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Some Approximations and Inequalities for Arc Tan and Ln

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RESUMEN

La aplicación de la aproximación Padé a $G = {}_2F_1(1, \alpha; \gamma; -k(z))$ *conduce a mejores aproximaciones y desigualdades para G y también para la función* $F = {}_2F_1(1, a; c; -z)$, *las cuales son considerablemente mejores que aquellas que se obtienen directamente de F. Se ilustran los resultados para arc tan z y ln (1 + z).*

SUMMARY

The application of the Padé approximations when applied to $G = {}_2F_1(1\alpha; \gamma; -k(z))$ *leads to powerful approximations and inequalities for G and so also for* $F = {}_2F_1(1, a; c; -z)$, *which are considerably improved over those obtained directly from F. The results are illustrated for arc tan z and ln (1 + z).*

1. SUMMARY AND INTRODUCTION

In previous works [1, 2] we showed that the first subdiagonal and main diagonal Padé approximations for $F = {}_2F_1(1, a; c; -z)$ give lower and upper bounds, respectively for F when $z > 0$ and suitable restrictions are placed on a and c. Now for certain values of a and c, there are available quadratic transformation formulas such that F is simpley related to the form $G = {}_2F_1(1, \alpha; \gamma; -k(z))$ where k (z) is a function of z. The convergence properties of G are vastly superior to those for F since $|k(z)| \leq |z|$, z suitably restricted. Thus application of the Padé approximations noted above when applied to G leads to powerful approximations and inequalities for G and so also for F which are noticeably improved over those obtained directly from F. The results are illustrated for arc tan z and ln $(1 + z)$.

2. APRRoXIMATIONS AND INEQUALITIES FOR ${}_2F_1(1, a; c; -z)$

Consider

$$F(z) = {}_2F_1(1, a; c; -z). \tag{1}$$

Let

$$F(z) = \{A_n(z)/B_n(z)\} + U_n(z), \tag{2}$$

where $\{A_n(z)/B_n(z)\}$ is the first subdiagonal Padé approximation for F(z) and $U_n(z)$ is the error. Again, let

$$F(z) = \{G_n(z)/D_n(z)\} + V_n(z), \tag{3}$$

where $\{G_n(z)/D_n(z)\}$ is the main diagonal Padé approximation for F(z) and $V_n(z)$ is the error. Numerous details concerning the polynomials in these Padé approximations and the errors are detailed in my volumes [1, 2]. For the most part these data will not be repeated here. For our present purposes, we need only the inequality

$$\{A_n(z)/B_n(z)\} < F(z) < \{C_n(z)/D_n(z)\},$$

$$z > 0, \ c \geq 1, \ c > a > 0, \tag{4}$$

with equality if z = 0 or a = 0 except that if z = 0, exclude the left hand inequality unless n > 0.

Other conditions for the validity of (4) as is or of (4) with inequality signs reversed are given in the sources cited.

In particular, we have

$$B_0(z) = C_0(z) = D_0(z) = 1, \quad A_0(z) = 0,$$

$$A_1(z) = 1, \quad B_1(z) = 1 + az/c$$

$$A_2(z) = 1 + \left(\frac{2(a+1)}{c+2} - \frac{a}{c}\right)z,$$

$$B_2(z) = 1 + \frac{2(a+1)z}{c+2} + \frac{(a+1)az^2}{(c+2)(c+1)},$$

$$C_1(z) = 1 + \frac{(c-a)z}{c(c+1)}, \quad D_1(z) = 1 + \frac{(a+1)z}{(c+1)},$$

$$C_2(z) = 1 + \left(\frac{2(a+2)}{(c+3)} - \frac{a}{c}\right)z + \left(\frac{(a+1)(a+2)}{(a+2)(a+3)} - \frac{2a(a+2)}{c(c+3)} + \frac{a(a+1)}{c(c+1)}\right)z^2$$

$$D_2(z) = 1 + \frac{2(a+2)z}{(c+3)} + \frac{(a+2)(a+1)z^2}{(c+3)(c+2)},$$

and further approximants can be found after the manner in the cited references.

3. APPROXIMATIONS AND INEQUALITIES FOR ARC TANZ

We first derive a quadratic transformation formula for

$$\arctan z = z \ F\left(\frac{1}{2}1; 3/2; -z^2\right). \tag{6}$$

Consider [1, Vol. 1, p. 92, Eq. (1)] or [2, p. 270, Eq. (1)] with

$$z \text{ replaced by} -\left(\frac{W-1}{z}\right) \text{ where } W = (1+z^2)^{\frac{1}{2}}. \tag{7}$$

Then

$$ {}_2F_1\left(a, a+\frac{1}{2}; c; -z^2\right) = \left(\frac{2}{1+W}\right)^{2a} {}_2F_1\left(2a, 2a-c+1; c; \frac{1-W}{1+W}\right) \tag{8}$$

$$= W^{-2a} \, {}_2F_1\left(2a, 2c-2a-1; c; \frac{W-1}{2W}\right);$$

in virtue of a Kummer transformation formula. Now put $a = \frac{1}{2}$. The ${}_2F_1$ on the right can be expressed as 1 plus another ${}_2F_1$ to which we apply a Kummer transformation formula. We so obtain the desired form

$$ {}_2F_1\left(1, \frac{1}{2}; c; -z^2\right) = \frac{1}{W} + \frac{(W-1)^2(c-1)}{z^2 Wc} \, {}_2F_1\left(1, 2-c; c+1; -\left(\frac{W-1}{z}\right)^2\right),$$

$$|\arg(1+z^2)| < \pi, \quad z^2 \neq -1, \tag{9}$$

which with c = 3/2 gives z^{-1} arc tan z. Note that $\{(W-1)/z\}$ vanishes if $z \to 0$ and increases monotonically to 1 as $z \to \infty$. The Padé approximations for the ${}_2F_1$ on the right hand side of (9) lead to powerful approximations for the ${}_2F_1$ on the left and these approximations are vastly superior to the corresponding Padé approximations for the ${}_2F_1$ on the left hand side of (9). Application of the results in Section 2 to (9) with c = 3/2 yields the following approximations and inequalities for arc tan z.

$$z^{-1} L_n \leq z^{-1} \arctan z \leq z^{-1} R_n, \quad z \geq 0, \tag{10}$$

$$L_1 = \frac{z}{w} + \frac{10v^2 z}{3w(v^2+5},$$

$$R_1 = \frac{z}{w} + \frac{2v^2 z(8v^2+35)}{3w(15v^2+35)}, \tag{11}$$

$$v = \frac{w-1}{z} = \frac{z}{w+1}, \quad w = (1+z^2)^{\frac{1}{2}}, \tag{12}$$

$$L_2 = \frac{z}{w} + \frac{14v^2 z(7v^2+15)}{15w(v^4+14v^2+21)},$$

$$R_2 = \frac{z}{w} + \frac{2v^2 z(64v^4+819v^2+1155)}{105w(5v^4+30v^2+33)} \tag{13}$$

The inequalities become exact as $z \to 0$. If $z = 1$, we find

$$L_1 = \frac{41(2)^{\frac{1}{2}} - 25}{42} = 0.78530\ 372 < \pi/4$$

$$= 0.78539\ 81634$$

$$< R_1 = \frac{811(2)^{\frac{1}{2}} - 605}{690} = 0.78540\ 174,$$

$$L_2 = \frac{325(2)^{\frac{1}{2}} - 224}{300} = 0.78539\ 80259 < \pi/4$$

$$< R_2 = \frac{53301(2)^{\frac{1}{2}} - 38104}{47460} = 0.78539\ 81687.$$

If $z \to +\infty$, $z/w \to 1$ and $v \to 1$. So

$$L_1 = 14/9 = 1.55556 < \pi/2$$

$$= 1.57079\ 633 < 118/75 = 1.57333,$$

$$L_2 = 212/135 = 1.57037\ 037 < \pi/2 < R_2$$

$$= \frac{2804}{1785} = 1.57086\ 835.$$

4. APPROXIMATIONS AND INEQUALITIES FOR $\ln(1+z)$

We first get a quadratic transformation formula for

$$\ln(1+z) = z\ {}_2F_1(1,1;2;-z). \tag{14}$$

Consider [1, Vol. 1, p. 93, Eq. (8)] or [2, p. 271, Eq. (8)] with $a = 1$ and z replaced by $-z/2$. Apply a Kummer transformation formula to the $_2F_1$ on the right hand side of the resulting equation, and so obtain

$${}_2F_1(1,b;2;-z) = \frac{(z+2)}{(2z+2)^{\frac{1}{2}(b+1)}}$$

$${}_2F_1\left(\frac{b+1}{2}, \frac{3-b}{2}; \frac{3}{2}; \frac{z^2}{4(z+1)}\right),$$

$$|\arg(1+z)| < \pi, \quad \left|\arg \frac{(z+2)^2}{(z+1}\right| < \pi, \tag{15}$$

and with $b = 1$, we have a formula for $z^{-1}\ln(1+z)$. Application of the results in Section 2 yields the following approximations and inequalities.

$$L_n \leq \ln(1+z) \leq R_n, \quad z > 0, \tag{16}$$

$$L_1 = \frac{3z(z+2)}{2(z+1)(2y+3)}, R_1 = \frac{z(z+2)(2y+15)}{6(z+1)(4y+5)}, \tag{17}$$

$$y = z^2/4(z+1), \tag{18}$$

$$L_2 = \frac{5z(z+2)(10y+21)}{6(z+1)(8y^2+40y+35)},$$

$$R_2 = \frac{z(z+2)(8y^2+210y+315)}{30(z+1)(8y^2+28y+21)} \tag{19}$$

These inequalities become exact as $z \to 0$.

In illustration, with $z = 2$, we have

$$L_1 = \frac{12}{11} = 1.09091 < \ln 3 = 1.09861\ 2289 <$$

$$R_1 = \frac{188}{171} = 1.09941$$

$$L_2 = \frac{1460}{1329} = 1.09857\ 0354 < \ln 3 < R_2$$

$$= \frac{13892}{12645} = 1.09861\ 6054.$$

We now get another quadratic transformation formula for the logarithm which is even more powerful than (15). Consider [1, Vol. 1, p. 93, Eq. (7)] or [2, p. 271, Eq. (7)] with z replaced by -z. Apply a Kummer transformation formula to the right hand side of this equation. Then

$${}_2F_1(a,b;2a;-z) = u^{-b}\ {}_2F_1$$

$$\left(b, 2a-b; a+\frac{1}{2}; -\frac{(1-u)^2}{4u}\right),$$

$$u = (1+z)^{\frac{1}{2}}, \quad |\arg(1+z)| < \pi,$$

$$|\arg(1+u)^2/u| < \pi, \tag{20}$$

and if $a = b = 1$, we have an expression for $z^{-1}\ln(1+z)$ Application of the results in Section 2 yields the following approximations and inequalities.

$$L_n \leq \ln(1+z) \leq R_n, \quad z > 0, \tag{21}$$

$$L_1 = \frac{6z}{u^2+4u+1}, \quad R_1 = \frac{z(u^2+28u+1)}{6u(u^2+3u+1)}, \tag{22}$$

$$L_2 = \frac{5z(5u^2+32u+5)}{3(u^4+16u^3+36u^2+16u+1)},$$

$$R_2 = \frac{z(u^4+101u^3+426u^2+101u+1)}{15u(u^4+10u^3+20u^2+10u+1)} \tag{23}$$

These inequalities become exact as $z \to 0$.

In illustration, with $z = 2$, we get

$$L_1 = \frac{3(3^{\frac{1}{2}} - 1)}{2} = 1.09807\ 6211 < \ln 3 = 1.09861\ 2289$$

$$< R_1 = \frac{4(59(3)^{\frac{1}{2}} - 75)}{99} = 1.09862\ 6168,$$

$$L_2 = \frac{20(312(3)^{\frac{1}{2}} - 473}{1227}$$

$$= 1.09861\ 2094 < \ln 3$$

$$< R_2 = \frac{4(1042(3)^{\frac{1}{2}} - 1743)}{225} = 1.09861\ 2293 .$$

Shafer [3] initiated the technique of quadratic approximation and in illustration shows that

$$\arctan x = \frac{8x}{3 + (25 + 80x^2/3)^{\frac{1}{2}}} + \varepsilon(x), x > 0. \qquad (24)$$

It can be proved that $\varepsilon'(x) > 0$ for $x > 0$ and since $\varepsilon(x) = 0$ for $x = 0$, (24) gives a left hand inequality for arc tan x when $x \geq 0$, $\varepsilon(x)$ increases monotonically for $0 \leq x < \infty$, and $\varepsilon(x) \to 1.5492$ as $x \to \infty$. For a comparable situation note that when $x \to \infty$, then from (11), $L_1 \to 1.5556$. Since the true value is $\pi/2$, we see that the left hand inequality (10) with $n = 1$ is superior.

REFERENCES

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