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ASYMPTOTIC FORMULAE FOR THE PERCENTILES AND C.D.F. OF HOTELLING'S TRACE UNDER VIOLATIONS

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ABSTRACT

Asymptotic expansions for the percentile and distribution of the statistic $T = mTr\, \underset{\sim}{S}_1\underset{\sim}{S}_2^{-1}$ have been studied where $m\underset{\sim}{S}_1$ has non-central Wishart distribution, $W(m,p,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $n\underset{\sim}{S}_2$ has central Wishart distribution, $W(n,p,\underset{\sim}{\Sigma}_2)$. Let $(\underset{\sim}{F}) = Tr\, \underset{\sim}{F} = Tr(\underset{\sim}{B}^{-1}\underset{\sim}{A}-\underset{\sim}{I})$, where $\underset{\sim}{\Sigma}_1^{-1} = \underset{\sim}{B}$ and $\underset{\sim}{\Sigma}_2^{-1} = \underset{\sim}{A}$. Chattopadhyay [3] obtained asymptotic expansions for the percentile and c.d.f. of T up to terms of order n^{-1} neglecting $f_{ij}f_{kl}$ terms where f_{ij} is the (i,j)th element of F. In this paper, the work of Chattopadhyay is extended to include the $f_{ij}f_{kl}$ terms neglected by him which improves the expansion a great deal as is shown in the tabulation of powers provided . The work is useful for robustness studies.

RESUMEN

El objeto de este trabajo es extender lo estudiado por Chattopadhyay sobre expansiones asintóticas para el percentil y la distribución del estadístico

$$T = mTr\, \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}.$$

Para el estadístico $T = mTr\, \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ donde $m\underset{\sim}{S}_1$ tiene distribución Wishart no-central $W(m,p,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ y $n\underset{\sim}{S}_2$ tiene distribución Wishart central $W(n,p,\underset{\sim}{\Sigma}_2)$ con $(\underset{\sim}{F}) = Tr\, \underset{\sim}{F} = Tr(\underset{\sim}{B}^{-1}\underset{\sim}{A}-\underset{\sim}{I})$, donde $\underset{\sim}{\Sigma}_1^{-1} = \underset{\sim}{B}$ y $\underset{\sim}{\Sigma}_2^{-1} = \underset{\sim}{A}$, Chattopadhyay obtuvo expansiones asintóticas para el percentil y la función de densidad acumulada de T sobre los términos de orden n^{-1}, despreciando los términos $f_{ij}f_{kl}$ donde f_{ij} es el elemento (i,j)-ésimo de F.

La extensión del trabajo de Chattopadhyay consiste en la inclusión de los términos $f_{ij}f_{kl}$ en el estudio de las expansiones asintóticas para el percentil y la distribución acumulada de T lo cual mejora la expansión.

1. INTRODUCTION

Asymptotic expansions for the distribution and percentile of the statistic $T = m\, Tr\, \underset{\sim}{S}_1 \underset{\sim}{S}_2^{-1}$ have been obtained by the authors [13] up to terms of the order $1/n^2$ where $m\underset{\sim}{S}_1$ and $n\underset{\sim}{S}_2$ have independent central Wishart distributions with m degrees of freedom and covariance matrix $\underset{\sim}{\Sigma}_1, W(m,p,\underset{\sim}{\Sigma}_1)$, and with n degrees of freedom and covariance matrix $\underset{\sim}{\Sigma}_2$, $W(n,p,\underset{\sim}{\Sigma}_2)$, respectively. Further, denoting the non-centrality by $(\underset{\sim}{F}) = Tr\underset{\sim}{F} = Tr(\underset{\sim}{B}^{-1}\underset{\sim}{A}-\underset{\sim}{I})$, where $\underset{\sim}{\Sigma}_1^{-1} = \underset{\sim}{B}$ and $\underset{\sim}{\Sigma}_2^{-1} = \underset{\sim}{A}$, terms involving $f_{ij}f_{kl}$ where f_{ij} is the (i,j)-th element of $\underset{\sim}{F}$, which were previously neglected by Chattopadhyay and Pillai [2], were also included. In this paper again we extend the work of Chattopadhyay [3] who derived an asymptotic expansion up to terms of order $1/n$, neglecting $f_{ij}f_{kl}$ terms for c.d.f. and percentile of the trace statistic when $m\underset{\sim}{S}_1$ has non-central Wishart distribution with m degrees of freedom, covariance matrix $\underset{\sim}{\Sigma}_1$ and non-centrality matrix $\underset{\sim}{\Omega}$, $W(m,p,\underset{\sim}{\Sigma}_1,\underset{\sim}{\Omega})$ and $n\underset{\sim}{S}_2$ distributed central Wishart $W(n,p,\underset{\sim}{\Sigma}_2)$. The extension in this case is to include the $f_{ij}f_{kl}$ terms neglected by him. It may be noted that these terms were found to improve the expansion as we have shown in [13]. The results are helpful for the study of the violation of a) the assumption of a common covariance matrix in the MANOVA test based on the trace statistic and b) normality assumption in testing $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$. When $\underset{\sim}{\Sigma}_1 = \underset{\sim}{\Sigma}_2$, asymptotic expansions of the non-central c.d.f. have been studied by several authors [6] and [17].

2. THE METHOD OF ASYMPTOTIC EXPANSION

The notations in this paper generally follow those of [13] and other papers referred to earlier [2], [5], but additional notations will be introduced here. The method herein is also to obtain an asymptotic expansion for the percentile of T first, and use it to derive an expansion for the c.d.f. of T, where T may be defined as follows:

Let $\underset{\sim}{Z} = (\underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_m)$ be a $p x m$ matrix of independently distributed columns vectors where $\underset{\sim}{z}_i$ has the density $N(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_1)$, $i = 1, \ldots, m$. Then we may define $T = Tr\, \underset{\sim}{S}_2^{-1} \underset{\sim}{Z}\underset{\sim}{Z}' = \sum_{i=1}^{m} \underset{\sim}{z}_i' \underset{\sim}{S}_2^{-1} \underset{\sim}{z}_i$ where $n\underset{\sim}{S}_2$ is distributed $W(n, p, \underset{\sim}{\Sigma}_2)$ independently of $\underset{\sim}{Z}$.

Now if $\underset{\sim}{S}_2^{-1}$ is replaced by $\underset{\sim}{B}$ in T, then $Tr\, \underset{\sim}{B}\underset{\sim}{Z}\underset{\sim}{Z}'$ is distributed as a non-central chi-square with mp degrees of freedom and non-centrality parameter ω^2, where,

$$\omega^2 = Tr\, \underset{\sim}{B}\underset{\sim}{M}\underset{\sim}{M}' = Tr\, \underset{\sim}{\Omega}$$

$$\underset{\sim}{M} = \{\mu_1, \ldots, \mu_m\} \neq 0, \; \rho = mp/2 \quad , \quad \text{i.e.,}$$

we may note that

$$Pr\{Tr\, \underset{\sim}{B}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = e^{-\omega^2/2} \sum_{J=0}^{\infty} \frac{(\omega^2/2)^J}{J!\, 2^{\rho+J}\, \Gamma(\rho+J)} \int_0^{\theta} x^{\rho+J-1} e^{-x/2}\, dx$$

$$= G_{mp}(\theta, \omega^2),$$

where, $G_{mp}(\theta, \omega^2)$ is the c.d.f. of non-central chi-square with mp degrees of freedom and the non-centrality parameter ω^2.
Let

$$G(\theta) = Pr\{Tr\, \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}.$$

As a first approximation, for large n we may replace $\underset{\sim}{A}^{-1}$ by $\underset{\sim}{S}_2$ in $G(\theta)$, and consider

$$G(\theta) = Pr\{Tr\, \underset{\sim}{S}_2^{-1}\, \underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}.$$

Furthermore, as suggested by Ito [5], obtain a function $h(\underset{\sim}{S}_2)$ of the elements of $\underset{\sim}{S}_2$ and n large enough such that

$$G(\theta) = Pr\{Tr\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{S}_2)\} \tag{2.1}$$

and then write $h(\underset{\sim}{S}_2)$ as a series with the first term being a linear function of independent non-central chi-square variables and terms of decreasing order of magnitude, (see Equation (2.8)).

Now (2.1) can be written such that,

$$G(\theta) = E_{\underset{\sim}{S}_2}\{Pr[Tr\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{S}_2)|\underset{\sim}{S}_2]\}, \tag{2.2}$$

By using Taylor expansion it is possible to expand $Pr[Tr\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{S}_2)|\underset{\sim}{S}_2]$ about an origin $(\sigma_{11},\ldots,\sigma_{pp},\sigma_{12},\ldots,\sigma_{p-1,p})$, where

$$\underset{\sim}{A}^{-1} = (\sigma_{ij}), \qquad (i,j) = (1,\ldots,p). \tag{2.3}$$

Thus,

$$Pr\{Tr\ \underset{\sim}{S}_2^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{S}_2)|\underset{\sim}{S}_2\} = \{exp[Tr(\underset{\sim}{S}_2-\underset{\sim}{A}^{-1})\underset{\sim}{\partial}]Pr\{Tr\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{A}^{-1})\}, \tag{2.4}$$

where

$$\underset{\sim}{\partial}(pxp) = \left(\frac{1}{2}(1+\delta_{ij})\frac{\partial}{\partial\sigma_{ij}}\right), \tag{2.5}$$

and δ_{ij} is the Kronecker delta. Hence

$$G(\theta) = O \cdot Pr\{Tr\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq h(\underset{\sim}{A}^{-1})\} \tag{2.6}$$

where

$$\Theta = exp[-Tr\ \underset{\sim}{A}^{-1}\underset{\sim}{\partial}]\left|\underset{\sim}{I} - \frac{2}{n}\underset{\sim}{A}^{-1}\underset{\sim}{\partial}\right|^{-(n/2)}$$

$$= 1 + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} + O(n^{-2}), \qquad (2.7)$$

where $\sum$ denotes the summation over all sufixes $r,s,\ldots,$ each of which range from 1 to p. Expanding $h(\underset{\sim}{S}_2)$ around θ to get

$$h(\underset{\sim}{S}_2) = \theta + h_1(\underset{\sim}{S}_2) + h_2(\underset{\sim}{S}_2) + \ldots, \qquad (2.8)$$

where $h_s(\underset{\sim}{S}_2)$ is $O(n^{-s})$. From equations (2.6) , (2.7) and expanding $h(\underset{\sim}{S}_2)$ around $h(\underset{\sim}{A}^{-1})$ we can get,

$$G(\theta) = \left[1 + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur} + O(n^{-2})\right]\left[1 + h_1(\underset{\sim}{A}^{-1})D + O(n^{-2})\right]$$

$$\cdot\ Pr\{Tr\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\},$$

where $D = \frac{\partial}{\partial\theta}$, and by equating terms of successive order we get

$$\left[h_1(\underset{\sim}{A}^{-1})D + \frac{1}{n}\sum \sigma_{rs}\sigma_{tu}\partial_{st}\partial_{ur}\right]Pr\{Tr\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = 0 \qquad (2.9)$$

For the purpose of evaluating $\partial_{st}\partial_{ur}\ Pr\{Tr\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}$ we will use the perturbation technique [8].

Let

$$J = Pr\{Tr(\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1}\ \underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}\ , \qquad (2.10)$$

where $\underset{\sim}{\varepsilon}(pxp)$ is a symmetric matrix sufficiently close to $\underset{\sim}{O}(pxp)$. Following [3], [5] and [19], we get

$$J = |\underset{\sim}{I} - \underset{\sim}{X}\Delta|^{-(m/2)}\ Exp[-\omega^2/2] \cdot Exp\{(1/2)E\ Tr(\underset{\sim}{I} - \underset{\sim}{X}\Delta)^{-1}\underset{\sim}{\Omega}\}G_{mp}(\theta,0)\ , \tag{2.11}$$

where $\Delta = E-1$, $E^r G_{mp}(\theta,\omega^2) = G_{mp+2r}(\theta,\omega^2)$ and

$$\underset{\sim}{X} = \underset{\sim}{B}^{-1}(\underset{\sim}{A}^{-1} + \underset{\sim}{\varepsilon})^{-1} - \underset{\sim}{I} = (\underset{\sim}{B}^{-1}\underset{\sim}{A} - \underset{\sim}{I}) - \sum \varepsilon_{rs}(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})$$

$$+ \sum \varepsilon_{rs}\varepsilon_{tu}(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\underset{\sim}{A}) - \ldots,$$

where $\underset{\sim}{A}_{rs}^{-1}$ is the pxp matrix with (i,j)-th element $(1/2)(\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si})$. Also by Taylor's theorem J can be expressed in the following form

$$J = \{1 + \sum \varepsilon_{rs}\partial_{rs} + \frac{1}{2!}\sum \varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu} + \ldots\}\ Pr(Tr\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}. \tag{2.12}$$

Now if $\underset{\sim}{B}^{-1}\underset{\sim}{A} - \underset{\sim}{I} = \underset{\sim}{F}$ such that $|ch_i(\underset{\sim}{F})| < 1$, $i = 1,\ldots,p$, using the notations

$$Tr(\underset{\sim}{A}_{rs}^{-1}\ \underset{\sim}{A}) = (rs)$$

$$Tr(\underset{\sim}{A}_{rs}^{-1}\ \underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\ \underset{\sim}{A}) = (rs|tu)$$

$$Tr(\underset{\sim}{F})(\underset{\sim}{A}_{rs}^{-1}\ \underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\ \underset{\sim}{A}) = (\underset{\sim}{F}|rs|tu)$$

$$Tr(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{rs}^{-1}\ \underset{\sim}{A})(\underset{\sim}{B}^{-1}\underset{\sim}{A})(\underset{\sim}{A}_{tu}^{-1}\ \underset{\sim}{A}) = (\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu)$$

$$Tr(\underset{\sim}{F}^2) = (\underset{\sim}{F}^2),\ Tr(\underset{\sim}{F}^s) = (\underset{\sim}{F}^2), \ldots\ etc.,$$

and substituting $\underset{\sim}{X}$ in (2.11) , then term by term comparison between the two expansions of J, (2.12) and (2.11) after substituting $\underset{\sim}{X}$ will give the second derivative $\partial_{st}\partial_{ur}Pr\{\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}$ which can be written in the following form,

$$\partial_{st}\partial_{ur} Pr\{\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = 2 \sum_{J=0}^{6} A_J' E^J G_{mp}(\theta,\omega^2), \tag{2.13}$$

where

$$\begin{aligned} A_0^* = {} & (\tfrac{m}{4})\{-2(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) + (\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu) \\ & + 2(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) - 2(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu) \\ & - 2(\underset{\sim}{F}|\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|tu)\} + (\tfrac{m^2}{8})\{(\underset{\sim}{I}+\underset{\sim}{F}|rs)(\underset{\sim}{I}+\underset{\sim}{F}|tu) \\ & + 2(\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) - (\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs|\underset{\sim}{I}+\underset{\sim}{F}|tu) - 2(\underset{\sim}{F})(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) \\ & - 2(\underset{\sim}{I}+\underset{\sim}{F}|rs)(\underset{\sim}{F}|\underset{\sim}{I}+\underset{\sim}{F}|tu) - (\underset{\sim}{F}^2)(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu)\} \\ & - (\tfrac{m^3}{16})\{(\underset{\sim}{F})(\underset{\sim}{I}+\underset{\sim}{F}|rs)(\underset{\sim}{I}+\underset{\sim}{F}|tu) + (\underset{\sim}{F})^2(\underset{\sim}{I}+\underset{\sim}{F}|rs|tu) , \end{aligned}$$

other A_j^* coefficients are available in Appendix C of [14].

3. AN ASYMPTOTIC EXPANSION FOR THE PERCENTILE OF $T = Tr\, \underset{\sim}{S}_2^{-1}\, \underset{\sim}{Z}\underset{\sim}{Z}'$.

Recalling that $G_{mp}(\theta,\omega^2)$ is the c.d.f. of the non-central chi-square distribution with mp degrees of freedom and non-centrality parameter ω^2 we may note that

$$E^r G_{mp}(\theta,\omega^2) = G_{mp+2r}(\theta,\omega^2) .$$

Hence, it is possible to rewrite (2.13) in the following form,

$$\partial_{st}\partial_{ur} Pr\{\underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = 2 \sum_{J=0}^{6} A_J^* G_{mp+2J}(\theta,\omega^2), \tag{3.1}$$

Again, we note

$$\sum_{r,s,t,u} \sigma_{st}\sigma_{ur}(rs|tu) = \frac{1}{2}p(p+1),$$

$$\sum \sigma_{rs}(rs) = p, \quad \sum \sigma_{st}\sigma_{ur}(rs)(tu) = p,$$

$$u = \sum \sigma_{st}\sigma_{ur}(\underset{\sim}{F}|rs)(tu) = (\underset{\sim}{F}),$$

$$v = \sum \sigma_{st}\sigma_{ur}(\underset{\sim}{F}|\underset{\sim}{F}|rs|tu) = \frac{1}{2}(\underset{\sim}{F}^2)(p+1),$$

$$w = \sum \sigma_{st}\sigma_{ur}(\underset{\sim}{\Omega}|rs|\underset{\sim}{F}|tu) = \frac{1}{2}[(\underset{\sim}{\Omega})(\underset{\sim}{F}) + (\underset{\sim}{\Omega}\underset{\sim}{F})], \ldots \text{ etc.},$$

As a check for the above relationships, let $\underset{\sim}{F}(pxp) = \underset{\sim}{I}(pxp)$ and $\Omega(pxp) = I(pxp)$, then u should equal p which is the value of $\sum \sigma_{st}\sigma_{ur}(rs)(tu)$. Similarly v and w will be reduced to $\sum \sigma_{st}\sigma_{ur}(rs|tu)$ equal to $1/2p(p+1)$. With the help of the above relationships, it is possible to evaluate A_J^*'s, $J = 0,\ldots,6$, after summing over all subscripts, r, s, t, u.

Now by using (2.9) and the results above we get

$$-h_1(\underset{\sim}{A}^{-1})D\, Pr\{Tr\, \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}$$

$$= \frac{1}{4n}\sum_{J=0}^{4} a_J(m,p)G_{mp+2J}(\theta,\omega^2)$$

$$+ \frac{1}{n}\sum_{J=0}^{6} A_J(m,p)G_{mp+2J}(\theta,\omega^2),$$

where

$$a_0 = mp(m-p-1)$$

$$a_1 = -2m(mp-\omega^2),$$

$$a_2 = mp(m+p+1)-2(2m+p+1)\omega^2 + tr\, \underset{\sim}{\Omega}^2,$$

$$a_3 = 2\{(m+p+1)\omega^2 - tr\, \underset{\sim}{\Omega}^2\},$$

$$a_4 = tr\, \underset{\sim}{\Omega}^2,$$

and $A_J = \sum_{r,s,t,u} \sigma_{rs}\sigma_{tu} A'_J$, $J = 0,\ldots,6,$

and the above can be simplified after tedious algebra and it can be written in the following expression,

$$A_J = \sum_{k=0}^{3} m^k A_{Jk}, \qquad J = 0,1,\ldots,6,$$

where

$$A_{00} = 0,$$

$$A_{01} = -(1/4)\left[2\left[(\tilde{F})^2+(\tilde{F}^3)\right](p+1)+\left[(\tilde{F})^2+(\tilde{F}^2)+2(\tilde{F}^2)(\tilde{F})+2(\tilde{F}^3)\right]\right],$$

$$A_{02} = (1/8)\left[\left[(\tilde{F})p-2(\tilde{F})^2-3(\tilde{F})(\tilde{F}^2)-(\tilde{F}^2)^2\right](p+1)\right.$$
$$\left. -\left[(\tilde{F})^3+(\tilde{F})(\tilde{F}^2)+6(\tilde{F}^2)+4(\tilde{F}^3)\right]\right],$$

$$A_{03} = -(1/8)\left[(1/2)\left[(\tilde{F})^2p+(\tilde{F})^3\right](p+1)+(\tilde{F})p+2(\tilde{F})^2+(\tilde{F})(\tilde{F}^2)\right],$$

$$A_{10} = \{-3(\tilde{\Omega})\left[(\tilde{F}^2)+(\tilde{F}^3)\right] - 3\left[(\tilde{\Omega}\tilde{F}^2) + (\tilde{\Omega}\tilde{F}^3)\right]\},$$

$$A_{11} = \{\left[24(\tilde{F}^2)+18(\tilde{F}^3)+(\tilde{\Omega})(\tilde{F}^3) + \left[(3/2)p+3(\tilde{F}^2)+3(\tilde{F}^3)\right].(\tilde{\Omega}\tilde{F})\right.$$
$$+ \left[-(3/2)p + 3(\tilde{F})+3(\tilde{F}^2)\right](\tilde{\Omega}\tilde{F}^2) + \left[3p+3(\tilde{F})\right](\tilde{\Omega}\tilde{F}^3)\left.\right](p+1)$$
$$+ \left[(9/2)(\tilde{F}^2)^2+(45/2)(\tilde{F})(\tilde{F}^2)+19(\tilde{F})(\tilde{F}^3)\right](\tilde{\Omega}) + \left[(3/2)(\tilde{F})^2\right.$$
$$\left.+ (9/2)(\tilde{F}^2)+3(\tilde{F})(\tilde{F}^2)+4(\tilde{F}^3)\right](\tilde{\Omega}\tilde{F})+\left[6+21(\tilde{F})+(3/2)(\tilde{F})^2\right.$$
$$\left.+ (9/2)(\tilde{F}^2)\right](\tilde{\Omega}\tilde{F}^2)+\left[18(\tilde{F})+54\right](\tilde{\Omega}\tilde{F}^3)+36(\tilde{\Omega}\tilde{F}^4)+12(\tilde{F})^2$$
$$+ 12(\tilde{F}^2)+18(\tilde{F})(\tilde{F}^2)+18(\tilde{F}^3)\}/(12),$$

$$A_{12} = \{\left[-(\tilde{F})p+3(\tilde{F}^2)p+8(\tilde{F})^2+9(\tilde{F})(\tilde{F}^2)+\left[(\tilde{F})(\tilde{F}^2)-(\tilde{F}^2)\right.\right.$$
$$\left.+ (\tilde{F})^2\right](\tilde{\Omega})/2+\left[-(\tilde{F})p+2(\tilde{F})^2+3(\tilde{F})(\tilde{F}^2)+(\tilde{F}^2)\right](\tilde{\Omega}\tilde{F})/2$$
$$+ \left[(\tilde{F})p+(\tilde{F})^2\right](\tilde{\Omega}\tilde{F}^2)\left.\right].(p+1)+\left[-2(\tilde{F})+(\tilde{F}^2)-(1/2).\right.$$

$$(\dot{\underset{\sim}{F}})(\underset{\sim}{F}^2)+(\underset{\sim}{F}^3)+2(\underset{\sim}{F}^2)(\underset{\sim}{F})^2+(\underset{\sim}{F})^3](\underset{\sim}{\Omega})+[-p+2(\underset{\sim}{F})$$

$$+\ (9/2)(\underset{\sim}{F}^2)+2(\underset{\sim}{F})^2+(\underset{\sim}{F})(\underset{\sim}{F}^2)+2(\underset{\sim}{F}^3)+(\underset{\sim}{F})^3/2](\underset{\sim}{\Omega}\underset{\sim}{F})$$

$$+\ [p+12(\underset{\sim}{F})+(3/2)(\underset{\sim}{F})^2+2(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F}^2)+6(\underset{\sim}{\Omega}\underset{\sim}{F}^3)(\underset{\sim}{F})$$

$$+\ 4(\underset{\sim}{F})+20(\underset{\sim}{F}^2)+12(\underset{\sim}{F}^3)\}/(8),$$

$$A_{13} = \{[9(\underset{\sim}{F})^2p+(\underset{\sim}{F}^2)(\underset{\sim}{F})^2+(1/2)(\underset{\sim}{\Omega})(\underset{\sim}{F})^3+[(\underset{\sim}{F})^2+(\underset{\sim}{F})^3].\ (3(\underset{\sim}{\Omega}\underset{\sim}{F})/2)].(p+1)$$

$$+\ [3(\underset{\sim}{F})^2+(1/2)(\underset{\sim}{F})^4](\underset{\sim}{\Omega})+[3(\underset{\sim}{F})p+12(\underset{\sim}{F})^2+3(\underset{\sim}{F})(\underset{\sim}{F}^2)$$

$$+\ (1/2)(\underset{\sim}{F})^3](\underset{\sim}{\Omega}\underset{\sim}{F})+3(\underset{\sim}{\Omega}\underset{\sim}{F}^2)(\underset{\sim}{F})^2+18(\underset{\sim}{F})p+36(\underset{\sim}{F})^2$$

$$+\ 18(\underset{\sim}{F}^2)(\underset{\sim}{F})\}/(48),$$

$$A_{20} = \{-(\underset{\sim}{\Omega}\underset{\sim}{F}^2)(p+1+[4(\underset{\sim}{F})+42(\underset{\sim}{F}^2)+36(\underset{\sim}{F}^3)](\underset{\sim}{\Omega})$$

$$+\ [4-3(\underset{\sim}{\Omega})(\underset{\sim}{F})-3(\underset{\sim}{\Omega})(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F})+[38-(\underset{\sim}{\Omega})(\underset{\sim}{F})](\underset{\sim}{\Omega}\underset{\sim}{F}^2)$$

$$-\ 3(\underset{\sim}{\Omega}\underset{\sim}{F})^2+28(\underset{\sim}{\Omega}\underset{\sim}{F}^3)-4(\underset{\sim}{\Omega}\underset{\sim}{F}^4)+(\underset{\sim}{\Omega}\underset{\sim}{F}\underset{\sim}{\Omega}')+(\underset{\sim}{\Omega}'\underset{\sim}{F}\underset{\sim}{\Omega})$$

$$+\ (\underset{\sim}{\Omega}'\underset{\sim}{F}^2\underset{\sim}{\Omega})-4(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}^2)\}/(4),$$

$$A_{21} = \{[-8(\underset{\sim}{F})-40(\underset{\sim}{F}^2)-24(\underset{\sim}{F}^3)+(\underset{\sim}{\Omega})[4(\underset{\sim}{F})-2(\underset{\sim}{F}^2)-(16/3)(\underset{\sim}{F}^3)]$$

$$+\ [-2p-20(\underset{\sim}{F}^2)-16(\underset{\sim}{F}^3)](\underset{\sim}{\Omega}\underset{\sim}{F})+[4p-20(\underset{\sim}{F})-16(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F}^3)-(16)$$

$$[p+(\underset{\sim}{F})](\underset{\sim}{\Omega}\underset{\sim}{F}^3)+[-p/2+(\underset{\sim}{F})+(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F})^2+[2p+2(\underset{\sim}{F})](\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{\Omega}\underset{\sim}{F}^2)$$

$$+\ (\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})(\underset{\sim}{F}^2)](p+1)-[8(\underset{\sim}{F})^2+134(\underset{\sim}{F})(\underset{\sim}{F}^2)+\frac{160}{3}(\underset{\sim}{F})(\underset{\sim}{F}^3)+24(\underset{\sim}{F}^2)^2](\underset{\sim}{\Omega})$$

$$-\ [8(\underset{\sim}{F})-16+10(\underset{\sim}{F})^2+36(\underset{\sim}{F}^2)+\frac{64}{3}(\underset{\sim}{F}^3)+16(\underset{\sim}{F})(\underset{\sim}{F}^2)](\underset{\sim}{\Omega}\underset{\sim}{F})$$

$$-\ [64+98(\underset{\sim}{F})+32(\underset{\sim}{F}^2)+8(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F}^2)-[312+96(\underset{\sim}{F})](\underset{\sim}{\Omega}\underset{\sim}{F}^3)$$

$$-\ 192(\underset{\sim}{\Omega}\underset{\sim}{F}^4)+[4+6(\underset{\sim}{F})+(3/2)(\underset{\sim}{F}^2)+(1/2)(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F})^2$$

$$+\ [-4+6(\underset{\sim}{F})^2+7(\underset{\sim}{F})(\underset{\sim}{F}^2)](\underset{\sim}{\Omega})(\underset{\sim}{\Omega}\underset{\sim}{F})-20(\underset{\sim}{F})^2-20(\underset{\sim}{F}^2)-24(\underset{\sim}{F})(\underset{\sim}{F}^2)$$

$$+\ 24(\underset{\sim}{\Omega}\underset{\sim}{F}')-24(\underset{\sim}{F}^3)+[-2(\underset{\sim}{F})+(\underset{\sim}{F})^2](\underset{\sim}{\Omega}\underset{\sim}{F}\underset{\sim}{\Omega}')+[6(\underset{\sim}{F})$$

$$+ (\underset{\sim}{F}^2)] (\underset{\sim}{\Omega}'\underset{\sim\sim}{F\Omega}) + [14(\underset{\sim}{F}) + (\underset{\sim}{F}^2)] (\underset{\sim}{\Omega}'\underset{\sim}{F}^2\underset{\sim}{\Omega}) + [28+6(\underset{\sim}{F})] (\underset{\sim\sim}{\Omega F}) . (\underset{\sim\sim}{\Omega F}^2)$$

$$+ 12(\underset{\sim\sim}{\Omega F})(\underset{\sim\sim}{\Omega F}^3) + [-2(\underset{\sim}{F}) + (\underset{\sim}{F}^2)] (\underset{\sim}{\Omega}'\underset{\sim}{\Omega}) + 8(\underset{\sim}{F})(\underset{\sim}{\Omega}'F^3\underset{\sim}{\Omega})$$

$$+ 4(\underset{\sim}{\Omega})(\underset{\sim\sim}{\Omega F}^2) + 4(\underset{\sim\sim}{\Omega F}^2)^2\}/(16),$$

$$A_{22} = \{[-4(\underset{\sim}{F})p - 40(\underset{\sim}{F})^2 - 36(\underset{\sim}{F})(\underset{\sim}{F}^2) - 12(\underset{\sim}{F}^2)p - [8(\underset{\sim}{F})(\underset{\sim}{F}^2)$$

$$+ 2(\underset{\sim}{F})^2] (\underset{\sim}{\Omega}) + [4(\underset{\sim}{F})p - 20(\underset{\sim}{F})^2 - 24(\underset{\sim}{F})(\underset{\sim}{F}^2) - 8(\underset{\sim}{F}^2)p] (\underset{\sim\sim}{\Omega F})$$

$$- 16[(\underset{\sim}{F})p + (\underset{\sim}{F})^2] (\underset{\sim\sim}{\Omega F}^2) + [(\underset{\sim}{F})p + (\underset{\sim}{F})^2] (\underset{\sim\sim}{\Omega F})^2 + (\underset{\sim}{\Omega})(\underset{\sim\sim}{\Omega F})(\underset{\sim}{F})^2] . (p+1)$$

$$+ [24(\underset{\sim}{F}) - 16(\underset{\sim}{F}^2) - 15(\underset{\sim}{F})^3 - 32(\underset{\sim}{F}^2)(\underset{\sim}{F})^2] (\underset{\sim}{\Omega}) + [12p - 8(\underset{\sim}{F})^3 - 32(\underset{\sim}{F}^3)$$

$$- 24(\underset{\sim}{F}) - 84(\underset{\sim}{F}^2) - 26(\underset{\sim}{F})^2 - 16(\underset{\sim}{F})(\underset{\sim}{F}^2)] (\underset{\sim\sim}{\Omega F}) - [16p + 200(\underset{\sim}{F})$$

$$+ 32(\underset{\sim}{F}^2) + 24(\underset{\sim}{F})^2] (\underset{\sim\sim}{\Omega F}^2) + [p + 10(\underset{\sim}{F}) + (\underset{\sim}{F}^2) + (\underset{\sim}{F})^2] (\underset{\sim\sim}{\Omega F})^2$$

$$- 32(\underset{\sim}{F}) - 88(\underset{\sim}{F}^2) - 12(\underset{\sim}{F})^3 - 12(\underset{\sim}{F})(\underset{\sim}{F}^2) - 48(\underset{\sim}{F}^3) - 32(\underset{\sim}{F})(\underset{\sim\sim}{\Omega F}')$$

$$- 64(\underset{\sim}{F})(\underset{\sim\sim}{\Omega F}^3) + (\underset{\sim}{\Omega}'\underset{\sim}{\Omega})(\underset{\sim}{F})^2 + (\underset{\sim\sim\sim}{\Omega F \Omega}')(\underset{\sim}{F})^2 + (\underset{\sim}{\Omega}'\underset{\sim\sim}{F\Omega})(\underset{\sim}{F})^2$$

$$+ (\underset{\sim}{\Omega}'\underset{\sim}{F}^2\underset{\sim}{\Omega})(\underset{\sim}{F})^2 + 4(\underset{\sim\sim}{\Omega F})(\underset{\sim}{F}) + 4(\underset{\sim\sim}{\Omega F})(\underset{\sim\sim}{\Omega F}^2)(\underset{\sim}{F})\}/(32),$$

$$A_{23} = -\{[[(3/2)(\underset{\sim}{F})^2 p + (3/2)(\underset{\sim}{F})^3] (\underset{\sim\sim}{\Omega F}) + (1/2)(\underset{\sim}{\Omega})(\underset{\sim}{F})^3] . (p+1)$$

$$+ [3(\underset{\sim}{F})p + 12(\underset{\sim}{F})^2 + 3(\underset{\sim}{F}^2)(\underset{\sim}{F}) + (1/2)(\underset{\sim}{F})^3] (\underset{\sim\sim}{\Omega F}) + [3(\underset{\sim}{F})^2 + (1/2)$$

$$(\underset{\sim}{F})^4] (\underset{\sim}{\Omega}) + 3(\underset{\sim}{F})^2 (\underset{\sim\sim}{\Omega F}^2)\}/(12),$$

and other A_{JK} coefficients are available in Appendix C of [14]. As an immediate result $h(\underset{\sim}{S}_2)$ can be expressed in the following manner:

$$h(\underset{\sim}{S}_2) = \theta - [\frac{1}{4n} \sum_{J=0}^{4} a_J(m,p) G_{mp+2J}(\theta,\omega^2)$$

$$+ \frac{1}{2} \sum_{J=0}^{6} A_J G_{mp+2J}(\theta,\omega^2)] [G'(\theta)]^{-1} + O(n^{-2}) . \tag{3.2}$$

Now θ is the appropriate percentile of the linear function of independent non-central chi-square variables of the form

$y = \sum_{j=1}^{p} \lambda_j \chi_j^2(m, \omega_i^2)$, $\sum_{i=1}^{p} \omega_i^2 = \omega^2$, λ_j's are the characteristic roots of $\underset{\sim}{A}\underset{\sim}{B}^{-1}$ and $G(\theta)$ is the c.d.f. of y in terms of the percentile. Finally, we can state the following theorem:

THEOREM 1 *Let $\underset{\sim}{Z} = (\underset{\sim}{z}_1, \ldots, \underset{\sim}{z}_m)$ be a pxm random matrix of independently distributed column vectors where $\underset{\sim}{z}_i$ has the density $N(\underset{\sim}{\mu}_i, \underset{\sim}{\Sigma}_1 = \underset{\sim}{B}^{-1})$, and $n\underset{\sim}{S}_2$ distributed central Wishart $W(n, p, \underset{\sim}{\Sigma}_2 = \underset{\sim}{A}^{-1})$. Under the assumption that $\underset{\sim}{B}^{-1}\underset{\sim}{A} = \underset{\sim}{I} + \underset{\sim}{F}$ and $|Ch_i(\underset{\sim}{F})| < 1$, $i = 1, \ldots, p$, then an asymptotic expansion for the percentile of T is given by Equation (3.2).*

The following are special cases of (3.2).

CASE 1. When terms involving $f_{ij}f_{kl}$ are negligible, where f_{ij} is the (i,j) element of $\underset{\sim}{F}$, terms like $(\underset{\sim}{F})^2$, $(\underset{\sim}{F}^2)$ and $(\underset{\sim}{\Omega})(\underset{\sim}{F}^3)$ could be dropped. Consequently A_6 disappears and A_0, A_1 up to A_5 are reduced drastically and finally (3.2) agrees with the result of Chattopadhyay [3] up to the indicated order.

CASE 2. Under the equality of the two covariance matrices, the deviation matrix is zero. Putting $(\underset{\sim}{F}) = 0$ in (3.2), we get Equation (6.4) of Siotani [19].

4. AN ASYMPTOTIC EXPANSION FOR THE C.D.F. OF $T = Tr\, \underset{\sim}{S}_2^{-1} \underset{\sim}{Z}\underset{\sim}{Z}'$.

In this section an asymptotic expansion for the c.d.f. of T up to $O(n^{-1})$ is derived using the method described in the last section. Also it is possible to write

$$Pr\{Tr\ \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = \int_R Pr\{Tr\ \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta | \underset{\sim}{S}_2\} Pr\{d\underset{\sim}{S}_2\}$$

$$= \circledR\ Pr\{Tr\ \underset{\sim}{A}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\}$$

and $\circledR$ is given by Equation (2.7). It follows that

$$Pr\{Tr\ \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = G(\theta) - \frac{1}{n}\left[h_1(\underset{\sim}{A}^{-1})\right]G'(\theta) + O(n^{-2})$$

Under the assumptions of Theorem 3.1, we get

$$Pr\{Tr\ \underset{\sim}{S}_2^{-1}\underset{\sim}{Z}\underset{\sim}{Z}' \leq \theta\} = G(\theta) + \frac{1}{4n}\sum_{J=0}^{4} a_J(m,p)G_{mp+2J}(\theta,\omega^2)$$

$$+ \frac{1}{n}\sum_{J=0}^{6} A_J G_{mp+2J}(\theta,\omega^2) + O(n^{-2}), \qquad (4.1)$$

a_j's and A_J's are the same as in Section 3. $G(\theta)$ and $G_{mp}(\theta,\omega^2)$ are as defined earlier. Then

THEOREM 2 *Under the assumptions of the previous theorem an asymptotic expansion for the* c.d.f. *of* T *is given by* (4.1).

Similarly we can get the two special cases as we pointed out in the previous Section.

5. NUMERICAL RESULTS

The expansion given by Equation (4.1) has been used here to compute the powers of the test when the departure from the null hypothesis occurs. The following table shows these powers. For tabulation of this table, the upper five percent points were taken from Pillai an Jayachandran [11]. Table 1 is in agreement with findings in Table 1 of [13] where neglecting $\delta_{ij}\delta_{kl}$ terms was found to exaggerate the powers.

TABLE 1

Powers of T Test Under Violations for $p = 2$, $m = 3$, $\alpha = 0.05$ while the Deviation Matrix has Equal Deviation Parameters

ω_1	ω_2	$(\underset{\sim}{F})$	Up to the order	$n = 83$
		.00001	$O(1)$	.0379288
			$O(n^{-1})$	.049277 (.049273)
0	.00001	.00015	$O(1)$	.037942
			$O(n^{-1})$	.049296 (.049344)
		.005	$O(1)$	.038399
			$O(n^{-1})$	.049433 (.04977)
		.00001	$O(1)$	.03793
			$O(n^{-1})$	.049279 (.049276)
0	.0001	.00015	$O(1)$	.037945
			$O(n^{-1})$	.049300 (.049347)
		.005	$O(1)$	.038403
			$O(n^{-1})$	.049636 (.05182)
		.00001	$O(1)$	.038085
			$O(n^{-1})$	.04944 (.049455)
0	.005	.00015	$O(1)$	.038098
			$O(n^{-1})$	.049437 (.049475)
		.005	$O(1)$	.038557
			$O(n^{-1})$	.0499188[1] (.05200)

The figures in () are computed using Chattopadhyay expansion [3].

[1]Further work in progress by Pillai and Y.S. Hsu for $O(n^{-2})$ order terms gives the value .05073.

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