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ON AN INTEGRAL INVOLVING H-FUNCTION

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ABSTRACT

In this paper an integral involving hypergeometric function and the H-function has been evaluated. The results are believed to be new. A few interesting particular cases have also been given.

RESUMEN

En este trabajo una integral que involucra la función hipergeométrica y la función H ha sido evaluada. Se cree que los resultados son nuevos. Mencionamos algunos casos especiales.

1. INTRODUCTION

The H-function introduced by Fox [3] will be defined and represented in the following manner:

$$H_{p\ q}^{m\ n}\left[z \left| \begin{matrix} (a_1, e_1), \ldots, (a_p, e_p) \\ (b_1, f_1), \ldots, (b_q, f_q) \end{matrix} \right.\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - f_j s) \prod_{j=1}^{n} \Gamma(1 - a_j + e_j s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + f_j s) \prod_{j=n+1}^{p} \Gamma(a_j - e_j s)} z^s \, ds$$

(1.1)

where L is a suitable contour of Mellin-Barnes type and the parameters are so restricted that the H-function has a meaning.

Braaksma [1] has proved that the integral on the right hand side of (1.1) is absolutely convergent when $\theta > 0$ and $|\arg z| < \frac{\theta\pi}{2}$, where

$$\theta = \sum_{j=1}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=1}^{m} f_j - \sum_{j=m+1}^{q} f_j \qquad (1.2)$$

Throughout this paper (1.1) will be denoted by $H_{p\ q}^{m\ n}\left[z \left| \begin{matrix} {}_1(a_j, e_j)_p \\ {}_1(b_j, f_j)_q \end{matrix} \right.\right]$

2. MAIN RESULTS

First we shall prove the following integral involving hypergeometric function. The result is believed to be new.

$$J = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} [ax + b(1-x)]^{-\alpha-\beta}$$

$$\times {}_2F_1 \left[\delta , \delta + \frac{1}{2} ; \gamma ; 4 \frac{abx(1-x)}{\{ax+b(1-x)\}^2} \right] dx = \frac{\Gamma(\alpha)\ \Gamma(\beta)}{a^{\alpha}\ b^{\beta}\ \Gamma(\alpha+\beta)}$$

$$\times {}_4F_3 \left(\delta , \delta + \frac{1}{2} , \alpha , \beta ; \gamma , \frac{\alpha+\beta}{2} , \frac{\alpha+\beta+1}{2} ; 1 \right) \tag{2.1}$$

where $R(\alpha) > 0$, $R(\beta) > 0$, $R(\gamma-2\delta) > 0$, a and b are non-zero constants and the expression $[ax + b(1-x)]$, where $0 \leqslant x \leqslant 1$ is not zero.

Proof: Express the hypergeometric function as a series, change the order of integration and summation which is justified due to the uniform convergence of the SERIES in the interval $(0,1)$, evaluate the integral with the help of the result [4, p.450], we then get

$$J = \sum_{r=0}^{\infty} \frac{(\delta)_r \left(\frac{1}{2} + \delta \right)_r 2^{2r}\ \Gamma(\alpha + r)\ \Gamma(\beta + r)}{(\gamma)_r\ \underline{|r}\ a^{\alpha}\ b^{\beta}\ \Gamma(\alpha+\beta+2r)} \tag{2.2}$$

Now if we use the results $(2z)_{2r} = 2^{2r}\,(z)_r\left(z+\frac{1}{2}\right)_r$ and $\Gamma(z+r) = (z)_r\,\Gamma(z)$, and sum the series, we finally get (2.1).

When $\delta = \frac{\alpha+\beta}{2}$, (2.1) gives

$$\int_0^1 x^{\alpha-1}\,(1-x)^{\beta-1}\,[ax+b(1-x)]^{-\alpha-\beta}$$

$$\times\ {}_2F_1\left[\frac{\alpha+\beta}{2},\ \frac{\alpha+\beta+1}{2};\ \gamma;\ 4\,\frac{ab\,x(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$= \frac{\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{a^{\alpha}\,b^{\beta}\,\Gamma(\alpha+\beta)\,\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \qquad (2.3)$$

where $R(\alpha) > 0$, $R(\beta) > 0$, $R(\gamma-\alpha-\beta) > 0$, a and b are non-zero constants and the expression $[ax+b(1-x)]$, where $0 \leqslant x \leqslant 1$ is not zero.

On the other hand if we take $\alpha = \beta = \gamma$ in (2.1), we have

$$\int_0^1 (x-x^2)^{\gamma-1}\,[ax+b(1-x)]^{-2\gamma}\ {}_2F_1\left[\delta,\ \delta+\frac{1}{2};\ \gamma;\ 4\,\frac{a\,b\,x\,(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\gamma-2\delta)}{a^{\gamma}\,b^{\gamma}\,(2\gamma-2\delta)}\,2^{-2\delta} \qquad (2.4)$$

where $R(\gamma) > 0$, $R(\gamma-2\delta) > 0$, a and b are non-zero constants and the expression $[ax+b(1-x)]$, where $0 \leqslant x \leqslant 1$ is not zero.

Now we evaluate the following integral involving hypergeometric function and the H-function:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\left[ax+b(1-x)\right]^{-\alpha-\beta}$$

$$\times\ {}_2F_1\left[\frac{\alpha+\beta}{2},\ \frac{\alpha+\beta+1}{2};\gamma;4\ \frac{a\,b\,x\,(1-x)}{\{ax+b(1-x)\}^2}\right]$$

$$\times\ H_{p\ q}^{m\ n}\left[z\left\{\frac{ax}{b(1-x)}\right\}^{\lambda}\left|\begin{matrix}{}_1(a_j,e_j)_p\\ {}_1(b_j,f_j)_q\end{matrix}\right.\right]dx$$

$$=\frac{\Gamma(\gamma)\ \Gamma(\gamma-\alpha-\beta)}{a^{\alpha}\ b^{\beta}\ \Gamma(\alpha+\beta)}\ H_{p+2\ q+2}^{m+1\ n+1}\left[z\left|\begin{matrix}(1-\alpha,\lambda), & {}_1(a_j,e_j)_p, & (\gamma-\alpha,\lambda)\\ (\beta,\lambda), & {}_1(b_j,f_j)_q, & (1-\gamma+\beta,\lambda)\end{matrix}\right.\right] \tag{2.5}$$

where $\lambda > 0$, $R(\gamma-\alpha-\beta) > 0$, $R[\alpha+\lambda(b_j/f_j)] > 0$, $j = 1,\ldots,m$, $R[\beta-\lambda(a_j-1/e_j)] > 0$, $j=1,\ldots n$, $\theta > 0$, $|arg\ z| < \frac{\theta\pi}{2}$, a and b are non-zero constants and the expression $[ax+b(1-x)]$, $(0 \leqslant x \leqslant 1)$ is not zero.

Proof: Denoting the left hand side of (2.5) by I, expressing the H-function in terms of Mellin-Barnes integral and changing the order of integration which is justified under the conditions stated above due to the absolute convergence of the integrals involved in the process, we get:

$$I = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-f_j s)\prod_{j=1}^{n}\Gamma(1-a_j+e_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+f_j s)\prod_{j=n+1}^{p}\Gamma(a_j-e_j s)} z^s$$

$$\times \left\{ \int_0^1 x^{\alpha+\lambda s-1}(1-x)^{\beta-\lambda s-1}\left[ax+b(1-x)\right]^{-\alpha-\beta}\right.$$

$$\left.\times {}_2F_1\left[\frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}; \gamma; 4\frac{a\,b\,x\,(1-x)}{\{ax+b(1-x)\}^2}\right]dx\right\} ds .$$

Now evaluating the inner integral with the help of the result (2.3), after a little simplification interpreting it with the help of (1.1) we finally get (2.5).

3. PARTICULAR CASES

(1). Taking $a_1 = 1-\gamma+\beta$, $b_1 = \gamma-\alpha$, $e_1 = f_1 = \lambda$ in (2.5), we get the following integral representation for the H-function:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\left[ax+b(1-x)\right]^{-\alpha-\beta}$$

$$\times {}_2F_1\left[\frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}; \gamma; 4\frac{a\,b\,x\,(1-x)}{[ax+b(1-x)]^2}\right]$$

$$\times H_{p\ q}^{m\ n}\left[z\left\{\frac{ax}{b(1-x)}\right\}^{\lambda}\left|\begin{matrix}(1-\gamma+\beta,\lambda), & {}_2(a_j,e_j)_p\\(\gamma-\alpha,\lambda), & {}_2(b_j,f_j)_q\end{matrix}\right.\right]dx$$

$$=\frac{\Gamma(\gamma)\ \Gamma(\gamma-\alpha-\beta)}{a^{\alpha}\ b^{\beta}\ \Gamma(\alpha+\beta)}\ H_{p\ q}^{m\ n}\left[z\left|\begin{matrix}(1-\alpha,\lambda), & {}_2(a_j,e_j)_p\\(\beta,\lambda), & {}_2(b_j,f_j)_q\end{matrix}\right.\right] \qquad (3.1)$$

where $\lambda > 0$, $R(\gamma) > 0$, $R(\gamma-\alpha-\beta) > 0$, $R[\alpha+\lambda(b_j/f_j)] > 0$, $j = 2, \ldots m$, $R[\beta-\lambda(a_j-1/e_j)] > 0$, $j = 2, \ldots, n$, a and b are non-zero constants and the expression $[ax+b(1-x)]$, where $0 \leqslant x \leqslant 1$ is not zero; $\phi > 0$, $|\arg z| < \frac{\phi\pi}{2}$ where ϕ is given by

$$\phi = 2\lambda + \sum_{j=2}^{n} e_j - \sum_{j=n+1}^{p} e_j + \sum_{j=2}^{m} f_j - \sum_{j=m+1}^{q} f_j$$

(2). If we take $m = n = p = q = 2$, $e_2 = f_2 = 1$, $1-a_2 = \delta$, $b_2 = 0$, $\lambda = 1$ in (3.1) and use the result [5, p.363], we get the following interesting result:

$$\int_0^1 x^{\alpha-1}\ (1-x)^{\beta-1}\ [ax+b(1-x)]^{-\alpha-\beta}$$

$$\times\ {}_2F_1\left[\frac{\alpha+\beta}{2}, \frac{\alpha+\beta+1}{2}; \gamma; 4\ \frac{a\,b\,x\,(1-x)}{[ax+b(1-x)]^2}\right]$$

$$\times\ {}_2F_1\left[\gamma-\beta, \delta; 2\gamma-\alpha-\beta+\delta; 1-z\ \frac{ax}{b(1-x)}\right]dx$$

$$= \frac{\Gamma(\gamma)\ \Gamma(\alpha)\ \Gamma(\beta+\delta)\ \Gamma(\gamma-\alpha-\beta)\ \Gamma(2\gamma-\alpha-\beta+\delta)}{a^{\alpha}\ b^{\beta}\ \Gamma(\gamma-\beta)\ \Gamma(\alpha+\beta+\delta)\ \Gamma(\gamma-\alpha+\delta)\ \Gamma(2\gamma-\alpha-\beta)}\ {}_2F_1\,(\alpha\,,\ \delta\,;\ \alpha+\beta+\delta\,;\ 1-z)$$

(3.2)

where $R(\gamma) > 0$, $R(\alpha) > 0$, $R(\beta+\delta) > 0$, $R(\gamma-\alpha-\beta) > 0$, $|\arg z| < 2\pi$, provided $2\gamma-\alpha-\beta$, $\gamma-\alpha+\delta \neq 0,-1,-2,\ldots$, a and b are non-zero constants and the expression $[ax+b(1-x)]$, where $0 \leqslant x \leqslant 1$ is not zero.

(3). Taking $m = 1$, $n = p = q = 2$, $e_2 = \delta_2 = 1$, $1-a_2 = u$, $1-b_2=v$, $\lambda = 1$ in (3.1) , use the result [2, 3 p.439], we get the following interesting result:

$$\int_0^1 x^{\gamma-1}\ (1-x)^{\alpha+\beta-\gamma-1}\ [ax+b(1-x)]^{-\alpha-\beta}$$

$$\times\ {}_2F_1\left[\frac{\alpha+\beta}{2}\,,\ \frac{\alpha+\beta+1}{2}\,;\ \gamma\,;\ 4\ \frac{a\,b\,x\,(1-x)}{[ax+b(1-x)]^2}\right]$$

$$\times\ {}_2F_1\left[2\gamma-\alpha-\beta\,,\ \gamma-\alpha+u\,;\ \gamma-\alpha+v\,;\ -z\ \frac{ax}{b(1-x)}\right] dx$$

$$= \frac{\Gamma(\gamma)\ \Gamma(\beta+u)\ \Gamma(\gamma-\alpha-\beta)\ \Gamma(\gamma-\alpha+v)}{a^{\alpha}\ b^{\beta}\ \Gamma(\beta+v)\ \Gamma(2\gamma-\alpha-\beta)\ \Gamma(\gamma-\alpha+u)}\ z^{\alpha+\beta-\gamma}\ {}_2F_1\,(\alpha+\beta\,,\ \beta+u\,;\ \beta+v\,;\ -z)$$

(3.3)

where $R(\gamma) > 0$, $R(\alpha) > 0$, $R(\beta+u) > 0$, $R(\gamma-\alpha-\beta) > 0$, $|\arg z| < \pi$, a

and b are non-zero constants and the expression $[ax + b(1-x)]$, where $0 \leqslant x \leqslant 1$ is not zero.

This paper is a generalization of the paper recently given by me [6].

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