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ON SELF SIMILARITY

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ABSTRACT

The self-similarity has been used in the solution of many problems related to transport phenomena . In this article a sistematic approach is given for the deduction of the similarity variables.

RESUMEN

La auto-semejanza ha sido utilizada en la solución de muchos problemas relacionados con el fenómeno de transporte . En este artículo se presenta un enfoque sistemático para la deducción de las variables de semejanza.

1. INTRODUCTION

Although the similarity variables have been used widely in the solution of many problems related to transport phenomena, a systematic procedure based on pure physical ground for their deduction has not been presented. Due to this sometimes as noted by V. Streeter [1] problems which have clearly self-similarity were excluded by some authors.

In this article a method for obtaining the similarity variables is presented, and some remarks are given so as to simplify the plotting of the solution to problems which have the self-similarity scheme.

Several problems are worked in order to illustrate the procedure and a step by step process is presented.

In order to illustrate the use of self-similarity procedure a well known problem was selected, namely the laminar flow over a flat plate of a viscous fluid.

The governing equation for the boundary layer with zero pressure gradient is [2] [3] :

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_y}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} \tag{a}$$

with the boundary conditions:

$y = 0 \qquad v_x = v_y = 0$

$y = \infty \qquad v_x = v_\infty$

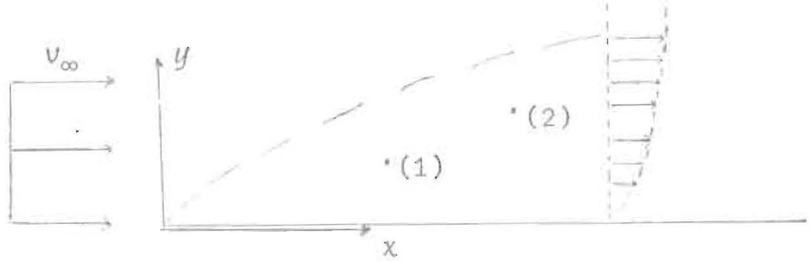


Figure 1

By self-similarity we expect the existence of points such as (1) and (2) (see Fig. 1) on which the velocity is the same. All we need is to find a relation between "x" and "y" such that all the points whose position coordinates follow this relation have the same velocity.

In order to acomplish that, let define the ratios

$$K_{v_x} = \frac{v_{x_1}}{v_{x_2}} \qquad ; \qquad K_{v_y} = \frac{v_{y_1}}{v_{y_2}}$$

$$K_x = \frac{x_1}{x_2} \qquad ; \qquad K_y = \frac{y_1}{y_2} \qquad \text{and} \qquad \text{(b)}$$

$$K = \frac{\nu_1}{\nu_2}$$

where subscript 1 stands for "at point (1)" and subscript 2 stands for "at point (2)", since (a) applies at (1) as well as at (2): at (1)

$$v_{x_1} \frac{\partial v_{x_1}}{\partial x_1} + v_{y_1} \frac{\partial v_{x_1}}{\partial y_1} = \nu_1 \frac{\partial^2 v_{x_1}}{\partial y_1^2} \qquad . \qquad \text{(c)}$$

solving (b) for conditions at point (1) in function of conditions at point (2) and substituting in (c) :

$$\frac{K_{v_x}^2}{K_x} v_{x_2} \frac{\partial v_{x_2}}{\partial x_2} + \frac{K_{v_x} K_{v_y}}{K_y} v_{y_2} \frac{\partial v_{x_2}}{\partial y_2} = K_\nu \frac{K_{v_x}}{K_y^2} \nu_2 \frac{\partial^2 v_{x_2}}{\partial y_2^2}$$

since ν does not changes with position:

$$K_\nu = 1$$

and since we are interested in points which have the same velocity

$$K_{v_x} = 1 \qquad \text{and} \qquad K_{v_y} = 1 \qquad \text{so}$$

$$\frac{1}{K_x} v_{x_2} \frac{\partial v_{x_2}}{\partial x_2} + \frac{1}{K_y} v_{y_2} \frac{\partial v_{x_2}}{\partial y_2} = \frac{1}{K_y^2} \nu_2 \frac{\partial^2 v_{x_2}}{\partial y_2^2}$$

multiplying by K_y^2 :

$$\frac{K_y^2}{K_x} v_{x_2} \frac{\partial v_{x_2}}{\partial x_2} + \frac{K_y^2}{K_y} v_{y_2} \frac{\partial v_{x_2}}{\partial y_2} = \nu_2 \frac{\partial^2 v_{x_2}}{\partial y_2^2} \qquad \text{(d)}$$

but from (a) :

$$v_{x_2} \frac{\partial v_{x_2}}{\partial x_2} + v_{y_2} \frac{\partial v_{x_2}}{\partial y_2} = \nu_2 \frac{\partial^2 v_{x_2}}{\partial y_2^2}$$

so a solution for (d) is:

$\frac{K_y^2}{K_x} = 1 = \frac{K_y^2}{K_y}$ and since we want a relation between x_1, y_1 and x_2, y_2, then from (b) :

$$\frac{\dfrac{y_1^2}{y_2^2}}{\dfrac{x_1}{x_2}} = 1 \qquad \text{or} \qquad \frac{y_1^2}{x_1} = \frac{y_2^2}{x_2}$$

then the desired relation is

$$\eta = c\,\frac{y}{\sqrt{x}} \quad \text{where } c \text{ is a constant.}$$

In order to simplify the solution let introduce the stream function:

$$v_x \equiv \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v_y \equiv -\frac{\partial \psi}{\partial x}$$

defining

$$K_\psi = \frac{\psi_1}{\psi_2} \quad \text{and using (b)}$$

$$K_{v_x} v_{x_2} = \frac{K_\psi}{K_y}\,\frac{\partial \psi_2}{\partial y_2} \quad \text{since } K_{v_x} = 1$$

then

$$\frac{K_\psi}{K_y} = 1 \qquad \text{or} \qquad \frac{\psi_1}{\psi_2} = \frac{x_1^{\frac{1}{2}}}{x_2^{\frac{1}{2}}}$$

or [3] [4]

$$\psi = c_2 x^{\frac{1}{2}} f(\eta)$$

and after substituting in (a) we will get [3]

$$2\,\frac{d^3 f}{d\eta^3} + f\,\frac{d^2 f}{d\eta^2} = 0$$

which is anordinary differential equation with boundary conditions:

$$\eta = 0 \qquad f = 0 \qquad \frac{df}{d\eta} = 0$$

$$\eta = \infty \qquad \frac{df}{d\eta} = 1$$

PROCEDURE

The procedure followed in the previous example may be summarized as follows:

1. Write down all the K's and substitute into the governing equation.
2. Specify all the K's values known.
3. Specify the K's values among similar points.
4. Substitute in differential equation. Manipulate the equation obtained until you get one of the terms free from K's.
5. Find the relation among the K's of interest.
6. Substitute K's by their definition and get the similarity relation.

APPLICATION

The steps indicated in the preceding section are applied to several problems (omiting the algebraic routine).

A) Free - Convection (Laminar flow on a vertical plate) [3]:
The governing equation is

$$v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = \nu \frac{\partial^2 v_x}{\partial y^2} + \beta g (T - T_\infty)$$

STEP 1

$$K_\nu = \frac{\nu_1}{\nu_2} \quad ; \quad K_\beta = \frac{\beta_1}{\beta_2} \quad ; \quad K_g = \frac{g_1}{g_2}$$

$$K_T = \frac{T_1}{T_2} \quad ; \quad K_{\upsilon_x} = \frac{\upsilon_{x_1}}{\upsilon_{x_2}} \quad ; \quad K_{\upsilon_y} = \frac{\upsilon_{y_1}}{\upsilon_{y_2}}$$

$$K_x = \frac{x_1}{x_2} \qquad \text{and} \qquad K_y = \frac{y_1}{y_2}$$

STEP 2

$$K_\nu = K_\beta = K_g = 1$$

STEP 3

Similar points are those with the same temperature, so:

$$K_T = 1$$

STEP 4

$$\frac{K_{\upsilon_x}^2}{K_x} \upsilon_{x_2} \frac{\partial \upsilon_{x_2}}{\partial x_2} + \frac{K_{\upsilon_x} K_{\upsilon_y}}{K_y} \upsilon_{y_2} \frac{\partial \upsilon_{x_2}}{\partial y_2} = \frac{K_{\upsilon_x}}{K_y^2} \nu_2 \frac{\partial^2 \upsilon_{x_2}}{\partial y_2^2}$$

$$+ \beta_2 g_2 (T_2 - T_\infty)$$

the last term is already free from K's.

STEP 5

One of the solution is

$$\frac{K_{v_x}^2}{K_x} = \frac{K_{v_x} K_{v_y}}{K_y} = \frac{K_{v_x}}{K_y^2} = 1$$

since we are interested in a relation between x and y, square the last term and get

$$\frac{K_{v_x}^2}{K_y^4} = \frac{K_{v_x}^2}{K_x} \quad \text{or} \quad K_y^4 = K_x$$

STEP 6

$$\frac{y_1^4}{y_2^4} = \frac{x_1}{x_2} \quad \text{or} \quad \eta = c\,\frac{y}{\sqrt[4]{x}}$$

where η is the desired similarity variable and c is a constant [3].

Also from step 5 may get a simplified presentation of

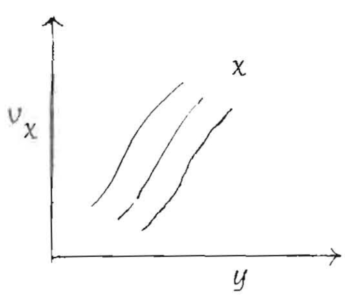


Figure 2

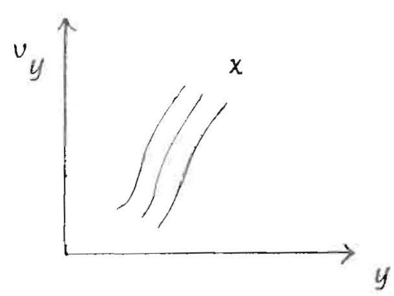


Figure 3

since from

$$\frac{K_{v_x} K_{v_y}}{K_y} = \frac{K_{v_x}}{K_y^2} = 1$$

we will get

$$\frac{\dfrac{v_{y_1}}{v_{y_2}}}{\dfrac{y_1}{y_2}} = \frac{1}{\dfrac{y_1^2}{y_2^2}} \qquad\qquad v_{y_1} y_1 = v_{y_2} y_2$$

and from

$$\frac{K_{v_x}^2}{K_x} = 1$$

$$\frac{v_{x_1}}{\sqrt{x_1}} = \frac{v_{x_2}}{\sqrt{x_2}}$$

that is instead of Fig. 2 and 3 get figures 4a and 4b (See Ref.3. Pp. 154-155).

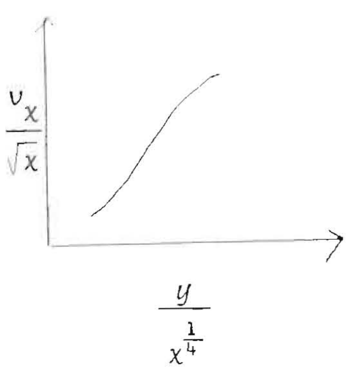


Figure 4a

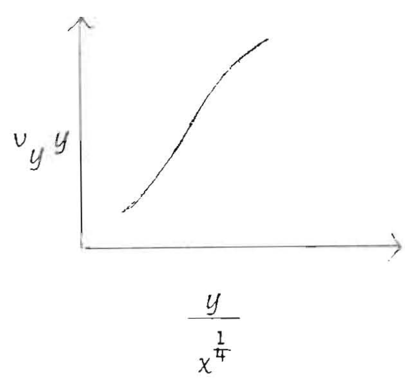


Figure 4b

A further simplification by introducing the stream function [3]:

$$v_x \equiv \frac{\partial \psi}{\partial y} \qquad \& \qquad v_y \equiv \frac{-\partial \psi}{\partial x}$$

using the procedure we will get

$$\frac{K_{v_x} K_y}{K_\psi} = 1 \qquad \text{and} \qquad \frac{K_{v_y} K_x}{K_\psi} = 1$$

and want a relation between ψ and x (not "y" because what is wanted is the distribution along "y" so we will integrate for "x" fixed).

Then:

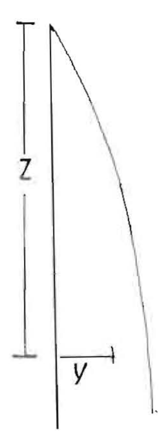
$$K_\psi = K_{v_x} K_y = \sqrt{K_x}\,\sqrt[4]{K_x}$$

So

$$\psi = cx^{\frac{3}{4}} \quad F(\eta)$$

B) Film Condensation on vertical Plates [3]
Governing equations:

$$\frac{\partial v_z}{\partial Z} + \frac{\partial v_y}{\partial y} = 0 \tag{b.1}$$



$$v_z \frac{\partial v_z}{\partial Z} + v_y \frac{\partial v_z}{\partial y} = g\left(1 - \frac{\rho_v}{\rho}\right) + \nu \frac{\partial^2 v_z}{\partial y^2} \qquad \text{(b.2)}$$

$$v_z \frac{\partial T}{\partial Z} + v_y \frac{\partial T}{\partial y} = \frac{K}{\rho\, c_p} \frac{\partial^2 T}{\partial y^2} \qquad \text{(b.3)}$$

The K's are:

$$K_{v_z} = \frac{v_{z_1}}{v_{z_2}} \quad ; \quad K_y = \frac{y_1}{y_2} \quad ; \quad K_z = \frac{Z_1}{Z_2}$$

$$K_g = \frac{g_1}{g_2} \quad ; \quad K_\rho = \frac{\left(1 - \frac{\rho_v}{\rho}\right)_1}{\left(1 - \frac{\rho_v}{\rho}\right)_2} \quad ; \quad K_\nu = \frac{\nu_1}{\nu_2}$$

$$K_T = \frac{T_1}{T_2}$$

Substituting in (b.2)

$$\frac{K_{v_z}^2}{K_z} v_{z_2} \frac{\partial v_{z_2}}{\partial Z_2} + \frac{K_{v_y} K_{v_z}}{K_y} v_{y_2} \frac{\partial v_{z_2}}{\partial y_2} = K_g K\, g_3 \left(1 - \frac{\rho_v}{\rho}\right)_2 + \frac{K_\nu K_{v_z}}{K_y^2} \nu_2 \frac{\partial^2 v_{z_2}}{\partial y_2^2}$$

with $K_g = K_\nu = 1$

and dividing by K_ρ get

$$\frac{K_{v_z}^2}{K_z K_\rho} = \frac{K_{v_y} K_{v_z}}{K_y K_\rho} = \frac{K_{v_z}}{K_y^2 K_\rho} = 1 \qquad \text{(b.4)}$$

$$\frac{K^2_{\upsilon_z}}{K_z K_\rho} = \frac{K^2_{\upsilon_z}}{K^4_y K^2_\rho}$$

$$K_z = K_\rho K^4_y \quad ; \quad \frac{z_1}{z_2} = \frac{\left(1 - \frac{\rho_\upsilon}{\rho}\right)_1}{\left(1 - \frac{\rho_\upsilon}{\rho}\right)_2} \frac{y_1^4}{y_2^4}$$

or $\eta = c\left(1 - \frac{\rho_\upsilon}{\rho}\right)^{\frac{1}{4}} \frac{y}{\sqrt[4]{z}}$ where c = constant

and η is the similarity variable.

If $v_z \equiv \frac{\partial \psi}{\partial y}$ and $v_y \equiv \frac{-\partial \psi}{\partial z}$ we will get

$$F(\eta) = c' \frac{\psi}{z^{\frac{3}{4}}} \left[\frac{\rho - \rho_\upsilon}{\rho}\right]^{-\frac{1}{4}}$$

If $\theta = \frac{T - T_{sat}}{T_w - T_{sat}}$ then after substitution in the original equations and by a proper selection of c and c' will result in:

$$F''' + 3FF''' - 2(F') + 1 = 0$$

$$\theta'' + 3PrF\,\theta' = 0$$

As a matter of fact from (b.4) it is possible to obtain a simplified relation between velocities and position.
So from:

$$\frac{K_{\nu_y} K_{\nu_z}}{K_y K} = \frac{K_{\nu_z}}{K_y^2 K}$$

get $y_1 \nu_{y_1} = y_2 \nu_{y_2}$ or plot : $y \nu_y \quad \nu_s . \eta$

and from

$$\frac{K_{\nu_z}}{K_y^2 K_\rho} = \frac{K_{\nu_z}^2}{K_z K_\rho} \quad : \quad \frac{1}{K_y^2} = \frac{K_{\nu_z}}{K_z}$$

plot : $\dfrac{\nu_z y^2}{z} \nu_s . \eta$

as suggested by Pohlhausen to Schmidt and Beckmann [5].

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