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ON AN INTEGRAL EQUATION

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ABSTRACT

The object of this paper is to obtain the solution of a general class of integral equations examples of which occur widely in boundary value problems in applied mathematics . In particular the solution of a generalization of an integral equation which arises in certain electrostatic problems is presented and shown to include as a special case an equation whose solution has been given by Kalla.

RESUMEN

El objeto de este trabajo es obtener solución de una clase general de ecuaciones integrales , cuyos ejemplos abundan en los problemas de valores de contorno de Matemáticas Aplicadas . En particular , la solución de una ecuación integral generalizada relacionada con algunos problemas de Electroestática se presenta aquí. La ecuación integral tratada es general y contiene como un caso especial resultados dados por Kalla.

1. THE INTEGRAL EQUATION

In this paper we apply some simple generalizations of the Erdélyi-Kober operators of fractional integration to obtain the general solution of the integral equation

$$\int_0^b f(y)\, K_z(x,y)\, dy = g(x)\ , \qquad 0 < x < b\ , \tag{1}$$

whose kernel is defined by

$$K_z(x,y) = \left(\frac{2\sigma}{\Gamma(1-\alpha)}\right)^2 \frac{y^{2\sigma(\alpha-\mu)-1}}{x^{2\sigma(1+\eta-\alpha)}} \int_0^z \frac{t^{2\sigma(\eta+\mu+1)-1}\ \phi(t)\ dt}{\left[\left(x^{2\sigma}-t^{2\sigma}\right)\left(y^{2\sigma}-t^{2\sigma}\right)\right]^{\alpha}}\ , \tag{2}$$

where $\sigma > 0$, $1-\alpha > 0$, μ, η are real parameters , $z = min(x,y)$, $g(x)$ and $\phi(t)$ are prescribed functions and $f(y)$ is the solution function to be determined.

Examples of this equation arise in the solution of a variety of boundary value problems in applied mathematics . Using formal analysis we show how the use of the operators of fractional integration enables the general solution of the integral equation to be determined in a compact form from which the solutions of special cases of the equation occurring in certain electrostatic problems and previously considered by Kalla [1] and Lebedev [2] can easily be deduced. In another paper the author [3] has solved several other integral equations, which are related to equation (1), by a similar technique.

2. THE INTEGRAL OPERATORS

We shall make use of some elementary extensions of the Erdélyi-

Kober operators defined in [3] by

$$I_{\eta,\alpha}(a,x:\delta)\,f(x) = \frac{\delta\, x^{-\delta(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^x \left(x^\delta - t^\delta\right)^{\alpha-1} t^{\delta(1+\eta)-1} f(t)\,dt, \quad \alpha > 0, \tag{3}$$

$$= x^{1-\delta(\alpha+\eta+1)}\, D_x^m \left\{ x^{\delta(\alpha+\eta+m+1)-1}\, I_{\eta,\alpha+m}(a,x:\delta)\, f(x) \right\}, \quad \alpha < 0, \tag{4}$$

$$K_{\eta,\alpha}(x,b:\delta)\,f(x) = \frac{\delta\, x^{\delta\eta}}{\Gamma(\alpha)} \int_x^b \left(t^\delta - x^\delta\right)^{\alpha-1} t^{\delta(1-\alpha-\eta)-1} f(t)\,dt, \quad \alpha > 0, \tag{5}$$

$$= (-1)^m\, x^{\delta(\eta-1)+1}\, D_x^m \left\{ x^{\delta(m-\eta+1)-1}\, K_{\eta-m,\alpha+m}(x,b:\delta)\, f(x) \right\}, \quad \alpha < 0, \tag{6}$$

when $0 \leqslant a < x < b$, $\delta > 0$ and

$$D_x^m = \left(\frac{1}{\delta} \frac{d}{dx}\, x^{1-\delta} \right)^m, \tag{7}$$

where m is the smallest positive integer such that $\alpha + m > 0$ when $\alpha < 0$.

The inverse operators are defined by

$$I^{-1}_{\eta,\alpha}(a,x:\delta)\, f(x) = I_{\eta+\alpha,-\alpha}(a,x:\delta)\, f(x), \tag{8}$$

$$K^{-1}_{\eta,\alpha}(x,b:\delta)\, f(x) = K_{\eta+\alpha,-\alpha}(x,b:\delta)\, f(x) \tag{9}$$

3. THE GENERAL SOLUTION OF THE EQUATION

Substituting for the kernel from equation (2) into equation (1) we find that the integral equation becomes

$$\int_0^x f(y)\, K_y(x,y)\, dy + \int_x^b f(y)\, K_x(x,y)\, dy = g(x), \qquad 0 < x < b, \quad (10)$$

Inverting the order of the integrations and using the definitions (3) and (5) we see that the above equation can be written in the operational form

$$I_{\eta,1-\alpha}(0,x:2\sigma)\ \phi(x)\ K_{\mu,1-\alpha}(x,b:2\sigma)\ f(x) = g(x), \qquad 0 < x < b, \quad (11)$$

where $1 - \alpha > 0$.

On applying the inverse operators (8) and (9) in turn to this equation we obtain the general solution of equation (1) as

$$f(y) = K_{p,\alpha-1}(y,b:2\sigma)\left[\frac{1}{\phi(y)}\, I_{q,\alpha-1}(0,y:2\sigma)\, g(y)\right], \qquad 0 < y < b, \quad (12)$$

where $p = \mu + 1 - \alpha$, and $q = \eta + 1 - \alpha$.

If m is the smallest positive integer such that $m + \alpha - 1 > 0$ we can by using the definitions (3) to (7), easily show that the above solution, when written out in detail, is given by

$$f(y) = (-1)^m\, y\left[\frac{2\sigma\, y^{\sigma(\mu-\alpha)}}{\Gamma(m+\alpha-1)}\right]^2 D_y^m\, y^{2\sigma-1}\int_y^b \frac{t^{-2\sigma(\eta+\mu)}\, G(t)}{\phi(t)\left(t^{2\sigma} - y^{2\sigma}\right)^{2-\alpha-m}}\, dt\,, \quad (13)$$

where $0 < y < b$, $\quad m + \alpha > 1$,

$$G(t) = D_t^m \, t^{2\sigma-1} \int_0^t \frac{x^{2\sigma(2+\eta-\alpha)-1}}{\left(t^{2\sigma} - x^{2\sigma}\right)^{2-\alpha-m}} \, g(x)\, dx \tag{14}$$

and

$$D_x^m = \left(\frac{1}{2\sigma} \frac{d}{dx} x^{1-2\sigma}\right)^m \tag{15}$$

4. EXAMPLES OF THE EQUATION

If in equation (2) we set

$$2\sigma\, \Gamma(\beta)\, \phi(t) = \Gamma(1-\alpha)\, \Gamma(1+\beta-\alpha)\, t^{2\sigma(\beta-\eta-\mu-1)}, \quad \beta > 0, \tag{16}$$

we find that a special case of the kernel is

$$K_z^{(1)}(x,y) = \frac{2\sigma\, \Gamma(1+\beta-\alpha)}{\Gamma(\beta)\, \Gamma(1-\alpha)} \, \frac{y^{2\sigma(\alpha-\mu)-1}}{x^{2\sigma(1+\eta-\alpha)}} \int_0^z \frac{t^{2\sigma\beta-1}}{\left[\left(x^{2\sigma} - t^{2\sigma}\right)\left(y^{2\sigma} - t^{2\sigma}\right)\right]^{\alpha}}, \tag{17}$$

where $1-\alpha > 0$, $\beta > 0$, $\sigma > 0$ and $z = min(x,y)$.

On using the properties of the hypergeometric function ${}_2F_1$(*) given in [4, pp.50 - 54] it can readily be shown that the above expression can be written in the form

$$K_z^{(1)}(x,y) = \frac{x^{2\sigma(\beta-\eta-1)}\, y^{-2\sigma(\mu-\beta)-1}}{\left(x^{\sigma} + y^{\sigma}\right)^{2\beta}} \; {}_2F_1\left[\beta,\ \beta-\alpha+\frac{1}{2};\ 2(\beta-\alpha)+1;\ \frac{4(x\,y)^{\sigma}}{\left(x^{\sigma}+y^{\sigma}\right)^2}\right]. \tag{18}$$

In this case the integral equation (1) assumes the form

$$\int_0^b \frac{\Phi(y)}{\left(x^\sigma + y^\sigma\right)^{2\beta}}\ {}_2F_1\left[\beta, \beta-\alpha+\frac{1}{2};\ 2(\beta-\alpha)+1;\ \frac{4(xy)^\sigma}{\left(x^\sigma+y^\sigma\right)^2}\right] dy =$$

$$= \Psi(x), \qquad 0 < x < b\,, \tag{19}$$

where $\sigma > 0,\ \beta > 0,\ 1-\alpha > 0$ and we have written

$$\phi(y) = y^{2\sigma(\beta-\mu)-1} f(y) \quad , \qquad \Psi(x) = x^{2\sigma(\eta+1-\beta)} g(x)\,. \tag{20}$$

Equation (19) is the same as that considered by Kalla [1] who gave its solution in the special case when $\beta = \alpha,\ 0 < \alpha < 1$ and $\sigma = 1$.

From the general solution given by equations (13) and (14) we have, on using the definitions (20), that the solution of the integral equation (19) is

$$\phi(y) = \frac{(-1)^m (2\sigma)^3}{\left[\Gamma(m+\alpha-1)\right]^2}\ y^{2\sigma(\beta-\alpha)}\ \mathcal{D}_y^m\ y^{2\sigma-1} \int_y^b \frac{t^{-2\sigma(\beta-1)}\ G(t)}{\left(t^{2\sigma}-y^{2\sigma}\right)^{2-\alpha-m}}\, dt\,, \tag{21}$$

with

$$G(t) = \frac{\Gamma(\beta)}{\Gamma(1-\alpha)\ \Gamma(1+\beta-\alpha)}\ \mathcal{D}_t^m\ t^{2\sigma-1} \int_0^t \frac{x^{2\sigma(\beta-\alpha+1)-1}}{\left(t^{2\sigma}-x^{2\sigma}\right)^{2-\alpha-m}}\ \Psi(x)\, dx\,, \tag{22}$$

where $\sigma > 0,\ \beta > 0,\ 1 > \alpha > 1-m,\ m = 1,2,3,\ldots,$ and $\mathcal{D}_x^m$ is the operator defined by equation (15).

Setting $\sigma = 1,\ \beta > 0,\ 0 < \alpha < 1$ we find from equation (19) that Kalla's integral equation

$$\int_0^b \frac{\phi(y)}{(x+y)^{2\beta}}\ {}_2F_1\left[\beta,\ \beta-\alpha+\frac{1}{2};\ 2(\beta-\alpha)+1;\ \frac{4xy}{(x+y)^2}\right] dy =$$

$$= \Psi(x), \qquad 0 < x < b, \tag{23}$$

has the solution , obtained from equations (21) and (22) with $\sigma = 1$, $m = 1$,

$$\Phi(y) = -\frac{2y^{2(\beta-\alpha)}}{[\Gamma(\alpha)]^2} \frac{d}{dy} \int_y^b \frac{t^{-2(\beta-1)}\, G(t)}{\left(t^2 - y^2\right)^{1-\alpha}}\, dt, \qquad 0 < y < b \tag{24}$$

where

$$G(t) = \frac{\Gamma(\beta)}{\Gamma(1-\alpha)\,\Gamma(1+\beta-\alpha)} \frac{d}{dt} \int_0^t \frac{x^{2(\beta-\alpha+1)-1}}{\left(t^2 - x^2\right)^{1-\alpha}}\, \Psi(x)\, dx \tag{25}$$

and $0 < \alpha < 1$, $\beta > 0$. When $\beta = \alpha$, $0 < \alpha < 1$ the above solution agrees completely with that given by Kalla [1]. As was pointed out by Kalla equation (23) is a generalization of the integral equation solved by Lebedev [2], for the case when $\beta = \alpha = \frac{1}{2}$, which is encountered in a number of boundary value problems in electrostatics.

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