Rev. Téc. Ing. Univ. Zulia. Vol. 37, Nº 2, 167 - 175, 2014

Certain fractional integral inequalities involving the Gauss hypergeometric function

***S.D. Purohit*[1]*, R.K. Raina*[2]**

[1]Department of Basic Sciences (Mathematics), College of Technology and Engineering, M.P. University of Agriculture and Technology, Udaipur-313001, India.
[2]10/11, Ganpati Vihar, Opposite Sector-5, Udaipur-313002, India.
sunil_a_purohit@yahoo.com , rkraina_7@hotmail.com

Abstract

By making use of the fractional integral operators involving the Gauss hypergeometric function, we establish certain new fractional integral inequalities for synchronous functions which are related to the Chebyshev functional. Some consequent results and special cases of the main results are also pointed out.

Keywords: integral inequalities, Gauss hypergeometric function, fractional integral operators.

Ciertas desigualdades integrales fraccionales que involucran la función hipergeométrica de Gauss

Resumen

Usando operadores integrales fraccionales que involucran la función hipergeométrica de Gauss, se establecen ciertas nuevas desigualdades integrales fraccionales para funciones sincrónicas, relacionadas con la funcional Chebyshev. Algunos resultados particulares y casos especiales de los resultados principales son también presentados.

Palabras clave: desigualdades integrales, función hipergeométrica de Gauss, operadores integrales fraccionales.

1. Introduction

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators. One may, for instance, refer to such type of works in the book [1], and the papers [2-11].

In the sequel, we use the following definitions and related details.

Definition 1

Two functions f and g are said to be synchronous on $[a,b]$, if

$$\left\{\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)\right\} \geq 0\,, \tag{1}$$

for any $x, y \in [a,b]$.

Definition 2

A real-valued function $f(t)\,(t>0)$ is said to be in the space $C_\mu\left(\mu \in \mathbb{R}\right)$ if there exists a real number $p>\mu$ such that $f(t)=t^p\phi(t)$.

Throughout this paper, we denote by $C(0,\infty)$, the space of all continuous functions from $(0,\infty)$ into $\mathbb{R}$ and $L_r([0,\infty))$ the space of all r[th] power Lebesgue integrable functions defined on the interval $[0,\infty)$.

Definition 3

Let $\alpha>0$, $\beta,\eta\in\mathbb{R}$ then the Saigo fractional integral $I_{0,t}^{\alpha,\beta,\eta}$ (in terms of the Gauss hypergeometric function) of order α for a real-valued continuous function $f(t)$ is defined by ([12], see also [13, p. 19, eqn. (1.1.l)], [14]):

$$I_{0,t}^{\alpha,\beta,\eta}\{f(t)\}=\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)f(\tau)d\tau, \tag{2}$$

where, the function ${}_2F_1(-)$ appearing as a kernel for the operator (2) is the Gaussian hypergeometric function defined by

$${}_2F_1(a,b;c;t)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}, \tag{3}$$

and $(a)_n$ is the Pochhammer symbol:

$$(a)_n=a(a+1)\cdots(a+n-1),\ (a)_0=1.$$

The operator (2) includes both the Riemann-Liouville and the Erdélyi-Kober fractional integral operators given by

$$R^{\alpha}\{f(t)\}=I_{0,t}^{\alpha,-\alpha,\eta}\{f(t)\}=$$

$$\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}f(\tau)d\tau\quad(\alpha>0) \tag{4}$$

and

$$I^{\alpha,\eta}\{f(t)\}=I_{0,t}^{\alpha,0,\eta}\{f(t)\}=$$

$$\frac{t^{-\alpha-\eta}}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}\tau^{\eta}f(\tau)d\tau\ \left(\alpha>0,\eta\in\mathbb{R}\right).$$

For $f(t)=t^{\mu}$ in (2), we get the image formula (see [12]):

$$I_{0,t}^{\alpha,\beta,\eta}\{t^{\mu}\}=\frac{\Gamma(\mu+1)\Gamma(\mu+1-\beta+\eta)}{\Gamma(\mu+1-\beta)\Gamma(\mu+1+\alpha+\eta)}t^{\mu-\beta}$$

$$(\alpha>0,\min(\mu,\mu-\beta+\eta)>-1,t>0). \tag{6}$$

Our aim in this paper is to obtain certain fractional integral inequalities for synchronous functions which are related to the Chebyshev functional ([15]) by using the Saigo fractional integral operator which involves in the kernel, the Gauss hypergeometric function (defined above). The concluding section gives some consequent results and some special cases of the main results.

2. Main results

We obtain in this section certain integral inequalities for the synchronous functions involving the Saigo fractional integral operator (2).

Theorem 1

Let p be a positive function, and f and g be two synchronous functions on $[0,\infty)$. If $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then (for all $t>0$, $\alpha>\max\{0,-\beta\}$, $\eta<0$):

$$2\Big|I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)g(t)\}-$$

$$I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)\}I_{0,t}^{\alpha,\beta,\eta}\{p(t)g(t)\}\Big|\le\frac{t^{-2\alpha-2\beta}\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma^2(\alpha)}$$

$$\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\left|\tau-\rho\right|d\tau d\rho$$

$$\le\left\|f'\right\|_r\left\|g'\right\|_s t\left(I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}\right)^2. \tag{7}$$

Proof: Let f and g be two synchronous functions, then using Definition 1, for all $\tau,\rho\in(0,t),t\ge0$, we define

$$H(\tau,\rho)=\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right). \tag{8}$$

Consider

$$F(t,\tau)=\frac{t^{-\alpha-\beta}\left(t-\tau\right)^{\alpha-1}}{\Gamma(\alpha)}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$(\tau\in(0,t);t>0) \tag{9}$$

$$=\frac{1}{\Gamma(\alpha)}\frac{(t-\tau)^{\alpha-1}}{t^{\alpha+\beta}}+\frac{(\alpha+\beta)(-\eta)}{\Gamma(\alpha+1)}\frac{(t-\tau)^{\alpha}}{t^{\alpha+\beta+1}}+$$

$$\frac{(\alpha+\beta)(\alpha+\beta+1)(-\eta)(-\eta+1)}{2\Gamma(\alpha+2)}\frac{(t-\tau)^{\alpha+1}}{t^{\alpha+\beta+2}}+\cdots.$$

We observe that each term of the above series is positive in view of the conditions stated with Theorem 1, and hence, the function $F(t,\tau)$ remains positive, for all $\tau\in(0,t)$ $(t>0)$.

Multiplying both sides of (8) by $F(t,\tau)\,p(\tau)$ (where $F(t,\tau)$ is given by (9)) and integrating with respect to τ from 0 to t, and using (2), we get

$$\frac{t^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)p(\tau)\times$$

$$\mathrm{H}(\tau,\rho)d\tau = I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)g(t)\}-$$

$$f(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)g(t)\}-g(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)\}+$$

$$f(\rho)g(\rho)\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}. \quad (10)$$

Next, on multiplying both sides of (10) by $F(t,\rho)\,p(\rho)$, where $F(t,\rho)$ is given by (9), and integrating with respect to ρ from 0 to t, we can write

$$\frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}\,(t-\rho)^{\alpha-1}\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)$$

$$\times\, p(\tau)p(\rho)\,\mathrm{H}(\tau,\rho)d\tau d\rho$$

$$=2\left(I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)g(t)\}\right.$$

$$\left.-I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)\}\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)g(t)\}\right). \quad (11)$$

In view of (8), we have

$$\mathrm{H}(\tau,\rho)=\int_\tau^\rho\int_\tau^\rho f'(y)g'(z)dydz. \quad (12)$$

Using the following Holder's inequality for the double integral:

$$\left|\int_\tau^\rho\int_\tau^\rho f(y)g(z)dydz\right|\le\left|\int_\tau^\rho\int_\tau^\rho |f(y)|^r\,dydz\right|^{r^{-1}}\times$$

$$\left|\int_\tau^\rho\int_\tau^\rho |g(z)|^s\,dydz\right|^{s^{-1}}\qquad (r^{-1}+s^{-1}=1), \quad (13)$$

we obtain

$$|\mathrm{H}(\tau,\rho)|\le\left|\int_\tau^\rho\int_\tau^\rho |f'(y)|^r\,dydz\right|^{r^{-1}}\left|\int_\tau^\rho\int_\tau^\rho |g'(z)|^s\,dydz\right|^{s^{-1}}. \quad (14)$$

Since

$$\left|\int_\tau^\rho\int_\tau^\rho |f'(y)|^r\,dydz\right|^{r^{-1}}=|\tau-\rho|^{r^{-1}}\left|\int_\tau^\rho |f'(y)|^r\,dy\right|^{r^{-1}} \quad (15)$$

and

$$\left|\int_\tau^\rho\int_\tau^\rho |g'(z)|^s\,dydz\right|^{s^{-1}}=|\tau-\rho|^{s^{-1}}\left|\int_\tau^\rho |g'(z)|^s\,dz\right|^{s^{-1}}, \quad (16)$$

then, (14) reduces to

$$|\mathrm{H}(\tau,\rho)|\le|\tau-\rho|\left|\int_\tau^\rho |f'(y)|^r\,dy\right|^{r^{-1}}\left|\int_\tau^\rho |g'(z)|^s\,dz\right|^{s^{-1}}. \quad (17)$$

It follows from (11) that

$$\frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}\,(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times|\mathrm{H}(\tau,\rho)|\,d\tau d\rho\le\frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\times$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}\,(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$\times\,{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|\times$$

$$\left|\int_\tau^\rho |f'(y)|^r\,dy\right|^{r^{-1}}\left|\int_\tau^\rho |g'(z)|^s\,dz\right|^{s^{-1}}d\tau d\rho. \quad (18)$$

Applying again the Holder's inequality (13) on the right-hand side of (18), we get

$$\frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}\,(t-\rho)^{\alpha-1}\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times|\mathrm{H}(\tau,\rho)|\,d\tau d\rho\le\left[\frac{t^{-r\alpha-r\beta}}{\Gamma^r(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}\,(t-\rho)^{\alpha-1}\times\right.$$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)\times $$

$$ p(\tau)p(\rho)\left|\tau-\rho\right|\left|\int_\tau^\rho \left|f'(y)\right|^r dy\right| d\tau d\rho\Big]^{r^{-1}} $$

$$ \times\left[\frac{t^{-s\alpha-s\beta}}{\Gamma^s(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}\times\right. $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right) $$

$$ p(\tau)p(\rho)\left|\tau-\rho\right|\left|\int_\tau^\rho \left|g'(z)\right|^s dz\right| d\tau d\rho\Big]^{s^{-1}}. \quad (19) $$

In view of the fact that

$$ \left|\int_\tau^\rho \left|f(y)\right|^p dy\right| \le \left\|f\right\|_p^p, \quad (20) $$

we get

$$ \frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}\times $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho) $$

$$ \times\left|\mathsf{H}(\tau,\rho)\right| d\tau d\rho \le \left[\frac{t^{-r\alpha-r\beta}\left\|f'\right\|_r^r}{\Gamma^r(\alpha)}\right. $$

$$ \int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho\Big]^{r^{-1}} $$

$$ \times\left[\frac{t^{-s\alpha-s\beta}\left\|g'\right\|_s^s}{\Gamma^s(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}\times\right. $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho\Big]^{s^{-1}}. \quad (21) $$

From (21), we obtain

$$ \frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1} $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho) $$

$$ \times\left|\mathsf{H}(\tau,\rho)\right| d\tau d\rho \le \frac{t^{-2\alpha-2\beta}\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma^2(\alpha)}\times $$

$$ \left[\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\right. $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho\Big]^{r^{-1}}\times $$

$$ \left[\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times\right. $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho\Big]^{s^{-1}}. \quad (22) $$

Since $r^{-1}+s^{-1}=1$, therefore, the above inequality yields

$$ \frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}\times $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho) $$

$$ \times\left|\mathsf{H}(\tau,\rho)\right| d\tau d\rho \le \frac{t^{-2\alpha-2\beta}\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma^2(\alpha)} $$

$$ \int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times $$

$$ {}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\left|\tau-\rho\right| d\tau d\rho, \quad (23) $$

which in view of (11) gives

$$ 2\left|I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)g(t)\}-\right. $$

$$ \left.I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)\}I_{0,t}^{\alpha,\beta,\eta}\{p(t)g(t)\}\right| \le $$

$$ \frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right) $$

$$ \times{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\left|\mathsf{H}(\tau,\rho)\right| d\tau d\rho. \quad (24) $$

Making use of (23) and (24), the left-hand side of the inequality (7) follows.

To prove the right-hand side of the inequality (7), we observe that $0 \le \tau \le t$, $0 \le \rho \le t$, and therefore, $0 \le |\tau - \rho| \le t$.

Evidently from (23), we get

$$\frac{t^{-2\alpha-2\beta}}{\Gamma^2(\alpha)} \int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}$$

$$_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)\times$$

$$_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times|\mathrm{H}(\tau,\rho)|\,d\tau d\rho \le \frac{t^{-2\alpha-2\beta}\,\|f'\|_r\,\|g'\|_s\,t}{\Gamma^2(\alpha)}$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$\times{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\rho}{t}\right)p(\tau)p(\rho)d\tau d\rho =$$

$$\|f'\|_r\,\|g'\|_s\,t\left(I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}\right)^2, \tag{25}$$

which completes the proof of Theorem 1.

The following gives a generalization of Theorem 1.

Theorem 2

Let p be a positive function and f and g be two synchronous functions on $[0,\infty)$. If $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty))$, $r > 1$, $r^{-1} + s^{-1} = 1$, then

$$\Big|I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{p(t)f(t)g(t)\} +$$

$$I_{0,t}^{\gamma,\delta,\zeta}\{p(t)\}\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)g(t)\}$$

$$-I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{p(t)g(t)\} -$$

$$I_{0,t}^{\gamma,\delta,\zeta}\{p(t)f(t)\}\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)g(t)\}\Big|$$

$$\le \frac{t^{-\alpha-\beta-\gamma-\delta}\,\|f'\|_r\,\|g'\|_s}{\Gamma(\alpha)\Gamma(\gamma)}$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|\,d\tau d\rho \le$$

$$\|f'\|_r\,\|g'\|_s\,t\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{p(t)\} \tag{26}$$

for all $t > 0$, $\alpha > \max\{0,-\beta\}$, $\eta < 0$, $\gamma > \max\{0,-\delta\}$, $\zeta < 0$.

Proof: To prove the above theorem, we use the inequality (10). Multiplying both sides of (10) by

$$\frac{t^{-\gamma-\delta}(t-\rho)^{\gamma-1}}{\Gamma(\gamma)}\,{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\rho)$$

$$(\rho \in (0,t); t > 0),$$

which remains positive in view of the conditions stated with (26) and integrating with respect to ρ from 0 to t, we get

$$\frac{t^{-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}\times$$

$$_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times\mathrm{H}(\tau,\rho)d\tau d\rho = I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{p(t)f(t)g(t)\} +$$

$$I_{0,t}^{\gamma,\delta,\zeta}\{p(t)\}\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)g(t)\}$$

$$-I_{0,t}^{\alpha,\beta,\eta}\{p(t)f(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{p(t)g(t)\} -$$

$$I_{0,t}^{\gamma,\delta,\zeta}\{p(t)f(t)\}\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)g(t)\}. \tag{27}$$

Now making use of (17), then (27) gives

$$\frac{t^{-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}\times$$

$$F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times|\mathrm{H}(\tau,\rho)|\,d\tau d\rho \le \frac{t^{-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha)\Gamma(\gamma)}$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$\times{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|\times$$

$$\left|\int_\tau^\rho |f'(y)|^r dy\right|^{r^{-1}} \left|\int_\tau^\rho |g'(z)|^s dz\right|^{s^{-1}} d\tau d\rho. \quad (28)$$

Applying the Holder's inequality (13) on the right-hand side of (28), we get

$$\frac{t^{-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times|\mathrm{H}(\tau,\rho)|\,d\tau d\rho \le \Bigg[\frac{t^{-r\alpha-r\beta}}{\Gamma^r(\alpha)}$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$\times{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|\times$$

$$\left|\int_\tau^\rho |f'(y)|^r dy\right| d\tau d\rho\Bigg]^{r^{-1}}$$

$$\times\Bigg[\frac{t^{-s\gamma-s\delta}}{\Gamma^s(\gamma)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$\times{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|\times$$

$$\left|\int_\tau^\rho |g'(z)|^s dz\right| d\tau d\rho\Bigg]^{s^{-1}} \quad (29)$$

which on using (20) readily yields the following inequality:

$$\frac{t^{-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times|H(\tau,\rho)|\,d\tau d\rho \le \frac{t^{-\alpha-\beta-\gamma-\delta}\,\|f'\|_r\,\|g'\|_s}{\Gamma(\alpha)\Gamma(\gamma)}$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$\times{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)|\tau-\rho|\,d\tau d\rho. \quad (30)$$

In view of (27) and (30), and the properties of modulus, one can easily arrive at the left-sided inequality of Theorem 2. Moreover, we have $0\le\tau\le t,\ 0\le\rho\le t$, hence

$$0\le|\tau-\rho|\le t.$$

Therefore, from (30), we get

$$\frac{t^{-\alpha-\beta-\gamma-\delta}}{\Gamma(\alpha)\Gamma(\gamma)}\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}\times$$

$${}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right){}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)$$

$$\times|\mathrm{H}(\tau,\rho)|\,d\tau d\rho \le \frac{t^{-\alpha-\beta-\gamma-\delta}\,\|f'\|_r\,\|g'\|_s\,t}{\Gamma(\alpha)\Gamma(\gamma)}$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$

$$\times{}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)p(\tau)p(\rho)\,d\tau d\rho \le$$

$$\|f'\|_r\,\|g'\|_s\,t\,I_{0,t}^{\alpha,\beta,\eta}\{p(t)\}\,I_{0,t}^{\gamma,\delta,\zeta}\{p(t)\}, \quad (31)$$

which completes the proof of Theorem 2.

Remark 1. For $\gamma=\alpha,\delta=\beta,\zeta=\eta$, Theorem 2 immediately reduces to Theorem 1.

3. Consequent results and special cases

As implications of our main results, we consider some consequent results of Theorems 1 and 2 by suitably choosing the function $p(t)$. To this end, let us set $p(t)=t^\lambda\ (\lambda\in[0,\infty)\ t\in(0,\infty)$, then on using (6), Theorems 2 yield the following result.

Corollary 1

Let *f* and *g* be two synchronous functions on $[0,\infty)$. If $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then

$$\Bigg|\frac{\Gamma(\lambda+1)\Gamma(\lambda+1-\beta+\eta)}{\Gamma(\lambda+1-\beta)\Gamma(\lambda+1+\alpha+\eta)}t^{\lambda-\beta}I_{0,t}^{\gamma,\delta,\zeta}\left\{t^\lambda f(t)g(t)\right\}+$$

$$\frac{\Gamma(\lambda+1)\Gamma(\lambda+1-\delta+\zeta)}{\Gamma(\lambda+1-\delta)\Gamma(\lambda+1+\gamma+\zeta)}t^{\lambda-\delta}$$

$$\times I_{0,t}^{\alpha,\beta,\eta}\left\{t^\lambda f(t)g(t)\right\}-I_{0,t}^{\alpha,\beta,\eta}\left\{t^\lambda f(t)\right\}I_{0,t}^{\gamma,\delta,\zeta}\left\{t^\lambda g(t)\right\}-$$

$$I_{0,t}^{\gamma,\delta,\zeta}\left\{t^\lambda f(t)\right\}I_{0,t}^{\alpha,\beta,\eta}\left\{t^\lambda g(t)\right\}\Bigg| \le \frac{t^{-\alpha-\beta-\gamma-\delta}\,\|f'\|_r\,\|g'\|_s}{\Gamma(\alpha)\Gamma(\gamma)}$$

$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$
$$\times {}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)\tau^{\lambda}\rho^{\lambda}\left|\tau-\rho\right|d\tau d\rho \le \left\|f'\right\|_r \left\|g'\right\|_s$$
$$\times\frac{\Gamma^2(\lambda+1)\,\Gamma(\lambda+1-\beta+\eta)\Gamma(\lambda+1-\delta+\zeta)}{\Gamma(\lambda+1-\beta)\Gamma(\lambda+1+\alpha+\eta)\Gamma(\lambda+1-\delta)\Gamma(\lambda+1+\gamma+\zeta)}t^{1+2\lambda-\beta-\delta}, \tag{32}$$

for all $t>0$, $\lambda\ge 0$, $\alpha>\max\{0,-\beta\}$, $\eta<0$, $\gamma>\max\{0,-\delta\}$, $\zeta<0$ and $\min(\lambda,\lambda-\beta+\eta,\lambda-\delta+\zeta)>-1$.

Next, setting

$$p(t)=(1-at)^{-m}\ (a,m\in[0,\infty),t\in(0,\infty)),$$

then upon using the relation (which can be easily computed) that

$$I_{0,t}^{\alpha,\beta,\eta}\left\{(1-at)^{-m}\right\}=\frac{\Gamma(1-\beta+\eta)t^{-\beta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)}\times$$
$${}_3F_2\left(m,1,1-\beta+\eta;1-\beta,1+\alpha+\eta;at\right), \tag{33}$$

where the function

$${}_3F_2(a,b,c;d,e;x)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!}x^n$$
$$(d,e\ne 0,-1,-2,\cdots;|x|<1), \tag{34}$$

then Theorem 2 give the following result.

Corollary 2

Let f and g be two synchronous functions on $[0,\infty)$. If $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty)), r>1$, $r^{-1}+s^{-1}=1$, then

$$\Bigg|\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)t^{\beta}}\times$$
$${}_3F_2\left(m,1,1-\beta+\eta;1-\beta,1+\alpha+\eta;at\right)\times$$
$$I_{0,t}^{\gamma,\delta,\zeta}\left\{(1-at)^{-m}f(t)g(t)\right\}+\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)t^{\delta}}\times$$
$${}_3F_2\left(m,1,1-\delta+\zeta;1-\delta,1+\gamma+\zeta;at\right)\times$$
$$I_{0,t}^{\alpha,\beta,\eta}\left\{(1-at)^{-m}f(t)g(t)\right\}-$$
$$I_{0,t}^{\alpha,\beta,\eta}\left\{(1-at)^{-m}f(t)\right\}I_{0,t}^{\gamma,\delta,\zeta}\left\{(1-at)^{-m}g(t)\right\}-$$
$$I_{0,t}^{\gamma,\delta,\zeta}\left\{(1-at)^{-m}f(t)\right\}I_{0,t}^{\alpha,\beta,\eta}\left\{(1-at)^{-m}g(t)\right\}\Bigg|$$
$$\le\frac{t^{-\alpha-\beta-\gamma-\delta}\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma(\alpha)\Gamma(\gamma)}$$
$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$
$$\times {}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)(1-a\tau)^{-m}(1-a\rho)^{-m}\times$$
$$\left|\tau-\rho\right|d\tau d\rho\le\left\|f'\right\|_r\left\|g'\right\|_s$$
$$<\frac{\Gamma(1-\beta+\eta)\Gamma(1-\delta+\zeta)t^{1-\beta-\delta}}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)}\times$$
$${}_3F_2\left(m,1,1-\beta+\eta;1-\beta,1+\alpha+\eta;at\right)$$
$$\times {}_3F_2\left(m,1,1-\delta+\zeta;1-\delta,1+\gamma+\zeta;at\right), \tag{35}$$

for all $t>0$, $\alpha>\max\{0,-\beta\}$, $m\in[0,\infty)$, $\beta-1<\eta<0$, $\gamma>\max\{0,-\delta\}$, $\delta-1<\zeta<0$.

We observe that if we put $\lambda=0$ in Corollary 1 (or $m=0$ in Corollary 2), we obtain the following integral inequality:

Corollary 3

Let f and g be two synchronous functions on $[0,\infty)$. If $f'\in L_r([0,\infty))$, $g'\in L_s([0,\infty))$, $r>1$, $r^{-1}+s^{-1}=1$, then

$$\Bigg|\frac{\Gamma(1-\beta+\eta)}{\Gamma(1-\beta)\Gamma(1+\alpha+\eta)\,t^{\beta}}I_{0,t}^{\gamma,\delta,\zeta}\left\{f(t)g(t)\right\}+$$
$$\frac{\Gamma(1-\delta+\zeta)}{\Gamma(1-\delta)\Gamma(1+\gamma+\zeta)\,t^{\delta}}I_{0,t}^{\alpha,\beta,\eta}\left\{f(t)g(t)\right\}$$
$$-I_{0,t}^{\alpha,\beta,\eta}\left\{f(t)\right\}I_{0,t}^{\gamma,\delta,\zeta}\left\{g(t)\right\}-I_{0,t}^{\gamma,\delta,\zeta}\left\{f(t)\right\}I_{0,t}^{\alpha,\beta,\eta}\left\{g(t)\right\}\Bigg|$$
$$\le\frac{t^{-\alpha-\beta-\gamma-\delta}\left\|f'\right\|_r\left\|g'\right\|_s}{\Gamma(\alpha)\Gamma(\gamma)}$$
$$\int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}{}_2F_1\left(\alpha+\beta,-\eta;\alpha;1-\frac{\tau}{t}\right)$$
$$\times {}_2F_1\left(\gamma+\delta,-\zeta;\gamma;1-\frac{\rho}{t}\right)\left|\tau-\rho\right|d\tau d\rho\le\left\|f'\right\|_r\left\|g'\right\|_s\times$$
$$\frac{\Gamma(1-\beta+\eta)\Gamma(1-\delta+\zeta)\,t^{1-\beta-\delta}}{\Gamma(1-\beta)\Gamma(1-\delta)\Gamma(1+\alpha+\eta)\Gamma(1+\gamma+\zeta)}, \tag{36}$$

for all $t > 0$, $\alpha > \max\{0, -\beta\}$, $\gamma > \max\{0, -\delta\}$, $\beta, \delta < 1$, $\beta - 1 < \eta < 0$, $\delta - 1 < \zeta < 0$.

We now consider some special cases of the results derived in the previous section. If we set $\beta = \delta = 0$, and make use of the relation (5), Theorem 2 yield the following integral inequalities involving the Erdélyi-Kober type fractional integral operator defined by (5):

Corollary 4

Let p be a positive function and f and g be two synchronous functions on $[0,\infty)$. If $f' \in L_r([0,\infty))$, $g' \in L_s([0,\infty))$, $r > 1$, $r^{-1} + s^{-1} = 1$, then

$$\Big| I^{\alpha,\eta}\{p(t)\} I^{\gamma,\zeta}\{p(t)f(t)g(t)\} + I^{\gamma,\zeta}\{p(t)\} I^{\alpha,\eta}\{p(t)f(t)g(t)\} - I^{\alpha,\eta}\{p(t)f(t)\} I^{\gamma,\zeta}\{p(t)g(t)\} - I^{\gamma,\zeta}\{p(t)f(t)\} I^{\alpha,\eta}\{p(t)g(t)\}\Big| \leq \frac{t^{-\alpha-\gamma-\eta-\zeta}\, \|f'\|_r \, \|g'\|_s}{\Gamma(\alpha)\Gamma(\gamma)} \times \int_0^t\int_0^t (t-\tau)^{\alpha-1}(t-\rho)^{\gamma-1}\tau^{\eta}\rho^{\zeta} p(\tau)p(\rho) \times |\tau-\rho|\, d\tau d\rho \leq \|f'\|_r \, \|g'\|_s \, t\, I^{\alpha,\eta}\{p(t)\} I^{\gamma,\zeta}\{p(t)\}, \quad (37)$$

for all $t > 0$, $\alpha > 0$, $\eta < 0$, $\gamma > 0$ and $\zeta < 0$.

Further, it may be noted that for $\gamma = \alpha, \delta = \beta, \zeta = \eta$, the Corollaries 1 to 3 would reduce to similar types of results which alternatively would also follow from Theorem 1 directly.

Again, if we replace β by $-\alpha$ (and δ by $-\gamma$ additionally for Theorem 2), and make use of the relation (4), then Theorems 1 to 2 correspond to the known integral inequalities due to Dahmani, Mechouar and Brahami [3, pp. 39-42, Theorems 3.1 to 3.2] involving the Riemann-Liouville type fractional integral operator defined by (4).

Finally, by suitably choosing the function $p(t)$, one can further easily obtain additional integral inequalities involving the Riemann-Liouville and Erdélyi-Kober type fractional integral operators from the main results.

Acknowledgements

The authors are thankful to the referees for a very careful reading and valuable suggestions.

References

1. Anastassiou, G.A.: "Advances on Fractional Inequalities", Springer Briefs in Mathematics, Springer, New York, 2011.
2. Belarbi, S. and Dahmani, Z.: "On some new fractional integral inequalities". J. Inequal. Pure Appl. Math., Vol. 10, No. 3 (2009) 86, 5 (electronic).
3. Dahmani, Z., Mechouar, O. and Brahami, S.: "Certain inequalities related to the Chebyshev's functional involving a Riemann-Liouville operator". Bull. Math. Anal. Appl., Vol. 3, No. 4 (2011) 38-44.
4. Denton, Z. and Vatsala, A.S.: "Monotonic iterative technique for finit system of nonlinear Riemann-Liouville fractional differential equations". Opuscula Mathematica, Vol. 31, No. 3 (2011) 327-339.
5. Dragomir, S.S.: "Some integral inequalities of Gruss type". Indian J. Pure Appl. Math., Vol. 31. No. 4 (2000), 397-415.
6. Kalla, S.L. and Rao, Alka: "On Grüss type inequality for hypergeometric fractional integrals". Le Matematiche, Vol. 66, No. 1 (2011) 57-64.
7. Lakshmikantham, V. and Vatsala, A.S.: "Theory of fractional differential inequalities and applications". Commun. Appl. Anal., Vol. 11 (2007), 395-402.
8. Öğünmez, H. and Özkan, U.M.: "Fractional quantum integral inequalities". J. Inequal. Appl., Vol. 2011, Article ID 787939, 7 pp.
9. Purohit, S.D. and Raina, R.K.: "Chebyshev type inequalities for the saigo fractional integrals and their q-analogues". J. Math. Inequal., Vol. 7, No. 2 (2013) 239-249.
10. Ramírez, J.D. and Vatsala, A.S.: "Monotonic iterative technique for fractional differential equations with periodic boundary conditions". Opuscula Mathematica, Vol. 29, No. 3 (2009) 289-304.

11. Sulaiman,W.T.: "Some new fractional integral inequalities". J. Math. Anal., Vol. 2, No. 2 (2011) 23-28.
12. Saigo, M.: "A remark on integral operators involving the Gauss hypergeometric functions". Math. Rep. Kyushu Univ., Vol. 11 (1978), 135-143.
13. Kiryakova, V.: "Generalized Fractional Calculus and Applications" (Pitman Res. Notes Math. Ser. 301), Longman Scientific & Technical, Harlow, 1994.
14. Raina, R.K.: "Solution of Abel-type integral equation involving the Appell hypergeometric function". Integral Transforms Spec. Funct., Vol. 21, No. 7 (2010) 515-522.
15. Chebyshev, P.L.: "Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites". Proc. Math. Soc. Charkov, Vol. 2 (1882) 93-98.

Recibido el 26 de Septiembre de 2013
En forma revisada el 21 de Abril de 2014