjectures and questions are established according to the affectation of individual chaos in an ecosystem and its chaotic behaviour within the dynamics of this ecosystem, but as a whole. Thus, in the first instance, an introduction to the conceptualization of topological transitivity, chaos in the Devaney sense and how they are specified in continuous linear operators arranged in a Fréchet space (hypercyclic operators) will be established. In addition, the different notions of chaos that can occur depending on the relationship of the function, and its hyperextension will be described, to finally corroborate the present chaos with greater force than Devaney's according to the strong periodic specification property, the same one that applies to both f and a ¯f with the purpose of verifying the directionality in which individual and collective chaos can occur.

KEYWORDS: Function, hyperextension, dynamic system, individual chaos, collective chaos.

*Matemático, Universidad de Costa Rica, Máster Universitario Investigación Matemática, Universitat Politècnica de València, España. Docente en matemática de la Universidad de Costa Rica, sede regional del Pacífico – Costa Rica. ORCID: https://orcid.org/0000-0002-0925-8310. E-mail: hector.mendez@ucr.ac.cr

** Ingeniero Electrónico, Universidad Politécnica Salesiana, Máster Universitario en Investigación Matemática – Universidad de Valencia. Docente en la Escuela Superior Politécnica del Chimborazo – Ecuador. ORCID: https://orcid.org/0000-0002-0646-3984. E-mail: jorge.yaulema@espoch.edu.ec

*** Ingeniera Electrónica, Escuela Superior Politécnica del Chimborazo, Magister en Sistemas de Control y Automatización Industrial - Escuela Superior Politécnica del Chimborazo – Ecuador, Máster Universitario en Estadística Aplicada – Universidad de Granada - España. Docente en la Escuela Superior Politécnica del Chimborazo – Ecuador. ORCID: https://orcid.org/0000-0003-3911-0461. E-mail: paulina.bolanos@espoch.edu.ec

**** Ingeniero Mecánico, Escuela Superior Politécnica de Chimborazo, Máster Universitario en Matemáticas y Computación – Universidad Internacional de la Rioja - España. Docente en la Escuela Superior Politécnica del Chimborazo – Ecuador. ORCID: https://orcid.org/0000-0001-5549-9572. E-mail: fernando.marquez@espoch.edu.ec

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Relaciones entre sistemas dinámicos discretos en espacios topológicos y sus respectivas hiperextensiones a conjuntos de espacios compactos

RESUMEN

Se han realizado varios estudios vinculados con el análisis de la relación con respecto a las propiedades dinámicas de f y su hiperextensión $\bar{f}$. Sin embargo, es escasa la literatura respecto al análisis de los efectos del caos individual y colectivo sobre sus comportamientos. Por lo tanto, en el presente artículo se establecen varias conjeturas e interrogantes de acuerdo a la afectación del caos individual en un ecosistema y su comportamiento caótico dentro de la dinámica de este ecosistema, pero en su conjunto. Así, se establecerá en primera instancia una introducción a la conceptualización de la transitividad topológica, el caos en el sentido de Devaney y cómo se especifican en operadores lineales continuos dispuestos en un espacio de Fréchet (operadores hipercíclicos). Además, se describirán las diferentes nociones de caos que pueden darse según la relación de la función y su hiperextensión, para finalmente corroborar el caos presente con mayor fuerza que el de Devaney según la propiedad de especificación periódica fuerte, la misma que aplica tanto a f como a $\bar{f}$ con el propósito de verificar la direccionalidad en la que puede ocurrir el caos individual y colectivo.

PALABRAS CLAVE: Función, hiperextensión, sistema dinámico, caos individual, caos colectivo.

Introduction

In the analysis of a dynamical system f established in a topological space of X, the development of conceptualizations such as topological transitivity and chaos will be determined. In the case of studying the individual chaos on the behavior of orbits in the long term, it is necessary to determine its behavior according to the changes that can be generated when they occur in nature, which can also be collective and establish an affectation between both (Peris, 2005).

When studying collective chaos, an analysis based on the dynamics of the function is developed where a family of sets predominates, which in particular will be for the study of non-empty compacts established in a topological space (Barnsley, 1993).

Thus, a question is presented according to the different notions of chaos and the relation that it specifies regarding a continuous function $f: X \longrightarrow X$ with its hyperextension $\underline{f}: K(X) \longrightarrow K(X)$. For this purpose we will define the hyperspace of non empty compact sets established in a topological space of X that will be established by (X) . At the same time if it

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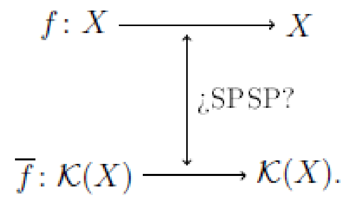
is presented in a metric space (X, d) we will be able to endow a metric Hausdorff space, same that will allow to measure the distance between the sets.

On the other hand, chaos is defined from different notions, for instance, the one established by Devaney and its variants both total chaos and exact chaos, as well as Li-Yorke, the ω-chaos and distributional chaos. All of them can specify the relation according to the function f and its hyperextension $\underline{f}$. The purpose is established on the comparative analysis between the individual and the collective chaos in each of these notions and vice versa.

Moreover, a stronger chaos than the one proposed by Devaney commonly known as the strong periodic specification property is established, which when it is applied to the function as its hyperextension, it will determine the directionality that is specified in rising or falling as shown in the following scheme:



1. Literature review

The description of the literature related to chaos in hyperspaces requires the initial analysis of preliminary conceptualizations established from topological dynamics with respect to dynamical systems and their objective on the behavior that is specified in the long term of the iterations of a certain function f on the domain points and starting on a certain topological point X (Peris, 2005). Dynamics, on the other hand, establishes its purpose according to the study of the behavior of the orbits within a system (Barnsley, 1993). This behavior may be equivalent to the dynamical system itself as to a conjugate system, a process called conjugacy (Román et al., 2018).

Conjugacy is related to the equivalences that can occur in dynamical systems, where a topological transitivity is established which is expressed by the theorem that states that when f is weakly mixing then it will also be topologically transitive (Bauer & Sigmund, 1975), (Ulcigrai, 2021) and (Banks, 2005).

Within the dynamics of operators, hypercyclicity is presented, which according to its universality criterion is defined by $\{T_n : n \in N\}$ being a succession of operators within a

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Fréchet space that is separated in E. If dense subsets X and Y of E are presented as an increasing succession of naturals $\{m_k\}_{k=1}^{\infty}$ and applications of $S_{mk} : Y \longrightarrow E,\ K \epsilon N$ (whether surely discontinuous or nonlinear, which these terms will be omitted when referring to T which will be described simply as an operator). Furthermore, an operator T on a locally convex space of E will be hypercyclic when the succession $\{T_n : n \epsilon N\}$ is universal, specified when there exists a vector $x \in X$ where the orbit $x\ by\ T$ is defined as: $Orb(T,x) := \{x, Tx, T^2x, ...\}$ being dense in E. Denoting in this case a vector x as a cyclic vector for T (Liao et al., 2006).

In relation with topology, convergence has been a topic of complete interest on the study of mathematicians, which has generated several classes of spaces, being Fréchet one of the pioneers in completely characterizing the classes of spaces by means of convergent sequences (Martínez, 2000).

From hyperspaces the topology can be analyzed from Vietoris that specifies the dense set of periodic points according to a dynamic system $: X \longrightarrow X$, being necessary for f to possess a chaotic condition conforming to that established by Devaney chaos where it is stated that a dense set of periodic points will be equivalent for the function f as for its hyperextension (Delgadillo & López, 2009).

There are several reformulations of Devaney's chaos such as the total chaos, which states that a dynamical system $f: X \longrightarrow X$ being in a metric space of X will establish that f is totally transitive if it is evident that the iterations of fn for each $n \epsilon N$ are topologically transitive. It is further mentioned that if f were fully transitive it will be f topologically transitive so it will suffice to consider $n = 1$.

Conversely, Devaney's exact chaos states that a dynamical system $f: X \longrightarrow X$ set in a topological space of X, will determine f to be topologically exact when the totality of the nonempty open subset $U \sqsubset X$ exists $m \epsilon N$ such that $fm(U) = X$.

2. Materials and methods

To establish transitivity requires the use of several definitions, examples, prepositions, lemmas and theorems of topological dynamics, operators and hyperspaces, for which the following sections are developed.

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2.1. Topological dynamics

Definition 1.1. The dynamical system is established according to the pair (f, X) where X represents the metric space and the continuous function determined by $f: X \longrightarrow X$..

In the research, it will be simply determined as a dynamical system to f ó $f: X \longrightarrow X$, or else (f, X). Starting from the point $x_0 \in X$,, where its iterations are defined as $f^n: X \longrightarrow X$, $n \geq 0$, $f^n(x_0) = \underbrace{f \circ \cdots \circ f}_{n-iteraciones}(x_0)$. Where f^0 is the identity function on X.

Definition 1.2. When the dynamical system is $f: X \longrightarrow X$ at the point $x_0 \in X$, the set of the orbit x under f will be:

$$Orb(x, f) = \{x, f(x), f^2(x), \dots\} = \{f^n(x): n \in N_0\}$$

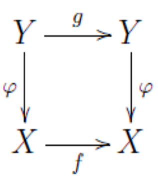
The study of the orbit with respect to the point $x_0 \in X$ under the function f establishes the succession $\left(f^n(x_0)\right)_n$, thus determining equivalence with the point x_0 under the function f.

In a metric space X is defined the $\omega - límite$ of $x \in X$ by means of the set $\omega(x, f)$ where it will be established the totality of the limit points present in the orbit x which is precise as a succession.

Example 1.3. When $f: C \longrightarrow C$ o $z \mapsto z^2$ we specify in the following formula the iterates of f, to denote: $f^n(z) = z^{2^n}$.

Where it is determined that if $|z| < 1$ la $Orb(z, f)$ will tend to 0. On the other hand if $|z| > 1$ the $Orb(z, f)$ will tend to ∞

Definition 1.4. The relation of two dynamical systems such (g, Y) and (f, X) can be presented by the continuous function $\varphi: Y \longrightarrow X$. Whereby, in the dynamical systems $g: Y \longrightarrow Y$ and $f: X \longrightarrow X$, f will be called a semiconjugate of g as there exists a continuous function as. If $\varphi: Y \longrightarrow X$, dense range were to be presented be $f \circ \varphi = \varphi \circ g$ the diagram will present commutativity, as presented below:



On the other hand, in the case that φ is presented as a homeomorphism both g and f will be called conjugates.

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Definition 1.5. Upon establishing a property P in a dynamical system a semiconjugate is preserved when in $g: Y \longrightarrow Y$ the property P is satisfied by making $f: X \longrightarrow X$ semiconjugate of g also satisfies the aforementioned property.

Therefore, a dynamical system can be defined by another system f by restricting a subset with $f - invariant$

Definition 1.6. If $f: X \longrightarrow X$ and a subset of $Y \subset X$ is determined $f - invariant$ or invariant under f when $f(Y) \subset Y$

In the case when $Y \subset X$ is $f - invariant$ we define $f|_Y: Y \longrightarrow Y$ as a dynamical system.

Definition 1.7. A dynamical system is set up with the function $f: T \longrightarrow T, z \mapsto z^2$ with $T = \{z \in C: |z| = 1\}$ since $f(T) \subset T$, same that will duplicate the argument of z.

Definition 1.8. In the dynamical system $f: X \longrightarrow X$ it is considered as topologically transitive when any of the pairs of U, V representing nonempty open subsets of X is presented $n \geq 0$, being $f^n(U) \cap V \neq \emptyset$, where f allows the connection of trivial parts of X, topological transitivity in existence can be deduced by means of a point $x \in X$ containing an orbit under f dense.

Proposition 1.9. The preservation of topological transitivity is presented according to semiconjugacy.

Proposition 1.10. If f represents a continuous function presenting a dense orbit within a metric space of X where no isolated points are evident, then f will be topologically transitive (Bauer & Karl, 1975).

Lemma 1.11. There exist certain equivalent statements to the dynamical system $f: X \longrightarrow X$ be:

(a) If f is considered topologically transitive.

(b) On a nonempty open set U of X let $\cup_{n=0}^{\infty} f^{-n}(U)$be dense in X.

Theorem 1.12. Birkhoff determines transitivity by means of the continuous function f present in a separate and complete metric space of X without presence of isolated points. By establishing the following equivalent statements:

(a) If f is considered topologically transitive.

(b) By evidencing $x \in X$ an orbit under f dense in X

If one of the two statements holds the set of points in X with an orbit under dense f will represent the set G_δ dense.

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Preposition 1.13. The property is established by possessing a dense orbit in preserving itself in semiconjugacy.

Definition 1.14. Defining a metric space without evidence of assortative points (X, d) expresses $f: X \longrightarrow X$ as a dynamical system possessing sensitive dependence on the initial conditions with $\delta > 0$ for each $x \in X$ y $\varepsilon > 0$ there will exist $y \in X$ with $d(x, y) < \varepsilon$ for $n \geq 0$, setting: $d(f^n(x), f^n(y)) > \delta$, where the sensitivity constant of f will be represented by the number δ.

All the fact previously described is known as the butterfly effect since it establishes that very small initial differences can lead to uncontrollable consequences. Furthermore, it specifies the stability to which a dynamic system is subject.

Definition 1.15. Considering the dynamical system $f: X \longrightarrow X$

(a) If the point $x \in X$ is set, it is considered as a fixed point of f when $f(x) = x$

(b) If the point $x \in X$ is set, it is considered as periodic point of f $n \geq 1$ as $f^n(x) = x$. Therefore, n represents the period of $f^k(x) \neq x$ for $k < n$.

Denoting the set of periodic points by $Per(f)$.

A point will be periodic only if it refers to a fixed point of any of the iterations of $f^n, n \geq 1$.

Preposition 1.16. Within the property of maintaining a dense set of periodic points is specified under conjugacy.

Definition 1.17. The initial version of Devaney's chaos states that a metric space without isolated points is denoted by (X, d) where $f: X \longrightarrow X$ represents the dynamical system that will settle down in a chaotic sense when the conditions described below are satisfied:

(a) f has a sensitive dependence on the conditions established initially.

(b) When f is considered topologically transitive.

(c) If f has a dense set of points that are periodic.

Example 1.18. To establish the dependence sensitive to initial conditions we require $f:]1, \infty[\longrightarrow]1, \infty[$ which is determined by $f(x) = 2x$. Sea $|f^n(x) - f^n(y)| = 2^n |x - y| \rightarrow \infty$ this when $x \neq y$ then f is defined to possess such a dependence relative to the usual metric $]1, \infty[$.

By defining $]1, \infty[$ with the metric $d(x, y) = |\, log\{x\} - log\{y\}|$, it will be equivalent to:

$$d(f\^n(x), f\^n(y)) = |\, log\{f\^n(x)\} - log\{f\^n(y)\}\,| = |\, log\{2\^nx\} - log\{2\^ny\}\,|$$

$$= |\, log\{x\} - log\{y\}\,| = d(x, y), \ \forall x, y \in X.$$

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Therefore, f will not have such a dependence on d. In both cases f is conjugate by the identity.

Theorem 1.19. Proposed by Banks, Brooks, Cains, Davis and Stacey specify that in a metric space X without isolated points with a dynamical system $f: X \longrightarrow X$ will be topologically transitive when it has a dense set of periodic points. Hence, f will have a sensitive dependence on the initial conditions with respect to the metric that is defined in the topology of X (Ulcigrai, 2021).

Definition 1.20. Devaney chaos describes a dynamical system $f: X \longrightarrow X$ as chaotic if it satisfies the following conditions:

(a) When f is considered topologically transitive.

(b) If f possesses a dense set of points that are periodic.

Both conditions are preserved under conjugacy.

Proposition 1.21. Devaney chaos is preserved by semiconjugacy.

Definition 1.22. The property of being mixable has greater force with respect to topological transitivity, the same being expressed by $f: X \longrightarrow X$ on any of the nonempty open subsets U, V belonging to X where there exists $N \geq 0$, determining: $f^n(U) \cap V \neq \emptyset, \ \forall n \geq N$.

Where it is observed that the function f is mixable as well as topologically transitive.

Preposition 1.23. The property to be preserved mixable will depend on the semiconjugacy. Having two metric spaces be X and Y set by their Cartesian product $X \times Y = \{(x, y): x \in X, y \in Y\}$ which will also be a metric space set by the metric: $d\big((x_1, y_1), (x_2, y_2)\big) = d_X(x_1, x_2) + d_Y(y_1, y_2)$, let d_X and d_Y be metrics that were defined respectively on X and Y. Within one of the bases of the topology states that the metric induced on the Cartesian product will be composed of the products $U \times V$ of the open subsets $U \subset X$ y $V \subset Y$.

Definition 1.24. The dynamical systems $f: X \longrightarrow X$ and $g: Y \rightarrow Y$ will have a function $f \times g$ defined by: $f \times g: X \times Y \longrightarrow X \times Y, \ (f \times g)(x, y) = \big(f(x), g(y)\big)$.

$f \times g$ is a continuous function whose iterations are set according to:

$$(f \times g)^n = f^n \times g^n.$$

Similarly, products with more than two spaces or functions will be defined.

Theorem 1.25. The dynamical $f: X \longrightarrow X$ and $g: Y \rightarrow Y$ shall satisfy the following statements:

(a) When $f \times g$ possesses a dense orbit f and g will also possess a dense orbit.

(b) When $f \times g$ is topologically transitive f and g will be as well.

(c) When $f \times g$ is chaotic f and g will be chaotic too.

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(d) When $fand\ g$ are topologically transitive and one of them is mixable $f \times g$ will be so too.

(e) When $f \times g$ is mixable only if $fand\ g$ are mixable as well.

Definition 1.26. $f: X \longrightarrow X$ will be weakly mixable when $f \times f$ is topologically transitive.

In the case of the products $U \times V$ of open sets $U, V \subset X$ generating a topological basis $X \times X$ will establish that the function f will be weakly mixable if and only if in any of the 4-tuple U_1, U_2, V_1, V_2 of nonempty open subsets of X where there exists $n \geq 0$,, it will be:

$$f^n(U_1) \cap V_1 \neq \emptyset \ \ y \ \ f^n(U_2) \cap V_2 \neq \emptyset.$$

Proposition 1.27. The property of weakly mixable will hold according to semiconjugation.

Moreover, established dynamical systems $f: X \longrightarrow X$ and $g: Y \rightarrow Y$, in the case where $f \times g$ is weakly mixable f and g will be weakly mixable as well.

Definition 1.28. The dynamical system $f: X \longrightarrow X$ on any pair of sets be $A, B \subset X$, we express the return set of both A and B as follows:

$$N_f(A, B) = N(A, B) = n \in N_0: f^n(A) \cap B \neq \emptyset.$$

Generally f will be omitted since there is no ambiguity. In this definition f will be topologically transitive or mixable if and only if the return set in any of the pairs of nonempty open sets U, V belonging to X is possessed.

$$N(U, V) \neq \emptyset \ (or\ cofinite, respectively).$$

Moreover, if and only if f for any of the 4-tuple U_1, U_2, V_1, V_2 of the non-empty open subsets of X, it will be obtained:

$$N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset.$$

Lemma 1.29. The 4-set trick is established according to the dynamical system $f: X \longrightarrow X$ and the nonempty open subsets of X, let U_1, U_2, V_1, V_2, by determining:

(a) As there exists a continuous function let $g: X \longrightarrow X$ which commutes with f, then:

$$g(U_1) \cap U_2 \neq \emptyset \ y \ g(V_1) \cap V_2 \neq \emptyset,$$

There will therefore exist non-empty open sets such as $U_1' \subset U_, V_1' \subset V_1$, let

$$N(U_1', V_1') \subset N(U_2, V_2) \ y \ N(V_1', U_1') \subset N(V_2, U_2).$$

Let f be topologically transitive, it is established that $N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset$.

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(b) As f is topologically transitive, it is determined:

$$N(U_1, U_2) \cap N(V_1, V_2) \neq \emptyset \Longrightarrow N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset.$$

Theorem 1.30. If $f: X \longrightarrow X$ represents a weakly mixable dynamical system, the $n - product\ f \times \cdots \times$, will be n times weakly mixable at $n \geq 2$ (Banks, 2005).

Proposition 1.31. The dynamical system $f: X \longrightarrow X$ will be weakly mixable when the nonempty open sets $U, V_1, V_2 \subset X$, are determined as:

$$N(U, V_1) \cap N(U, V_2) \neq \emptyset.$$

Preposition 1.32. The dynamical system $f: X \longrightarrow X$ will be weakly mixable when any pair of nonempty open sets $U, V \subset X$, is determined as:

$$N(U, U) \cap N(U, V) \neq \emptyset.$$

What will characterize the weakly mixable property with respect to the size terms present in the return sets $N(U, V)$ with equivalence in topological transitivity in certain subsubsessions $(f^{n_k})_k$

Definition 1.33. The syndetic of a strictly increasing sequence of positive integers $(n_k)_k$, is defined according to:

$$(n_{k+1} - n_k) < \infty.$$

On the other hand, the syndetic in a set $A \subset N$ is established according to the succession of positive integers if A is syndetic or in turn if its complement does not present intervals of extremely large length.

Theorem 1.34. According to a dynamical system $f: X \longrightarrow X$ its equivalence is established by the following conditions:

(a) If it is weakly mixable f

(b) In the case of any pair of nonempty open sets $U, V \subset X$ where $N(U, V)$ possesses extremely large intervals of length.

(c) In the case of any syndetic sequence $(n_k)_k$ let the sequence $(f^{n_k})_k$ be topologically transitive.

2.2. Dynamics of operators

The dynamics to be used in this section is set according to the operators $T: X \longrightarrow X$, where X will represent the Fréchet space.

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Definition 1.35. The pair (T, X) represents a linear dynamical system consisting of a separable Fréchet space of X and an operator $T: X \longrightarrow X$ both linear and continuous. Both terms will henceforth be omitted simply referred to T as the operator.

Definition 1.36. The operator $T: X \longrightarrow X$ is considered hypercyclic when $x \in X$ in whose orbit under T is dense. In this case x will be a hypercyclic vector for T. Hypercyclic vectors for T will be represented by $HC(T)$.

Definition 1.37. The operator $T: X \longrightarrow X$ and a vector $x \in X$ is called cyclic for T when the space evolved by the orbits is dense in X, being:

$$\underline{spanT^n x: n \geq 0} = X.$$

The vector $x \in X$ is called supercyclic for T when its projective orbit is represented as:

$$\{\lambda T^n x: n \geq 0, \lambda \in K\}$$

Being dense in X.

In Fréchet spaces when there are no isolated points by means of Birkhoff's transitivity theorem, it will facilitate the determination when an operator T is hypercyclic.

Theorem 1.38. Birkhoff's transitivity theorem determines that an operator T is considered hypercyclic only when it is topologically transitive, where the hypercyclic vector set $HC(T)$ occurs in a dense set G_δ. Several examples are described in (4).

Example 1.39. The Rolewicz operators, consider $X = l^p, 1 \leq p < 'oX = c_0$ on $\lambda \in K$, is specified:

$$T: X \longrightarrow X, (x_1, x_2, x_3, \dots) \longmapsto \lambda(x_2, x_3, x_4, \dots).$$

When $|\lambda| \leq 1$ we obtain $\|T^n x\| = |\lambda|^n \|(x_{n+1}, x_{n+2}, \dots.)\| \leq \|x\|$ for the totality of $x \in X$ as in $n \geq 0$, which determines that T may not be considered as hypercyclic.

However, when $|\lambda| > 1$ considers T to be hypercyclic. In the open and nonempty subsets U, V of X one may obtain $x \in U$ and in turn $y \in V$ as follows:

$$x = (x_1, x_2, \dots, x_N, 0{,}0, \dots),\ y = (y_1, y_2, \dots, y_N, 0{,}0, \dots),\ N \in N.$$

Proposition 1.40. By means of Proposition 1.9 the hypercyclicity to be preserved by semiconjugation is established.

Definition 1.41. The definition stated for chaos was described in Theorem 1.38 which establishes a new statement of chaos but with a linear perspective. Where an operator T will be chaotic from Devaney's perspective if the following conditions are satisfied:

(a) Let T be hypercyclic.

(b) When T has a dense set of periodic points.

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Proposition 1.42. Considering T a cyclic operator will have T sensitive dependence with respect to the initial conditions of any of the invariant metrics defined by the translations of a topology X.

Lemma 1.43. On a nonempty open set $U \subset X$ where X is a Fréchet space there will exist a nonempty open set $U_1 \subset U$, besides a 0-neighborhood, W, which sets $U_1 + W \subset U$. When W presents a neighborhood of 0 there will exist a 0-neighborhood W_1 let $W_1 + W_1 \subset W$. The proof can be evidenced in Lemma 2.36.

Within the mixable property it is required that the return sets $N(U, V)$, referred to U, V as nonempty open sets of X are presented as cofinite.

Proposition 1.44. The operator T is considered to be mixable if or only if the return sets are cofinite on any nonempty set U of X or 0-neighborhood W.

$$N(U, W) \ y \ N(W, U)$$

Definition 1.46. If X and Y ares Fréchet space then the following space is Fréchet space: $X \oplus Y = \{(x, y) : x \in X, y \in Y\}$. The operators $S: X \longrightarrow X$ y $T: Y \longrightarrow Y$ determined on Fréchet spaces X and Y are set to the operator $S \oplus T$ which is defined by:

$$S \oplus T: X \oplus Y \longrightarrow X \oplus Y, \ (S \oplus T)(x, y) = (Sx, Ty).$$

Proposition 1.47. The operators $S: X \longrightarrow X$ and $T: Y \longrightarrow Y$, when the operator $S \oplus T$ is defined to be hypercyclic both T and S will be hypercyclic.

Proposition 1.48. The operators $S: X \longrightarrow X$ y $T: Y \longrightarrow Y$, will be hypercyclic when at least one of them is mixable determining that $S \oplus T$ is hypercyclic or in turn if and only if S and T were mixable.

In the contextualization of the investigation an operator $T: X \longrightarrow X$ is defined as weakly mixable only when $T \oplus T$ were hypercyclic or in turn if and only if for the non-empty open subsets U_1, U_2, V_1, V_2 of ; $N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset$ will be obtained.

From the observation, therefore, a chain of implications concerning the operators is precise:

" mixable"⟹ weakly mixable"⟹ hypercyclic."

Theorem 1.49. For hypercyclic operators inside Banach spaces there will exist several which do not occur as weakly mixable (Liao et al., 2006).

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Lemma 1.50. T being a hypercyclic operator any pair of nonempty open sets U, V of X or a 0-neighborhood W will establish both a nonempty open set $U_1 \subset U$ and a 0-neighborhood$W_1 \subset W$, let:

$$N(U_1, W_1) \subset N(V, W) \; y \; N(W_1, U_1) \subset N(W, V).$$

Theorem 1.51. Since T is a hypercyclic operator and on any nonempty open set $U \subset X$ or 0-neighborhood W there exists a continuous operator $S: X \longrightarrow X$ commuting with T, as:

$$S(U) \cap W \neq \emptyset \; y \; S(W) \cap U \neq \emptyset$$

Where it is established that T will be weakly mixable.

Theorem 1.52. Since T is a weakly mixable operator when on any pair of nonempty open sets $U, V \subset X$ or 0-neighborhood W, such that:

$$N(U, W) \cap N(W, V) \neq \emptyset.$$

Theorem 1.53. When T represents a hypercyclic operator and a dense subset X_0 of X according to the orbit of each $x \in X_0$ is bounded T in a weakly mixable way.

Corollary 1.54. Among the operators the following are considered weakly mixable:

(a) Chaotic operators.

(b) Those hypercyclic operators that have a dense set of points where the orbits converge.

(c) Hypercyclic operators that have a dense generalized kernel.

Proposition 1.55. $T: X \longrightarrow X$ being an operator, it follows that:

(a) $T \oplus T$ will be weakly mixable only when T is as well.

(b) $T \oplus T$ will be chaotic only when T is as well.

In contrast to (a) with respect to the result of the more general (b), the following proposition is specified.

Proposition 1.55. Chaotic operators S and T will be chaotic operators if and only if $S \oplus T$ is as well.

2.3. Hyperspaces

This section specifies the study of a collective dynamics, which means the described conceptualizations of topological dynamics and of operators that will be applied to subsets of X or to the dynamics that are established by means of functions evaluated on subsets belonging to a metric space of X.

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Definition 1.57. In a topological space X the corresponding hyperspace of nonempty compact subsets of X is denoted by:

$$K(X) = \{K \subset X: K \ es \ compacto \ y \ no \ vacío\}.$$

It is denoted $K(X)$ according to the Vietoris topology which is established according to the sets of form

$$V(U_1, \ldots, U_k) = \left\{K \in K(X): K \subset \cup_{i=1}^{k} U_i \ y \ K \cap U_i \neq \emptyset, i = 1, \ldots k\right\},$$

Being $U_1, \ldots, U_k$ the non empty open subsets of X.

Within the hyper space of non empty compact sets of R^n is where the fractals (8) and (9) which are generally compact are accommodated.

Moreover, when X represents a metric space we will call $K(X)$ as the Hausdorff metric which is also a complete metric space, where the topology of it coincides with that of Vietoris.

Definition 1.58. The Hausdorff metric is established when (X, d) is a metric space by endowing $K(X)$, where:

$d_H(A, B) :=sup \ sup \{ d(x_1, B): x_1 \in A\}, sup \ sup \{ d(x_2, A): x_2 \in B\}$, with $A, B \in K(X)$ where $d(x, A) =inf \ inf \{ d(x, y): y \in A\}, x \in X, A \in K(X)$.

By defining the Hausdorff metric on neighborhoods of sets it is established that A will be an empty set of a metric space (X, d) with ε-neighborhood being the set

$$N_\varepsilon(A) = \{x \in X: d(x, A) < \varepsilon\}.$$

Since A and B are non-empty subsets of X, it will then be defined:

$$d_H(A, B) =inf \ inf \ \varepsilon > 0: A \subseteq N_\varepsilon(B) \ y \ B \subseteq N_\varepsilon(A) .$$

Both definitions have overlaps but in some cases the use of one is more accurate than the other.

Definition 1.59. The set $K \subseteq X$ is considered fully bounded when in all $\varepsilon > 0$ there is a finite subset $\{x_i: 1 \leq i \leq n\}$ belonging to K as $K \subset \cup_{i=1}^{n} B_d(x_i, \varepsilon)$. Denote $B_d(x, \varepsilon)$ as the ball of center x and radius ε referring to the metric d.

Proposition 1.60. A metric space $(K(X), d_H)$ is considered when it is exposed to the Hausdorff metric d_H. This proof follows the ideas put forward in (Banks et al., 1992) and (Grosse-Erdmann & Peris-Manguillot, 2011).

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Lemma 1.61. When $x \in X$ and $A, K(X)$ it will be said that there exists $a_x \in A$ such that $d(x, A) = d(x, a_x)$.

Set $A \in K(X)$ y $\varepsilon > 0$ we define:

$$A + \varepsilon := x \in X : d(x, A) \leq \varepsilon.$$

Proposition 1.62. For all $\varepsilon > 0$ as in $A \in K(X)$ the set $A + \varepsilon$ will be closed. By the result and the convergence of Cauchy sequences will provide the completeness proof of $(K(X), d_H)$.

Theorem 1.63. Since $A, B \in K(X)$ y $\varepsilon > 0$ will therefore establish $d_H(A, B) \leq \varepsilon$ only when $A \subseteq B + \varepsilon \; and \; B \subseteq A + \varepsilon$.

Lemma 1.64. Application Lemma which states that it will be $(A_n)_n$ a Cauchy sequence when it is in $K(X)$ and in turn $(n_k)_k$ represents an increasing sequence of positive integers. For its part $\left(x_{n_k}\right)_k$ being a Cauchy sequence in X such that $x_{n_k} \in A_{n_k}$ in all k, hence, a Cauchy sequence will exist when $(y_n)_n$ with respect to X such that $y_n \in A_n$ for its totality n and $y_{n_k} = x_{n_k}$ in all k.

Lemma 1.65. When $(A_n)_n$ is a sequence present in $K(X)$ and A represents the set that groups all points puntos $x \in X$ in such a way there will exist a sequence $(x_n)_n$ where it converges to x and furthermore satisfies $x_n \in A_n$ at all n. Since $(A_n)_n$ is a Caunchy sequence then A will be closed and nonempty.

To prove that $A \in K(X)$ it must be verified that A is completely bounded, then in the following lemma is specified a tool for that purpose which is to prove $(K(X), d_H)$.

Lemma 1.66. When $(B_n)_n$ is a sequence of sets that is totally bounded in X and furthermore represents A a subset of any of X. Moreover if for $\varepsilon > 0$ a positive integer N is presented as $A \subseteq B_N + \varepsilon$ then A will be fully bounded. The primary result of this section is described in the following theorem.

Theorem 1.67. It is therefore established if (X, d) is a complete metric space, $(K(X), d_H)$ will be as well.

Theorem 1.68. The following theorem represents the completion of this section which can also be visualized in (Furstenberg, 1967). Where it is expressed that if (X, d) is considered as a separable metric space then $(K(X), d_H)$ will be as well.

3. Results

Having established the literature review as well as the methods described previously, the necessary approaches are established for the demonstration by means of the analysis of the properties of transitivity and its influence on the different chaos that develop in hyperspace.

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3.1. Transitivity and chaos in hyperspaces

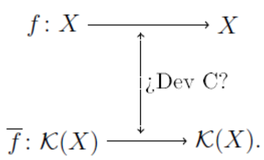
In a metric space (X, d) the concepts concerning topological transitivity and Davaney's chaos to be abbreviated by DEV C were studied by means of the application of functions defined on the hyperspace $K(X)$.

With the continuous function $f: X \longrightarrow X$ we consider the function $\underline{f}$ to be called hyperextension f set in hyperspace with the totality of nonempty compact subsets of X denoted by $K(X)$ with Vietoris topology, where $\underline{f}: K(X) \longrightarrow K(X)$ is naturally generated by $\underline{f}(K) = \{f(x): x \in K\}$, where $\underline{f}(K)$ represents the image of the nonempty compact set of K under f. It can be seen that $\underline{f}$ is correctly defined since f is continuous. So also $\underline{f}$ is continuous for which it was observed: that the nonempty open subset U of X and the continuity of f concocts that $f^{-1}(U)$ is recognized as an open subset of X, if $K(X)$ is considered open by $V(U) = \{K: K \subset U\ y\ K \cap U \neq \emptyset\}$, it is seen:

$$\underline{f}^{-1}\big(V(U)\big) = \{K: f(K) \subset U\ y\ f(K) \cap U \neq \emptyset\}$$
$$= \{K: K \subset f^{-1}(U)\ y\ K \cap f^{-1}(U) \neq \emptyset\}$$
$$= V(f^{-1}(U)).$$

The set $V(f^{-1}(U))$, is also presented as open $K(X)$ as the $\underline{f}$ will remain continuous.

With both functions it is necessary to know when it can go up or down as shown in the following scheme:



3.1.1. Topological transitivity

Topological transitivity is established according to Definition 1.20 which mentions Devaney's chaos, considering the following:

Theorem 2.1. As is $f: X \longrightarrow X$ a continuous function inside a topological space X, equivalent statements will be stated, as mentioned.

(a) The function f is weakly mixable.

(b) The hyperextension of the function $\underline{f}$ is weakly mixable.

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(c) The hyperextension of the function $\underline{f}$ is topologically transitive.

With nonempty open sets with respect to the Vietoris-based canonical basis in $K(X)$, where $n \in N$ is evident as specified in:

$$\underline{f}^n\left(V\left(U_1^i, \dots, U_k^i\right)\right) \cap V\left(V_1^i, \dots, V_k^i\right) \neq \emptyset, i = 1,2.$$

According to Theorem 1.30 applied to $m = 2k$ where there exists $n \in N$,, it is determined:

$$f^n\left(U_j^i\right) \cap V_j^i \neq \emptyset,\ i = 1,2,\ k = 1,2, \dots, k.$$

Where it is considered $x_{i,j} \in U_j^i$ such that $f^n\left(x_{i,j}\right) = y_{i,j} \in V_j^i$ is for $i = 1, 2$ y $j = 1, \dots, k$.. Establishing in this way the compacts $K_1 = \{x_{1,1}, \dots, x_{1,k}\}, K_2 = \{x_{2,1}, \dots, x_{2,k}\}$. It will be $\underline{f}(K_i) = \{y_{i,1}, \dots, y_{i,k}\}$, determining that:

$$f^n(K_i) \in \underline{f}^n\left(V\left(U_1^i, \dots, U_k^i\right)\right) \cap V\left(V_1^i, \dots, V_k^i\right).$$

The hyperextension $\underline{f}$ is found to be weakly mixable.

In $(b) \Rightarrow (c)$ in its hyperextension determines a weakly mixable dynamical system, which leads it to be topologically transitive as specified in Definition 1.26.

On the other hand, in $(c) \Rightarrow (a)$ when $\underline{f}$ is topologically transitive and with Proposition 1.31 it will be proved that by setting $U, V_1, V_2 \subset X$ nonempty open, it is determined that:

$$N(U, V_1) \cap N(U, V_2) \neq \emptyset.$$

The nonempty open subsets U, V_1, V_2 of X, since $\underline{f}$ is topologically transitive it is established that $V(U)$ and $V(V_1, V_2)$ are open in $K(X)$, where $n \geq 0$, specifying:

$$f^n\big(V(U)\big) \cap V(V_1, V_2) \neq \emptyset.$$

Thus, we obtain a non-empty compact set $K \in V(U)$ that determines

$$f^n(K) \in V(V_1, V_2),$$

Let $f^n(K) \subset V_1 \cup V_2$ with $f^n(K) \cap V_1 \neq \emptyset$ and $f^n(K) \cap V_2 \neq \emptyset$, where there exist $x, y \in K \subset U$ such that $f^n(x) \in V_1$ and $f^n(y) \in V_2$, which implies:

$$n \in N(U, V_1) \cap N(U, V_2).$$

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Proposition 2.2. A continuous function $f: X \longrightarrow X$ on a topological space X refers that f is mixable only when $\underline{f}: K(X) \longrightarrow K(X)$ is also mixable.

On arbitrary open sets based on Vietoris of $K(X)$ as f is mixable, it will be possible to find $N \in N$ for $i = 1, \dots, k$ and $n \geq N$, which results:

$$f^n(U_i) \cap V_i \neq \emptyset.$$

It is taken $u_i \in U_i$ where $f^n(u_i) \in V_i, i = 1, \dots, k$. Hence, it is set $K = \{u\_i: i = 1, \dots, k\} \in V(U\_1, \dots, U\_k)$ and $\underline{f}^n(K) \in V(V_1, \dots, V_k)$, determining that it is mixable the hyperextension $\underline{f}$.

When it occurs in the opposite direction ($\Leftarrow$) on nonempty open subsets U, V of X it is set $N \in N$ such that $f^n(K_n) \in V(V)$ at some $K_n \in V(U), n \geq N$. It is therefore determined that for any $x_n \in K_n \subset U$ let $f^n(x_n) \in V$, $n \geq N$ be satisfied, proving that it is mixable function f.

To summarize, the result of:

$\underline{f}$ topologically transitive $\Longrightarrow f$ topologically transitive.

It represents a false result according to that specified by Roman Flores in (Román-Flores, 2003) where it is stated that a topologically transitive function f is not so for its hyperextension $\underline{f}$.

Theorem 2.4. and Theorem 2.5. Referring to the Hypercyclicity criterion where it is stated that an operator $T: X \rightarrow X$ within a separable Banach space of X when the following statements are equivalent:

(a) T satisfies the Hypercyclicity criterion.

(b) When $\underline{T}: K(X) \longrightarrow K(X)$ is topologically transitive.

Assertions that states the following:

$\underline{f}$ topologically transitive $\Longrightarrow f$ topologically transitive.

f topologically transitive $\not\Rightarrow f$ topologically transitive.

3.1.2. Dense set of periodic points

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The dense set of periodic points is set to bound f to be chaotic according to the notion established by Devaney, for this purpose, the present section is described below:

Theorem 2.6. Given the dynamical system $f: X \longrightarrow X$ in a metric space of X, if f possesses a dense set of periodic points its hyperextension $\underline{f}$ will also have one (Ulcigrai, 2021).

Definition 2.7. Since $f: X \longrightarrow X$ is a dynamical system, the point $x \in X$ will be regularly recurrent if in the whole neighbourhood V of x there exists $n \in N_0$ let $f^{nk}(x) \in V$ at $k = 0,1,2, \ldots$

Lemma 2.8. Given the dynamical system $f: X \longrightarrow X$ on a compact metric space of (X, d) in the case where regularly in f the set of all recurrent points is dense in X, it will be established that its hyperextension $\underline{f}$ will have a dense set of periodic points.

It is therefore proved that every nonempty open subset $U \subseteq X$ possesses a periodic point for $\underline{f}$ arranged in $V(U)$. According to the compactness of X there will exist a nonempty open set $V \subseteq X$ where $\underline{V} \subset U$ determining that $\underline{V}$ will be the closure of V. A regularly recurring point $x \in V$ may be found. Subsequently a positive integer in n as $Orb(x, f^n) \subset V$. Whereby it is established that $\underline{Orb(x, f^n)} \subset \underline{V}$ with all limit points of $Orb(x, f^n)$ arranged in $\underline{V}$, which represents $\omega(x, f^n) \subset \underline{V}$. As for $\omega(x, f) \in K(X)$ and also in $\underline{f}^n(\omega(x, f^n)) = \omega(x, f^n)$ which allows to find the periodic point for the $\underline{f}$ arranged in $V(U)$.

Example 2.9. In (5) consider the Cantor space $\{0,1\}^N$ from a topological abelian group structure to define the following sum:

$$(x_1, x_2, \ldots) + (y_1, y_2, \ldots) = (z_1, z_2, \ldots),$$

Setting $z_i = x_i + y_i + c_i\, mód 2, c_1 = 0$ and $c_{i+1} = x_i + y_i + c_i (div 2) para i > 1$ to determine that:

$$f(x_1, x_2, x_3 \ldots) = (x_1, x_2, x_3, \ldots) + (1,0,0, \ldots)$$

Hence, the property of having a dense set of periodic points for f and $\underline{f}$ will not be equivalent.

$$f \text{ } es\ Dev\ C \nRightarrow \underline{f}\ es\ Dev\ C.$$

The described direction is not fulfilled because it has been established previously that being f topologically transitive will not necessarily determine that $\underline{f}$ is topologically transitive. It is further held that:

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$$\underline{f}\ es\ Dev\ C\ \not\Rightarrow f\ es\ Dev\ C,$$

Particularly arranged in Example 2.9 since it was determined that $\underline{f}$ has a set of periodic points, which is not the case with f. Which would establish that Devaney's causal properties of equivalence between the function and its hyperextension are not always satisfied.

Theorem 2.10. The Shauder-Tychonoff fixed point is determined when let K be a nonempty, compact t convex subset arranged in a convex space of X and $f: K \longrightarrow K$ being continuous it will thus be established that $f(p) = p$ for any $p \in K$. The proof of the same is described in (Rudin, 1991).

Theorem 2.11. T being a continuous linear operator arranged in a locally convex and complete space, the following equivalences are determined:

(a) It will be chaotic T in the sense established by Devaney.

(b) It will be chaotic $\underline{T}$ in the sense established by Devaney.

(c) It will be chaotic $\tilde{T}$ in the sense established by Devaney.

For the demonstration $(a) \Rightarrow (b)$ in the chaos of T in the sense established by Devaney, it will be weakly mixable as stated in Corollary 1.54. It is therefore determined and in accordance with Theorem 2.1 that $\underline{T}$ will be topologically transitive. Furthermore, the relation of Theorem 2.6 will be followed. Which determines the following:

By setting $(b) \Rightarrow (c)$ as Devaney chaos, the semiconjugacy will be established where $\underline{T}$ will be chaotic in the Devaney sense as well as $\tilde{T}$.

When $(c) \Rightarrow (a)$ will be topologically transitive T. Which entails that if U and V are presented as nonempty open subsets of X the following sets will be considered:

$$U' = V(U) \cap C(X)\ y\ V' = V(V) \cap C(X)$$

In $C(X)$ they will be nonopen nonempty. It will be topologically transitive $\tilde{T}$ which therefore determines that $n \geq 0$ and $K' \in U'$ as $\widetilde{T^n}\left(K'\right) \in V'$. Established that $x \in K' \subset U$ such that $T^n x \in V$. It is thus proved that T possesses a dense set of periodic points which proves the existence of periodic $K \in U'$ for $\tilde{T}$, which states that $\widetilde{T^n}(K) = K$ at some $n \geq 1$.

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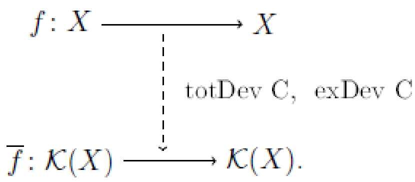
Being K nonempty, convex and compact in addition to establishing the Schauder-Tychonoff fixed point theorem it is concluded that there exists the point $y \in U \cap Per(T)$.

3.1.3. Reformulations of Devaney Chaos

In generating reformulations on the notion of Devaney chaos, conditions will be established that will formulate the total Devaney chaos (totDev C) and the exact one (exDev C), obtaining:



Which indicates that totDev C and exDev C with respect to the function f will possess the same notions of chaos and respectively with their hyperextension ¯f but not reciprocally.

3.1.3.1. Total Devaney Chaos

Definition 2.12. When $f: X \longrightarrow X$ is a dynamical system within a metric space X it will be set that f is totally transitive if for the iterations of f^n of each $n \in N$ to be topologically transitive, it will suffice to take $n = 1$..

Proposition 2.13. Since $f: X \longrightarrow X$ is a weakly mixable dynamical system it will therefore be f fully transitive. Revising in (Ulcigrai, 2021), the following proof is established:

It is demonstrated that by establishing that $m \geq 1$ and U, V as nonempty open subsets of X, it will be the nonempty open set $f^{-i}(V)$ be in $i = 0, \dots, m$ where f is considered to be weakly mixable by existing $k \geq 1$ as $f^k(U) \cap f^{-i}(V) \neq \emptyset parai = 0, \dots, m$. By the algorithm established by division, integers $q \geq 0$ and r will appear such that $k = qm + r$, with $0 \leq r \leq m - 1$,which will continue with $0 < m - r \leq m$. That is $f^k(U) \cap f^{-(m-r)}(V) \neq \emptyset$ therefore:

$$U \cap f^{-(k+m-r)}(V) = U \cap f^{-(q+1)m}(V) \neq \emptyset,$$

This leads to $f^{m(q+1)}(U) \cap V \neq \emptyset$

Therefore, the following chain will be fulfilled in any dynamic system f

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mixable

$\Downarrow$

weakly mixable

$\Downarrow$

fully transitive

$\Downarrow$

topologically transitive

Lemma 2.14. When be a dynamical system $f: X \longrightarrow X$ chaotic in the sense of Devaney and fully transitive one will establish f as weakly mixable.

By setting U and V to be nonempty open subsets of X it is determined that,

$$N(U,U) \cap N(U,V) \neq \emptyset.$$

Therefore, proposition 1.32 will conclude the above theorem.

Since f is chaotic in the sense of Devaney it is stated to be topologically transitive and to have a dense set of periodic points. In this function by its characteristic of topological transitivity is therefore determined the existence of $x \in U$ and $n_1 \in N$ with $f^{n_1}(x) \in V$. Having periodic points f will form a dense set in X which will determine the existence of a periodic point $y \in U$ of f be $f^{n_1}(y) \in V$ as also $f^p(y) = y$ for any $p \in N$, where p represents the period of y, which states,

$$f^n(U) \cap V \neq \emptyset, con\ n = n_1 + jp,\ j = 0,1,2$$

Where let $f^{n_1}(U) = U_1$, even though it is not open U_1 in general, it can be ensured that it possesses a nonempty interior. Then, as it is known about the existence of $x \in U$ let $f^{n_1}(x) \in V$ and V being open, an open ball $B_1 \subset V$ having center $f^{n_1}(x)$ will be determined. As U is also open with the continuity of f^{n_1} there will appear an open ball $B_2 \subset U$ whose center in X is determined by$f^{n_1}(B_2) \subset B_1$. Which denotes that U_1 will have a nonempty interior and f will be fully transitive, where by applying topological transitivity $f^p\ a\ U_1$ and U in search of an integer $n_2 \geq 0$ be,

$$f^{pn_2}(U_1) \cap U \neq \emptyset.$$

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By establishing that $j = n_2$ with the definition of n it is obtained $n = n_1 + n_2 p$ thus appreciating the following:

$$f^n(U) \cap V = f^{n_1+n_2p}(U) \cap V \neq \emptyset.$$

In addition, it can be observed that,

$$f^n(U) \cap U = f^{n_1+n_2p}(U) \cap U = f^{pn_2}(U_1) \cap U \neq \emptyset.$$

Where it is presented that $n \in N(U,U) \cap N(U,V)$ and where further with Proposition q.32 it is proved that f will be weakly mixable.

Definition 2.15. When be a dynamical system $f: X \longrightarrow X$, f is said to possess total chaos in the Devaney sense when the conditions described below are satisfied:

(a) When f is fully transitive.

(b) When f has a dense set of periodic points.

Then Leman 2.14 may be described according to the following Theorem.

Theorem 2.16. As a dynamical system $f: X \longrightarrow X$ possessing total chaos in the Devaney sense, it will be f weakly mixable.

Theorem 2.17. (Related to Theorem 2.1). When is a dynamical system $f: X \longrightarrow X$ within a topological space X, the following statements will be equivalent,

(a) As f is weakly mixable.

(b) As $\underline{f}$ is weakly mixable.

(c) As $\underline{f}$ is fully transitive.

(d) As $\underline{f}$ is topologically transitive.

It is demonstrated that $(a) \Rightarrow (b)$ results from the immediate Theorem 2.1.

$(b) \Rightarrow (c)$ with the application of Proposition 2.13 it is established that the function ¯f be fully transitive.

$(c) \Rightarrow (d)$ be immediate by establishing that ¯f is fully transitive and by definition is topologically transitive.

$(d) \Rightarrow (a)$ again with Theorem 2.1 will establish an immediate result.

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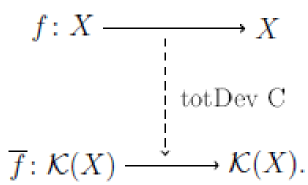
Furthermore in Theorem 2.17 the equivalence of the topological transitivity of $\underline{f}$ as the total transitivity of $\underline{f}$ in $K(X)$ is evidenced as a new result that determines the equivalence in the functions set on the hyperspace $K(X)$ of both total chaos and only Devaney chaos.

Theorem 2.19. In this sense by setting $f: X \longrightarrow X$ in a topological space of X as a dynamical system, it holds that f possesses total chaos in the Devaney sense as its hyperextension $\underline{f}: K(X) \longrightarrow K(X)$.

Since f possesses total chaos in the Devaney sense then it will also be totally transitive with a dense set of periodic points. If such a set of f is dense in X, the set of periodic points in $\underline{f}$ will be dense in $K(X)$ as Theorem 2.6 states to it. On the other hand, according to Theorem 2.17 which determines that $\underline{f}$ continues to be totally transitive so it will have therefore total Devaney chaos. Which will establish according to the following scheme the sense of going down, as shown below,



3.1.4. Exact Devaney Chaos

In the present section it will be proved that when there exists dynamical system f:X⟶X with exact chaos in the Devaney sense (exDev C), its hyperextension ¯f will also have it.

Definition 2.20. When be a dynamical system $f: X \longrightarrow X$ within a topological space X, it is determined that f will be topologically exact upon the occurrence in the entire nonempty open subset $U \subset X$ of the existence of $m \in N$ as $f^m(U) = X$.. It is worth mentioning the particular simplicity in determining that every topologically exact function is overjective.

Proposition 2.21. Since $f: X \longrightarrow X$ is a dynamical system inside a topological space X, f will be presented as topologically exact so it will also be topologically transitive.

It is demonstrated for U, V as nonempty open subsets of X that f is topologically exact when there exists $m \in N$ as $f^m(U) = X$, establishing for m the following,

$$f^m(U) \cap V = X \cap V = V \neq \emptyset.$$

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Showing that f is topologically transitive.

Definition 2.22. As $f: X \longrightarrow X$ is a dynamical system, it is determined to possess exact chaos in the sense of Devaney on f by satisfying the following conditions,

(a) As f is topologically exact.

(b) As f has a dense set of periodic points.

Theorem 2.23. As $f: X \longrightarrow X$ is a dynamical system, within a compact metric space (X, d), the following statements will be equivalent:

(a) As f is topologically exact.

(b) As $\underline{f}$ is topologically exact.

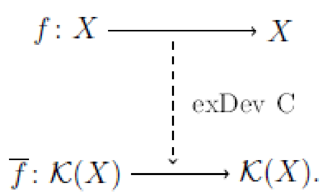
Which is demonstrated that $(b) \Rightarrow (a)$ by assuming $\underline{f}$ to be topologically exact where U will be a nonempty open subset of X. Moreover, $V(U)$ is nonempty and open in $K(X)$ thus determining the existence of $m \in N$ as $\underline{f}^m(V(U)) = K(X)$. Particularly it is defined $X \in \underline{f}^m(V(U)) \subset V(f^m(U))$ which determines that $f^m(U) = X$.

In turn $(a) \Rightarrow (b)$ in its reciprocal assumes that f will be topologically exact, which must be proved for any open set $U = V(U_1, \dots, U_k) \subset K(X)$ where there exists $m \in N$, as $\underline{f}^m(U) = K(X)$. In a sense of compactness on X for each $i = 1, \dots, k$ there will exist a nonempty open set V_i such that $V_i \subset U_i$ and $\underline{V_i} \subset U_i$ where $\underline{V}$ establishes the closure of V. Since f is topologically exact, it is found $m_1, \dots, m_k$ as $f^{m_i}(V_i) = X$ for each $i = 1, \dots k$, where $m = \{m_1, \dots, m_k\}$ will therefore be established,

$$\underline{f}^m\left(V\left(\underline{V_1}, \dots, \underline{V_k}\right)\right) = V\left(f^m\left(\underline{V_1}\right), \dots, f^m\left(\underline{V_k}\right)\right) = K(X).$$

Where $\underline{f}^m(U) = K(X)$ which proves that $\underline{f}$ is topologically exact.

Theorem 2.24. Since $f: X \longrightarrow X$ is a dynamical system, when f exhibits exact chaos in the Devaney sense it is determined that $\underline{f}: K(X) \longrightarrow K(X)$will also exhibit it. Which will determine the lowering power as shown in the following scheme,



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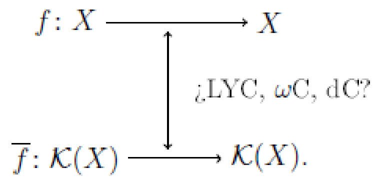
3.2. Transitivity and chaos in hyperspaces

The present section establishes an introduction to other types of chaos notions such as Li-Yorke chaos (LUC), ω-chaos (ωC) and distributional chaos (dC), which will be established according to comparisons resulting from the function f defined within a topological space X and its hyperextension $\underline{f}$ that is stated in the hyperspace of non-empty compact subsets of X, $K(X)$. From this perspective we will try to answer if it is possible to obtain the following scheme



It is specified that in the development of this section we present (X,d) in a compact metric space. Determining for this purpose that $f: X \longrightarrow X$ be a continuous function.

3.2.1. Li-Yorke chaos and ω-chaos

For the introduction to Li-Yorke chaos the following definitions are expressed:

Definition 3.1. When a pair of points $x, y \in X$ are set to be a Li-Yorke pair for f if it is present,

(a) $d\big(f^n(x), f^n(y)\big) \geq 0$
(b) $d\big(f^n(x), f^n(y)\big) = 0$

Definition 3.2. When a scrambled Li-Yorke subset S of X is set to f if $\#S \geq 2$ in all pairs of points other than S will be a Li-Yorke pair. Where $\#S$ establishes the cardinality of S.

Definition 3.3. $f: X \longrightarrow X$ being a Dynamical System it will be established that f happens to be chaotic in the Li-Yorke sense if it possesses a non-enumerable scrambled Li-Yorke set.

Example 3.4. According to (18) it is determined that $f: [0,1] \longrightarrow [0,1]$ will be a continuous function having a periodic point of period 3, so that f will be chaotic in the Li-Yorke sense.

Definition 3.5. A dynamical system be $f: X \longrightarrow X$, shall be stated,

(a) It shall be determined as proximal when a pair of points $x, y \in X$ is expressed as $d\big(f^n(x), f^n(y)\big) = 0$ and if a pair is not proximal so it shall be said to be distal.

(b) It will be determined as asymptotic when a pair of points $x, y \in X$ is expressed as $d\big(f^n(x), f^n(y)\big) = 0$.

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Proposition 3.6. With the definitions described previously, it is determined that, a pair of points $x, y \in X$ will be a Li-Yorke pair only when they are proximal and non-asymptotic and are denoted according to the following sets respectively.

$$Prox(f) = \{(x, y): d(f^n(x), f^n(y)) \ = 0\},$$

$$Asym(f) = \{(x, y): d(f^n(x), f^n(y)) = 0\,\}$$

For any $x \in X$ is called the proximal cell of the point x of a set that has all proximal points conforming to x denoted by $Prox(f)(x)$.

Definition 3.7. According to the previous definition it is determined that a point x is distal when $Prox(f)(x) = \{x\}$. Therefore, a dynamical system is said to be distal when its points are also distal. A dynamical system f is proximal when every pair of points of X are also proximal. Thus establishing the following, $Prox(f)(x) = X$ for all $x \in X$.

Definition 3.8. A dynamical system $f: X \longrightarrow X$ is determined to be Li-Yorke sensitive when $\delta > 0$ is present such that any point $x \in X$ y $\varepsilon > 0$ where $y \in X$ is evident with $d(x, y) < \varepsilon$ and with the proximal pair x, y in the case of some $n \geq 0$ will obtain $d(f^n(x), f^n(y)) > \delta$. The same manifests itself in (Akin & Kolyada, 2003).

Theorem 3.9. When is a dynamical system $f: X \longrightarrow X$ and f Li-Yorke sensitive it will be established that it possesses sensitive dependence on initial conditions. From this perspective and furthermore for any point $x \in X$ when the proximal cell $Prox(f)(x)$ is dense in X it will be presented that f is Li-Yorke sensitive.

Theorem 3.10. As a dynamical system $f: X \longrightarrow X$ is weakly mixable it will be established at all $x \in X$ as the proximal cell $Prox(f)(x)$ representing a dense subset of X.

Theorem 3.11. When $f: X \longrightarrow X$ is established as a dynamical system the following statements will be equivalent:

(a) f will be weakly mixable.

(b) In the case of any $x \in X$ when the proximal cell $Prox(f)(x)$ is dense in X

(c) In the case of the existence of $x \in X$ with a proximal cell $Prox(f)(x)$ dense in X

(d) $Prox(f)$ will be dense in $X \times X$.

Definition 3.12. As a dynamical system $f: X \longrightarrow X$ presents a subset S of X with at least two points, it is called a set $\omega - scrambled$ if for f any two distinct points $x, y \in S$ the following conditions will be satisfied:

(a) When it is numerable $\omega(x, f) \setminus \omega(y, f)$.

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(b) $\omega(x, f) \cap \omega(y, f) \neq \emptyset$

$$(c)\ \omega(x, f) \setminus Per(f) \neq \emptyset$$

Proposal established in (Li, 1993).

Definition 3.13. When a dynamical system $f: X \longrightarrow X$ is set up, it is determined that f will be $\omega - chaotic$ upon the existence of $\omega - scrambled$ non-numerable.

Theorem 3.14. Since a dynamical system $f: X \longrightarrow X$ will be $\omega - chaotic$ it is determined that f will be chaotic in the Li-Yorke sense.

The proof in (Lampart, 2003) is carried out as the evidence that the reciprocal is not true.

Theorem 3.15. The relation of the Li-Yorke chaos and the $\omega - chaos$ between the continuous function f and its hyperextension as asserted in (Guirao et al., 2009). In order to support the scheme shown at the beginning of the section. Then, being $f: X \longrightarrow X$ a dynamical system and $\underline{f}: K(X) \longrightarrow K(X)$the hyperextension of f in the hyperspace $K(X)$. Therefore, the following statements will be satisfied,

(a) When a set S is presented as revolved Li-Yorke ($\omega - scrambled$, respectively) there will then exist for f, a scrambled Li-Yorke ($\omega - scrambled$, respectively) set for $\underline{f}$ with equal cardinality of S.

(b) When f is chaotic in the Li-Yorke sense ($\omega - chaotic$, respectively) it will then also be $\underline{f}$.

For (a) it is demonstrated by means of the function, $\varphi: X \longrightarrow K(X), x \longmapsto \{x\}$ where it is denoted that,

$$d_H\big(\varphi(x), \varphi(y)\big) = d_H(\{x\}, \{y\}) = d(x, y).$$

Being φ an isometry, the effect $\underline{f}$ will be conjugate by f according to the path φ. Thus $\varphi \circ f = \underline{f} \circ \varphi$. will be obtained. Hence, $\underline{f}(\{x\}) = \{f(x)\}$.

When S is a scrambled Li-Yorke subset of X, it will be established that $\#S \geq 2$ and every pair of points that are distinct from S will be a Li-Yorke pair. Therefore, for $x, y \in S$ it will be held that,

$$lím\ sup_{n \to \infty}\ d_H\left(\underline{f}^n(\{x\}), \underline{f}^n(\{y\})\right) = lím\ sup_{n \to \infty}\ d_H(\{ f^n(x)\}, \{ f^n(y)\})$$

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$$= l\acute{\imath}m\ sup\ d_{n\to\infty}\left(f^n(x),\ f^n(y)\right) \geq 0.$$

It is also determined that,

$$l\acute{\imath}m\ inf_{n\to\infty}\ d_H\left(\underline{f}^n(\{x\}), \underline{f}^n(\{y\})\right) = l\acute{\imath}m\ inf_{n\to\infty}\ d_H(\{f^n(x)\}, \{f^n(y)\})$$

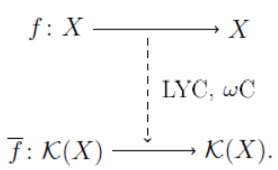
$$= l\acute{\imath}m\ inf\ d_{n\to\infty}\left(f^n(x),\ f^n(y)\right) = 0.$$

Proving that if x, y represents a Li-Yorke pair for f, it establishes $\{x\}, \{y\}$ as a Li-Yorke pair for f. So the following set is considered,

$$\underline{S} = \{\{x\} \in K(X): x \in S\}.$$

With respect to (b), the proof establishes that since f has chaos in the Li-Yorke sense it will also possess a set S which will be Li-Yorke scrambled. What would be established by (a), a continuation where $\underline{f}$ possesses a scrambled Li-Yorke set with equal cardinality to S, the latter being non-numerable will determine that the scrambled Li-Yorke set of $\underline{f}$ will be as well. Thus proving that $\underline{f}$ possesses chaos in the Li-Yorke sense.

According to Theorem 3.15 it will be obtained,



3.2.2. Distributional chaos

Distributional chaos had its notions at the introductory level in (23) and was later generalised in (Balibrea et al., 2005) and (Smítal, J & Stefánková, 2004).

Definition 3.16. Since (X, d) is a compact metric space. For $x, y \in X$ as for $t \in R$ and $n \in N$ will be,

$$\xi(x, y, n, t) = \#i: 0 \leq i < n\ y\ d\left(f^i(x), f^i(y)\right) < t.$$

Which defines the upper distribution function for x, y to be

$$F^*_{xy}(t) = \frac{1}{n}\,\xi(x, y, n, t).$$

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In addition, the lower distribution function for x, y will be

$$F_{xy}(t) = \frac{1}{n} \xi(x, y, n, t).$$

The two functions described are non-decreasing and also $0 \leq F_{xy} \leq F_{xy}^* \leq 1$ When $t < 0$ it will be determined that $F_{xy}^*(t) = 0$,, or in turn when $t > diam(X)$ will be $F_{xy}(t) = 1$.

Both functions describe both upper and lower bounds depending on how many times the distance $d\left(f^i(x), f^i(y)\right)$ that develop on the x and y trajectories are less than t over the course of n iterations.

Definition 3.17. Since $f: X \longrightarrow X$ is a dynamical system and if there exist a pair of points $x, y \in X$ such that,
(d_1C) $F_{xy}^* \equiv 1$ and $F_{xy}(t) = 0$ at some $t > 0$ it will be said that f possesses distributional chaos of type 1.

(d_2C) $F_{xy}^* \equiv 1$ and $F_{xy}(t) < F_{xy}^*(t)$ at some $t > 0$ it will be said that f possesses distributional chaos of type 2.

(d_3C) $F_{xy}(t) < F_{xy}^*(t)$ at all $t \in J$, where will be a non-degenerate interval J it will therefore be established that f will possess distributional chaos of type 3.

It is therefore denoted by $d_1C \Rightarrow d_2C \Rightarrow d_3C$ but not reciprocally.

Theorem 3.18. When let both $g: Y \longrightarrow Y$ and $f: X \longrightarrow X$ be conjugate dynamical systems via $\varphi: Y \longrightarrow X$, setting (X, d_X) and (Y, d_Y) to be metric space. It is determined that g possesses distributional chaos of type 1 and type 2 respectively only if f possesses the same types.

It is therefore demonstrated that since $\varphi: Y \longrightarrow X$ is a homeomorphism conjugate to g y f it will be established that $\varphi \circ g = f \circ \varphi$. By presenting continuity in φ given $\varepsilon > 0$ if there exists $\delta > 0$ be in $x, y \in Y$ if $d_Y(x, y) < \delta$ it is determined that $d_X\big(\varphi(x), \varphi(y)\big) < \varepsilon$. Such that,

$$d_Y\big(g^n(x), g^n(y)\big) < \delta \Rightarrow d_X\big(\varphi \circ g^n(x), \varphi \circ g^n(y)\big) < \varepsilon,$$

And being $\varphi \circ g^n = f^n \circ \varphi$, it generates,

$$d_Y\big(g^n(x), g^n(y)\big) < \delta \Longrightarrow d_X\big(f^n \circ \varphi(x), f^n \circ \varphi(y)\big) < \varepsilon.$$

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Therefore,

$$G^*_{xy}(\delta) \leq F^*_{\varphi(x)\varphi(y)}(\varepsilon)$$

Let G^*_{xy} and F^*_{xy} be upper distribution functions with respect to g and f respectively.

Similar case is presented by the continuity present in φ^{-1}, in any of the cases of $\varepsilon > 0$ where there exists arbitrarily small $\delta > 0$ being for $x, y \in Y$ $d_X(\varphi(x), \varphi(y)) < \delta$ which establishes that $d_Y(x, y) < \varepsilon$ Determining,

$$F_{\varphi(x)\varphi(y)}(\delta) \leq G_{xy}(\varepsilon),$$

F_{xy} and G_{xy} being lower distribution functions with respect to f and g.

In the case where $G^*_{xy} \equiv 1$ according to Definition 3.1 we obtain $F^*_{\varphi(x)\varphi(y)} \equiv 1$. On the other hand, if $G_{xy}(\varepsilon) = 0$ according to Definition 3.2 we will have $F_{\varphi(x)\varphi(y)}(\delta) = 0$. Establishing as a consequence that if g possesses distributional chaos of type 1 then f will also possess it.

So also, when let $G_{xy}(\varepsilon) < 1$ and by Definition 3.2 $F_{\varphi(x)\varphi(y)}(\delta) < 1$ is generated it will show that as g holds distributional chaos of type 2 then f will also possess it.

Finally, in (Balibrea et al., 2005) it is proved that distributional chaos of type 3 is not preserved under conjugacy.

Definition 3.19. Being a dynamical system $f: X \longrightarrow X$, the subset S de X will be set distributively scrambled for f when $\#S \geq 2$, as for any pair of distinct points $x, y \in S$ by holding that,

Let $F^*_{xy}(t) = 1$ in the totality of $t > 0$

Let $F_{xy}(t) = 0$ at some $t > 0$

The pair x, y will be considered to be distributionally chaotic for f. Moreover, it is specified that f will possess distributional chaos when there exists a distributionally scrambled set but it is not numerable for the same. Denoting the similarity between distributional chaos and type 1 chaos.

Theorem 3.20. Since the system $f: X \longrightarrow X$ is a dynamical system, it follows,

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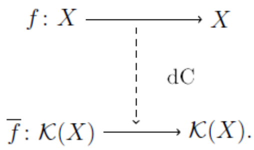
(a) When a distributionally scrambled set $S \subset X$ is evident for f there will exist a similar set for $\underline{f}$ and it will have the same cardinality as S.

(b) When f exhibits distributional chaos, $\underline{f}$ will also have it.

According to Theorem 3.20 it establishes a proof that is corroborated by Theorem 3.15 where it is stated that the function $\varphi: X \longrightarrow K(X),\ x \longmapsto \{x\}$ will be an isometry.

So also with respect to Theorem 3.20 it may be lowered as expressed in the following scheme,



Asserting according to Theorems 3.20 and 3.15 that they are not true in general terms.

In the theorem it is stated the compactification of integers, where it is considered a discrete topological space $X_\infty = Z \cup \{\infty\}$ defining the function $f: X_\infty \longrightarrow X_\infty$in,

$$f(n) = \{n + 1, n \in Z, \infty, n = \infty\}$$

3.2.3. Li-Yorke Chaos for Linear Operators

When T is a continuous linear operator within a Fréchet space X possessing chaos in the Li-Yorke sense, $\underline{T}$ will also possess chaos in the sense of Li-Yorke according to Theorem 3.15. With this and in accordance with (Bernardes et al, 2017) as well as the Banach-Steinhaus theorem, the following lemma is developed.

Lemma 3.22. As T is a continuous linear operator inside a Fréchet space X and $\underline{T}$ possesses a Li-Yorke pair a residual subset Z of X will be established being unbounded the $Orb(x, T)$ for each $x \in Z$. In addition to the detailed with the Li-Yorke chaos criterion and (Bernardes et al, 2015) the following Theorem will be proved.

Theorem 3.23. As T is a continuous linear operator inside a Fréchet space X and is defined as $NS(T) = x \in X: (T^n x)_{n \in N}$, it has subsequences converging to 0. In the case that $span\big(NS(T)\big)$ is dense, equivalence in the following statements will be determined,

(a) When T possesses Li-Yorke chaos

(b) When $\underline{T}$ possesses Li-Yorke chaos

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(c) When $\underline{T}$ possesses a Li-Yorke pair

Corollary 3.24. When X represents a Fréchet space with successions let be the basis $(e_n)_{n\in Z}$.. Assuming a translation with weights to the left, it is determined,

$$B_w(x_1, x_2, x_3, \dots) = (w_2x_2, w_3x_3, w_4x_4, \dots),$$

X being an operator, equivalences are established in the following statements,

When B_w possesses Li-Yorke chaos

When $\underline{B_w}$ possesses Li-Yorke chaos

When $\underline{B_w}$ possesses a Li-Yorke pair.

In the demonstration, it is established the set $NS(B_w)$ which has a dense subspace of X with all finite support successions, following as detailed in the previous theorem.

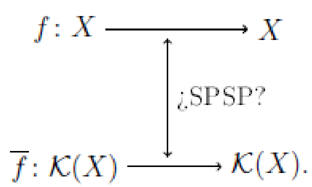
Conforming to the classical Banach spaces l_p , $1 \le p < \infty$ and c_0 furthermore by (Bermúdez et al., 2011) it is determined that B_w will contain Li-Yorke chaos only when,

$$sup\ sup\ |w_n \cdot \dots \cdot w_m| : n \in Z_+, m > n\ = \infty.$$

Thus establishing the characterization in relation to the succession of weights.

3.3. Specification properties

In the present scheme, the study is made in order to know the specification properties, which are determined as notions with greater strength than chaos in the sense of Devaney. Therefore, it is established as an objective to analyze the possibility of rising and falling in the scheme shown below,



SPSP being the acronym given to the strong periodic specification property.

3.3.1. Specification properties in hyperspace

We will work with $f: X \longrightarrow X$ as a continuous function where the compact metric space will be represented by (X, d).

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Definition 4.1. f:X⟶X as a continuous function will be able to satisfy the strong periodic specification property when in the entire $\varepsilon > 0$ an integer $N_\varepsilon > 0$ is possessed be any integer $s \geq 2$, such as the points $y_1, y_2, \dots, y_s \in X$ and the integers,

$$0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s,$$

For $a_j - b_{j-1} \geq N_\varepsilon$ con $j = 2, \dots, s$, there will therefore exist a point $x \in X$ that will satisfy the following conditions,

$$(a)\ d\left(f^i(x), f^i(y_l)\right) < for\ a_l \leq i \leq b_l yl = 1, \dots, s,$$

$$(b)\ \ f^{b_s + N_\varepsilon}(x) = x.$$

By satisfying this definition in the special case especial $s = 2$, it will only be determined that f satisfies the periodic specification property (PSP).

By omitting condition (b) in the periodic specification property, it is established that there exists a strong specification property in what is abbreviated as SSP (weak specification property-WSP).

Replacing both conditions in the strong periodic specification property by,

$$d\left(f^{i+k(b_s+N)}(x), f^{i+k(b_s+N)}(y_l)\right) < \varepsilon,$$

When $k \in N, a_l \leq i \leq b_l\ y\ l = 1, \dots, s$, it is possible to conceptualize the recurrent strong recurrent specification property (RSSP) and in the specific case $s = 2$ it will be said that f satisfies the recurrent weak periodic specification property (RWSP).

Proposition 4.2. Let $f: X \longrightarrow X$ be a continuous function defined on (X, d) as a metric space. When f satisfies SSP one will have that the set of periodic points of f is dense in X and furthermore f is mixable.

Proposition 4.3. When defining continuous functions $f: X \longrightarrow X$ and $g: Y \longrightarrow Y$ on compact metric spaces respectively as (X, d_X) and on (Y, d_Y), the following statements must be satisfied,

(a) When f satisfies SSP it is determined that f^k will also satisfy SSP at any $k \geq 1$.

(b) When f and g satisfy SSP it is determined that $f \times g$ will also satisfy SSP.

Theorem 4.4. In a dynamical system $f: X \longrightarrow X$ satisfying SSP it will be established that $\underline{f}: K(X) \longrightarrow K(X)$ also satisfies it. Allowing by this theorem to go down according to the scheme posed at the beginning of this section as observed in (Bauer & Sigmund, 1975).

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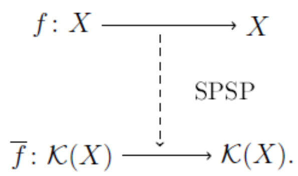
It will be obtained therefore,

$$d_H\left(\underline{f}^i(K),\underline{f}^i(K_l)\right) < d_H\left(\underline{f}^i(K),\underline{f}^i(L_l)\right) + d_H\left(\underline{f}^i(K_l),\underline{f}^i(L_l)\right) < \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

Where $a\_l \leq i \ \leq b\_l\$\ y\ \$l = 1, \dots, s$, which shows that $\underline{f}$ satisfy the strong periodic specification property.

Determining therefore that it will be possible to go down with the strong periodic specification property, as shown below,



3.3.2. Example

The following example is established in relation to (Guirao et al., 2009) which starts from a function f with its hyperextension that satisfies the property of strong periodic specification and also possesses chaos in the sense of Devaney but f does not have the same or the described property.

Established the dynamical system succession $\left((f_n, X_n)\right)_{n=1}^{\infty}$ being a metric space X_n compact in the totality of n will be considered $(\prod_{n=1}^{\infty} \ f_n, \prod_{n=1}^{\infty} \ X_n)$ as a dynamical System product.

The following lemmas will allow us to prove the mentioned.

Lemma 4.5. Let any succession $\left((f_n, X_n)\right)_{n=1}^{\infty}$ of dynamical systems which will feature every positive integer n in the whole set of recurring points, frequently of f_n which will be dense in X_n, it will therefore be established, $(\prod_{n=1}^{\infty} \ f_n, \prod_{n=1}^{\infty} \ X_n)$ as a product dynamical system which will contain a dense set of recurring points in a regular manner.

Lemma 4.6. Let any succession $\left((f_n, X_n)\right)_{n=1}^{\infty}$ of dynamical systems which will present the property of strong periodic specification, it will be established $(\prod_{n=1}^{\infty} \ f_n, \prod_{n=1}^{\infty} \ X_n)$ as a dynamical system, a product which will have the property of strong or weak recurrent specification.

It is proved in accordance with $\varepsilon > 0$ upon the existence of a positive integer M such that,

$$\sum_{i=M+1}^{\infty} \frac{1}{2^i} < \frac{\varepsilon}{4}.$$

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Being the constant N_n determined by the periodic specification property for f_n, when it is $1 \leq n \leq M$ and $N = N_n$. When being integers $x, y \in \prod_{n=1}^{\infty} X_n$ and $a_1 \leq b_1 < a_2 \leq b_2$ as $b_1 - a_1 > N \; and \; b_2 - a_2 > N$. So also, when are periodic points $p_1, \dots, p_M$ at each p_j the specification property on f_j, x_j, y_j will be obtained. Which defines the following point,

$$p = (p_1, \dots, p_M, q_{M+1}, q_{M+2}, \dots) \in \prod_{n=1}^{\infty} X_n,$$

Let $q_j \in X_j$ be any points. With direct calculation it will be verified how p satisfies the specification properties.

Lemma 4.7. When f is the continuous function with the strong recurrent (weak recurrent) specification properties, its hyperextension $\underline{f}$ will possess the strong periodic specification property (or in turn the periodic specification property).

The demonstration is established according to the satisfaction of f with respect to the recurrent strong specification property. When $\varepsilon > 0$ and the constant N are set by the above property at $\varepsilon/2$. It is set $s \geq 2$, sets $A_1, \dots, A_s \; y \; \leq b_1 < a_2 \leq \; b_2 < \cdots < a_s \leq b_s \; con \; b_j - a_j > N.$

Being open sets $U_{i,1}^l, \dots, U_{i,r}^l$ that need to cover $\underline{f}^i(A)$ set by balls having diameter less than $\varepsilon/4$ with centers A_i, let $a_l \leq i \leq b_l$. Therefore, it is assumed that r will be equally presented at i and l. Since $y_{i,j}^l \in A_l$ is the center of the ball $U_{i,1}^l$. Moreover, if the point $x^{i,j}$ is subject to the strong recurrent specification property at the points $y_{i,j}^1, \dots, y_{i,j}^s$ as in $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_s \leq b_s$. According to Lemma 2.8 which represents a similar argument, it finds a point $P_{i,j} \in K(X)$ periodic within a period $b_s + N$ and specifies that,

$$d_H\left(\underline{f}^k(P_{i,j}), \underline{f}^k(\{y_{i,j}^l\})\right) \leq \frac{\varepsilon}{2} \; para \; a_l \leq i \leq b_l \; y \; l = 1, \dots, s.$$

Considering the set

$$P = \cup_{i,j} P_{i,j}$$

It is therefore established that P represents a desired periodic point.

When $n \in N$ and Z_{n+1} is a cyclic group with $n + 1$ elements. Z_{n+1} will be endowed with the discrete topology. Being a topological product space $X_n = (Z_{n+1})^{\infty} = \{(x_m)_{m=1}^{\infty} : x_m \in X_{n+1}, m \in N\}$ corresponding to the nonnumerable number that can be

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presented of copies of Z_{n+1} it will result that X_n is homeomorphic with respect to the Cantor set. Which determines that X_n is a compact, perfect space with a basis of both closed and open numberable sets. Base that develops according to the cylinder sets in the following way,

$$[z_1, \dots, z_k] = (x_m)_{m=1}^{\infty} \in X_n: x_1 = z_1, \dots, x_k = z_k,$$

Let $k \in N$ and $z_1, \dots, z_k$ be an arbitrary succession of elements belonging to Z_{n+1} according to length k.

In the definition of the function $f_n: X_n \longrightarrow X_n$ by the $f((x_m)_{m=1}^{\infty}) = (y_m)_{m=1}^{\infty}$ where it will be established that,

$$y_m\{x_{m+1}, \ x_1 \neq \ X_{n+1} \ \ 1 + x_{m+1}, \ x_1 = \ X_{n+1}$$

In all $m \in N$.

Lemma 4.8. When $n \in N$ is given for $f_n: X_n \longrightarrow X_n$ it will follow that,

(a) f_n is a continuous function

(b) f_n will not contain periodic points in a period equal to n.

(c) When $n \geq 3$ is presented, the function f_n will be able to satisfy the strong periodic specification property.

(d) f_n will be topologically exact.

In the proof of (a) it is proved that by setting $z \in X_n$, it will be the preimage in any open neighbourhood of z conforming to f_n will be open. That is, being $[z_1, \dots, z_k]$ $(k \geq n)$ we will have,

$$f^{-1}([z_1, \dots, z_k]) = \left(\cup_{a \in Z_{n+1} \setminus \{z_n\}} [a, z_1, \dots, z_k]\right) \cup [z_n - 1, z_1 - 1, \dots, z_k - 1]$$

Which will be open. In case it is $k < n$ the disjunct decomposition will be set as,

$$[z_1, \dots, z_k] = \cup_{a_1, \dots, a_{n-k}} [z_1, \dots, z_k, a_1, \dots, a_{n-k}].$$

In the case of (b) by assuming the existence of the sequence $(x_m)_{m=1}^{\infty} \in X_n$ be,

$$(y_m)_{m=1}^{\infty} = f((x_m)_{m=1}^{\infty}) = (x_m)_{m=1}^{\infty}.$$

By definition of f_n it is appreciated that $k + x_{m+n} = x_m$ in the totality of $m \in N$, where $k = \#\{j \in \{1, \dots, n\}: x_j = x_{j+n}\}$ clearly specifying $0 \leq k \leq n$ where the following cases are considered:

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1. Being $k > 0$ it will present $j \in \{1, \dots, n\}$ in such a way that $x_j = x_{j+n}$ and considering the equality in $k + x_{j+n} = x_j$ it is appreciated $k = 0$ determining a contraindication.

2. Being $k = 0$ it will appear in particular $x_{n+1} = x_1$ having subsequently $k \geq 1$ determining a contraindication.

For (c) it is proved that as there exists $N \in N$ in each $s, t \in N$ in the case of two finite successions of points be $u_1, \dots, u_s$ and $v_1, \dots, v_t$ elements of Z_{n+1} a succession $w_1, \dots, w_N$ will be found conforming to,

$$f_n^{s+N}([u_1, \dots, u_s, w_1, \dots, w_N, v_1, \dots, v_t]) = [v_1, \dots, v_t].$$

It is then stated that $N = 2n$.

In (d) it is analyzed according to the same techniques described previously, that since $[u_1, \dots, u_k]$ is a non-empty cylindrical set, it will be defined that $f_n^{k+2n}([u_1, \dots, u_k]) = X_n$.

Theorem 4.9. Since f:X⟶X exists as a topologically exact Dynamical System, it does not satisfy the strong periodic specification property by presenting "Per" (f) as nondense in X. However, its hyperextension ¯f containing exact Devaney chaos will.

It is demonstrated according to $n \in \{2, \dots\}$ when presented (f_n, X_n) within a Dynamical System stated in Lemma 4.8 where each f_n contains exact Devaney chaos and further satisfies the strong periodic specification property, it will therefore be established as a Cartesian product where f is topologically exact including the recurrent strong specification property as provided in Lemma 4.6. Mentioning again Lemma 4.8 it is further determined that f possesses non-periodic points in a period equal to or greater than 2. Furthermore, the function being topologically transitive and different in its identity function establishes that the set $Per(f)$ will not be dense. This means that f does not possess exact chaos in the sense of Devaney and also does not satisfy the property of strong periodic specification, on the contrary its hyperextension $\underline{f}$ does possess such chaos and fulfils the property mentioned in Lemmas 4.5 and 4.7 in addition to Theorem 2.23.

If exact Devaney chaos is present in $\underline{f}$, it will also possess Devaney chaos which, according to Theorem 2.17, will be equivalent to almost total chaos in the Devaney sense, which determines that,

$$\underline{f}\ totDev\ C \nRightarrow f\ totDev\ C,$$

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And in turn

$$\underline{f}\ Dev\ C\ \nRightarrow f\ Dev\ C,$$

Since f does not have a dense set of periodic points.

Conclusions

With respect to the function f and its hyperextension $\underline{f}$ and in accordance with the different properties of partial chaos type analyzed, the following is established,

(a) In topological transitivity $\nRightarrow$; $\Leftarrow$

(b) Fully transitive $\Rightarrow$; $\Leftarrow$

(c) Exactly topological $\Rightarrow$; $\Leftarrow$

(d) Weakly mixable $\Rightarrow$; $\Leftarrow$

(e) Mixable $\Rightarrow$; $\Leftarrow$

(f) In the density of periodic points $\Rightarrow$; $\nLeftarrow$

With the various notions of chaos as the periodic point density, it can be established that,

(a) Devaney chaos (Dev C) $\nRightarrow$; $\nLeftarrow$

(b) In the cases: totDev C, exDev C, LYC, ω C, dC, and SPSP $\Rightarrow$; $\nLeftarrow$

Which determines that collectively the different notions of chaos have no implication with those individually. But vice versa they are consistently related, except for chaos in Devaney's sense.

References

Akin, Ethan y Kolyada, Sergiy (2003). Li-Yorke sensitivity. 4, 2003, *Nonlinearity*, Vol. 16, págs. 1421-1433.

Aliprantis, Charalambos y Border, Kim (2006). Infinite Dimensional Analysis. Berlin : A hitchhiker's guide, 2006. ISBN: 978-3-540-29587-7.

Balibrea, F, Smítal, J y Stefánková, M. (2005). The three versions of distributional chaos. 4, 2005, *Amer. Math.* Monthly, Vol. 99.

Banks, J, y otros (1992). On Devaney's definition of chaos. 4, 1992, *American Mathematical Society*, Vol. 99, págs. 332-334.

DOI: https://doi.org/10.46925//rdluz.40.04

Banks, John (2005). Chaos for induced hyperspace maps. *Chaos Solitons Fractals* Vol. 25, no. 3, págs. 681-685.

Barnsley, Michael (1993). Fractals everywhere. Second. Boston: Academic Press Professional, 1993. ISBN: 0-12-079061-0.

Bauer, W y Sigmund, K. (1975). Topological dynamics of transformations induced on the space of probability measures. 1975, *Monatsh. Math.*, Vol. 79, págs. 81-92

Bauer, Walter y Sigmund, Karl (1975). Topological dynamics of transformations induced on the space of probability measures. 1975, *Monatshefte für Mathematik*, Vol. 79, págs. 81-92.

Bermúdez, A, y otros (2011). Li-Yorke and distributionally chaotic operators. 1, 2011, J. Math. Anal. Appl., Vol. 373, págs. 83-93.

Bernardes, N, Peris, A y Rodenas, F. (2017). Set-valued chaos in linear dynamics. 4, 2017, Integral Equations Operator Theory, Vol. 87, págs. 451-463.

Bernardes, N, y otros (2015). Li-Yorke chaos in linear dynamics. 6, 2015, *Ergodic Theory Dynam. Systems*, Vol. 35, págs. 1723-1745.

De la Rosa, Manuel y Read, Charles (2009). A Hypercyclic Operator Whose Direct Sum T ⊕ T is not Hypercyclic.. 2, 2009, *Journal Operator Theory*, Vol. 61, págs. 369-380.

Delgadillo, Gerardo y López, Miguel (2009). *Espacios de Fréchet Urysohn*. No 1, 2009, Vol. 8, págs. 29-56.

Edgar, Gerald (2008). Measure, topology, and fractal geometry. Second. New York : Springer, 2008. ISBN: 978-0-387-74748-4.

Furstenberg, Harry (1967). Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation. 1, 1967, *Mathematical Systems Theory*, Vol. 1, págs. 1-49.

Grosse-Erdmann, Karl y Peris-Manguillot, Alfred (2011). Linear chaos. London : Universitext Springer, 2011. ISBN: 978-1-4471-2170-1.

Guirao, J, y otros (2009). Chaos on hyperspaces. 1-2, 2009, *Nonlinear Anal*, Vol. 71, págs. 1-8.

Lampart, Marek (2003). Two kinds of chaos and relations between them. 1, 2003, *Acta Math. Univ. Comenian*, Vol. 339, págs. 119-127.

Li, Shihai (1993). ω-Chaos and Topological Entropy. 1, 1993, *Transactions of the American Mathematical Society*, Vol. 3389, págs. 243-249.

Li, Shoumei; Ogura, Yukio y Kreinovich, Vladik (2022). Limit Theorems and Applications of Set-Valued and Fuzzy Set-Valued Random Variables. Dordrecht : Springer-Science+Bussiness media, B.V., 2022. Vol. 43. ISBN: 978-94-015-9932-0.

Li, Tien y Yorke, James (1975). Period three implies chaos. 10, 1975, The American

DOI: https://doi.org/10.46925//rdluz.40.04

Mathematical Monthly, Vol. 82, págs. 985-992.

Liao, Gongfu, Wang, Lidong y Zhang, Yucheng (2006). Transitivity, mixing and chaos for a class of set-valued mappings. 1, 2006, *SCI. China*, Vol. Ser. A 49, págs. 1-8.

Martínez, Félix (2000). Universalidad, Hiperciclicidad y Caos en Espacios de Frechet. Universidad Politécnica de Valencia. 2000. Tesis doctoral.

Peris, Alfredo (2005). Set-valued discrete chaos. 1, 2005, Chaos, Solitons & Fractals, Vol. 26, págs. 19-23.

Román-Flores (2003), *A note on transitivity in set-valued discrete systems,* Chaos Solitons Fractals **17,** no 1, 99-104.

Román, Heriberto, Aguirre, Iván y Ayala, Víctor (2018). Algunas conexiones entre dinámica individual y colectiva. 11, 2018, *Interciencia*, Vol. 43, págs. 744-750. ISSN: 0378-1844.

Rudin, W. (1991). Functional analysis. Second. New York : Mcgrawhill, 1991. ISBN 0-07-100944-2 .

Rudin, W.(1991). Functional analysis. New York : Mcgrawhill Inc., 1991.

Smítal, J y Stefánková, M. (2004). Distributional chaos for triangular maps. 5, 2004, *Chaos Solitons Fractals,* Vol. 21, págs. 1125-1128.

Ulcigrai, Corinna (2021) Slow chaos in surface flows. *Bollettino dell'Unione Matematica Italiana,* Vol. 14, págs. 231-255.