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A GENERATING FUNCTION FOR GENERALIZED HYPERGEOMETRIC POLYNOMIAL

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P.C.Munot - P.A. Padmanabham
Department of Mathematics
University of Jodhpur
Jodhpur - 342001
India.

ABSTRACT

The object of the present paper is to establish a general expansion formula, which provides us a generating function for the generalized hypergeometric polynomials.

RESUMEN

El objeto del presente trabajo, es el de establecer una fórmula de expansión, que nos da una función generadora para los polinomios generalizados hipergeométricos.

1. INTRODUCTION

In what follows (a) shall denote the sequence of a parameters a_1, $a_2, \ldots, a_A$, with a similar interpretation for (b), (c'),(d') etc.,whereas $[(a)]_n$ denotes the product

$$\prod_{J=1}^{A} (a_J)_n$$

with the usual meaning of the Pochhamer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1 & : \text{if } n=0 \\ a(a+1) \ldots (a+n-1) & : \text{if } n=1,2,\ldots,n \end{cases}$$

Srivastava ([7], p.76) has obtained a generating function for the generalised hypergeometric polynomial given by,

$$\sum_{n=o}^{\infty} \frac{[(a)]_n [(g)]_n}{[(b)]_n [(h)]_n} \; {}_{C+H+1}F_{D+G} \begin{bmatrix} -n, 1-(h)-n, (c): \\ 1-(g)-n, (d): \end{bmatrix} x \; \frac{t^n}{n!} \tag{1.1}$$

$$= F \begin{bmatrix} (a):(g);(c); \\ (b):(h);(d); \end{bmatrix} t, (-1)^{G-H+1} xt \Bigg]$$

where the function F on the right of (1.1) is the Kampé de Fériet function of two variables ([1],p.150)in the notation of Burchnall and Chaundy([2],p. 112).

Srivastava and Daoust([6],p.454)in an attempt to give a generalization of an expansion formula of the generalized Kampé de Fériet function defined by the following multiple hypergeometric series (generalized Lauricella's hypergeometric series)

$$F\begin{matrix}A:C';\dots;C^{(n)}\\B:D';\dots;D^{(n)}\end{matrix}\left[\begin{matrix}[(a):\theta',\dots,\theta^{(n)}]:[(c'):\psi'];\dots\\ [(b):\phi',\dots,\phi^{(n)}]:[(d'):\delta'];\dots\end{matrix}\right. \tag{1.2}$$

$$\left.\begin{matrix};[(c^{(n)}):\psi^{n}];\\;[(d^{(n)}):\delta^{n}];\end{matrix}\; x_1,x_2,\dots,x_n\right]$$

$$=\sum_{m_1\dots m_n=0}^{\infty}\frac{\prod_{J=1}^{A}(a_J)_{m_1\theta_j^1+\dots+m_n\theta_j^n}\prod_{J=1}^{C'}(c'_J)_{m_1\psi_j'}\dots}{\prod_{J=1}^{B}(b_J)_{m_1\phi_j^1+\dots+m_n\phi_j^1}\prod_{J=1}^{D'}(d_s')_{m_1\delta_J'}\dots}$$

$$\frac{\prod_{J=1}^{C^{(n)}}(c_J^{(n)})_{m_1\psi_J^{(n)}}}{\prod_{J=1}^{D^{(n)}}(d_j^{(n)})_{m_n\delta_J^{(n)}}}\frac{x_1^{m_1}}{m_1!}\dots\frac{x_n^{m_n}}{m_n!}$$

The object of the present paper is to establish a generalisation of the formula (1.1) in the form:

$$\sum_{n=o}^{\infty}\frac{[(a)]_n\,[(g)]_n}{[(b)]_n\,[(h)]_n}F\begin{matrix}A:C^1\dots C^{(r)}\\B:D^1\dots D^{(r)}\end{matrix}\left[\begin{matrix}[(a)+n:\theta_1',\dots,\theta^{(r)}]:\\ [(b)+n:\phi_1',\dots,\phi^{(r)}]:\end{matrix}\right. \tag{1.3}$$

$$\left.\begin{matrix}[(c'):\psi'];\dots;[(c^{(r)}):\psi^{(r)}]\\ [(d'):\delta'];\dots;[(d^{(r)}):\delta^{(r)}]\end{matrix}\; z_1,z_2,\dots,z_r\right]$$

$$ {}_{C+H+1}F_{D+G}\left[\begin{matrix}-n,1-(h)-n,(c): \\ 1-(g)-n,(d):\end{matrix} x\right]\frac{t^n}{n!} $$

$$ = F^{A:C^1,\ldots,C^{(r)},G,C}_{B:D^1,\ldots,D^{(r)},H,D}\left[\begin{matrix}[(a):\theta',\ldots,\theta^{(r)},1,1]:[(c'):\psi'];\ldots \\ [(b):\phi',\ldots,\phi^{(r)},1,1]:[(d'):\delta'];\ldots\end{matrix}\right. $$

$$ \left.\begin{matrix};[(c^{(r)}):\psi^{(r)}];[(g):1]\ [(c):1] \\ ;[(d^{(r)}):\delta^{(r)}];[(h):1]\ [(d):1]\end{matrix}\ z_1,\ldots,z_r,t,(-1)^{G-H+1}xt\right] $$

The formula(1.3) is not only a generalisation of the result (1.1) but is also a generalisation of many others such as Saxena([5],p.345),Manocha ([4],p.457).

2. PROOF OF THE FORMULA (1.3)

The left hand member of (1.3) can be written as

$$ \sum_{m_1,\ldots,m_r,n=o}^{\infty}\frac{\prod_{j=1}^{A}(a_j)_{n+\sum_{i=1}^{r}m_i\theta_j^i}[(g)]_n\prod_{j=1}^{C'}(c'_j)_{m_1\psi_j'}\cdots\prod_{j=1}^{C^{(r)}}(c_j^{(r)})_{m_r\psi_j^{(r)}}}{\prod_{j=1}^{B}(b_j)_{n+\sum_{i=1}^{r}m_i\phi_j^i}\prod_{j=1}^{D'}(d'_j)_{m_1\delta_j^1}\cdots\prod_{j=1}^{D^{(r)}}(d_j^{(r)})_{m_r\delta_j^{(r)}}[(h)]_n} $$

$$ {}_{C+H+1}F_{D+G}\left[\begin{matrix}-n,1-(h)-n;(c); \\ 1-(g)-n;(d);\end{matrix} x\right]\frac{z_1^{m_1}}{m_1!}\frac{z_2^{m_2}}{m_2!},\ldots\frac{z_r^{m_r}}{m_r!}\frac{t^n}{n!} $$

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j)_{\sum_{i=1}^{r} m_i\theta_j^i} \prod_{j=1}^{C'} (c'_j)_{m_1\psi'_j} \cdots \prod_{j=1}^{C^{(r)}} (c_j^{(r)})_{m_r\psi_j^{(r)}}}{\prod_{j=1}^{B} (b_j)_{\sum_{i=1}^{r} m_i\phi_j^i} \prod_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{m_r\delta_j^{(r)}}}$$

$$\frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j+\sum_{i=1}^{r} m_i\theta_j^i)_n [(g)]_n}{\prod_{j=1}^{B} (b_j+\sum_{i=1}^{r} m_i\phi_j^i)_n [(h)]_n}$$

$$_{C+D+1}F_{D+G} \left[\begin{matrix} -n, 1-(h)-n; (c); \\ \\ 1-(g)-n; (d); \end{matrix} \; x\right] \frac{t^n}{n!}$$

which in view of (1.1) becomes

$$= \sum_{m_1,\ldots,m_r=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j)_{\sum_{i=1}^{r} m_i\theta_j^i} \prod_{j=1}^{C'} (c'_j)_{m_1\psi'_j} \cdots \prod_{j=1}^{C^{(r)}} (c_j^{(r)})_{m_r\psi_j^{(r)}}}{\prod_{j=1}^{B} (b_j)_{\sum_{i=1}^{r} m_i\phi_j^i} \prod_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{m_r\delta_j^{(r)}}}$$

$$F \left[\begin{matrix} (a_j+\sum_{i=1}^{r} m_i\theta_j^i):(g);(c); \\ \\ (b_j+\sum_{i=1}^{r} m_i\phi_j^i):(h);(d); \end{matrix} \; t, (-1)^{G-H+1}xt\right] \frac{x_1^{m_1}}{m_1!} \cdots \frac{x_r^{m_r}}{m_r!}$$

$$= \sum_{m_1,\ldots,m_r,p,q=0}^{\infty} \frac{\prod_{j=1}^{A} (a_j)_{p+q+\sum_{i=1}^{r} m_i\theta_j^i} \prod_{j=1}^{C'} (c'_j)_{m_1\psi'_j} \cdots \prod_{j=1}^{C^{(r)}} (c_j^{(r)})_{m_r\psi_j^{(r)}}}{\prod_{j=1}^{B} (b_j)_{p+q+\sum_{i=1}^{r} m_i\phi_j^i} \prod_{j=1}^{D'} (d'_j)_{m_1\delta'_j} \cdots \prod_{j=1}^{D^{(r)}} (d_j^{(r)})_{m_r\delta_j^{(r)}}}$$

$$\frac{[(g)]_p[(c)]_q}{[(h)]_p[(d)]_q} \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!} \frac{t^p}{p!} \frac{[(-1)^{G-H+1}xt]^q}{q!}$$

whence on interpreting this multiple series as the generalized Lauricella's function of (r+2) variable,

$$z_1, z_2, \ldots, z_r, t, (-1)^{G-H+1}xt$$

provides us the right member of (1.3).

3. PARTICULAR CASES.

It can be easily seen that if the positive constants θ's, ϕ's, ψ's and δ's are all chosen as unity then, the generalized Lauricella's hypergeometric function $F\,(z_1, z_2, \ldots, z_r)$ of r variable may be made to reduces to $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ ([3], p.113), viz.

$$F^{1:1,1,\ldots,1}_{0:1,1,\ldots,1}(z_1, z_2, \ldots, z_r) \tag{3.1}$$

$$= F_A^{(r)}(a:c', c'', \ldots, c^{(r)}; d', d'', \ldots, d^{(r)}; z_1, z_2, \ldots, z_r)$$

$$F^{0:2;\ldots;2}_{1:0;\ldots;0}(z_1, z_2, \ldots, z_r) \tag{3.2}$$

$$= F_B^{(r)}(c_1, c_1' : c_2, c_2'; \ldots, c_r, c_r'; b; z_1, z_2, \ldots, z_r)$$

$$F^{2:0;\dots;0}_{0:1;\dots;1}(z_1,z_2,\dots,z_r) = F_C^{(r)}(a,b:-;d',d'',\dots,d^{(r)};z_1,z_2,\dots,z_r) \tag{3.3}$$

and

$$F^{1;1,\dots,1}_{1;0,\dots,0}(z_1,z_2,\dots,z_r) = F_D^{(r)}(a:c',c'',\dots,c^{(r)};b;z_1,z_2,\dots,z_r) \tag{3.4}$$

Thus on setting $A=C'=\dots C^{(r)}=D'=\dots D^{(r)}=C=D=G=H=1$; $g=-\alpha, h=-\alpha-\beta; B=0$ in (1.3) we arrive at the elegant formula

$$\sum_{n=o}^{\infty} \frac{(a)_n}{(-\alpha-\beta)_n} F_A\,[a+n;(c_r):(d_r):(z_r)]\,H_n^{(\alpha-n,\beta-n)}(\rho,\sigma,x)t^n \tag{3.5}$$

$$= F_A[a;(c_r),-\alpha,\rho:(d_r),-\alpha-\beta,\sigma;(z_r),t,-xt]$$

where $H_n^{(\alpha,\beta)}(\rho,\sigma,x)$ on the left of (3.5) is the generalised Rices polynomial defined by

$$H_n^{(\alpha,\beta)}(\rho,\sigma,x) = \frac{(1+\alpha)_n}{n!}\,{}_3F_2\,[-n,1+\alpha+\beta+n,\rho,1+\alpha,\sigma;x]$$

while on setting $A = D' = \dots = D^{(r)} = 0$, $B = C = D = G = H = 1$, $C' = \dots = C^{(r)} = 2$, $g = -\alpha$, $h = -\alpha - \beta$ in (1.3), we obtain,

$$\sum_{n=o}^{\infty} \frac{1}{(b)_n(-\alpha-\beta)_n} F_B\,[(c_r),(c_r');b+n;(z_r)]\,H_n^{(\alpha-n,\beta-n)}(\rho,\sigma,x)t^n \tag{3.6}$$

$$= F^{0:2,2,\ldots,2,1,1}_{1:0,0,\ldots,0,1,1}\left[\begin{matrix} - \\ [(b):1,1,\ldots,1,1,1]\end{matrix} : [(c_1):1],[(c_1'):1],\ldots\right.$$

$$\left.\begin{matrix}[(c_r):1] & [(c_r'):1] & [-\alpha:1] & [\rho:1] \\ - & - & [-\alpha-\beta:1] & [\sigma:1]\end{matrix} z_1,\ldots z_r, t_1 - xt\right]$$

If we let $A=2, B=C'=\ldots=C^{(r)}=0, D'=\ldots=D^{(r)}=C=D=G=H=1, g=-\alpha, h=-\alpha-\beta$ in (1.3), we get,

$$\sum_{n=o}^{\infty} \frac{(a_1)_n (a_2)_n}{(-\alpha-\beta)_n} F_C\,[a_1+n, a_2+n; (d_r); (z_r)]\; H_n^{(\alpha-n,\beta-n)}(\rho,\sigma,x)t^n \qquad (3.7)$$

$$= F^{2:0,0,\ldots,0,1,1}_{0:1,1,\ldots,1,1,1}\left[\begin{matrix}[(a_1):1,1,\ldots,1,1,1]\ ,\ [(a_2):1,1,1,\ldots,1,1,1] : \\ - \qquad :\end{matrix}\right.$$

$$\left.\begin{matrix} - & \ldots & - & [-\alpha:1] & [\rho:1] \\ [(d'):1] & \ldots & [(d^{(r)}):1] & [-\alpha-\beta:1] & [\sigma:1]\end{matrix}\ (z_r), t, -xt\right]$$

Lastly on setting $A=B=C'=\ldots=C^{(r)}=C=D=H=G=1, D'=\ldots=D^{(r)}=0, g=-\alpha, h=(-\alpha-\beta)$, we get,

$$\sum_{n=o}^{\infty} \frac{(a)_n}{(b)_n(-\alpha-\beta)_n} F_D\,[(a+n);(c_r);(b+n);(z_r)]H_n^{(\alpha-n,\beta-n)}(\rho,\sigma,x)t^n \qquad (3.8)$$

$$= F^{1:1:1,\ldots,1,1,1}_{1:0,0,\ldots,0,1,1}\left[\begin{matrix}[(a):1,1,\ldots,1,1,1] : [(c_1):1];\ldots;[(c_r):1] \\ [(b):1,1,\ldots,1,1,1] : \quad - \quad ; - ; \quad -\end{matrix}\right.$$

$$\left.\begin{matrix}[-\alpha:1] & [\rho:1] \\ [-\alpha-\beta:1] & [\sigma:1]\end{matrix}\ z_1, z_2, \ldots, z_r, t, -xt\right]$$

It is well known that

$$H_n^{(\alpha,\beta)}(\rho,\rho,x) = P_n^{(\alpha,\beta)}(1-2x)$$

where $P_n^{(\alpha,\beta)}(x)$ is the Jacobi Polynomial, thus setting $\sigma=\rho$ in (3.5), we get

$$\sum_{n=o}^{\infty} \frac{(a)_n}{(-\alpha-\beta)_n} F_A\left[a+n;(c_r);(d_r);(z_r)\right] P_n^{(\alpha-n,\beta-n)}(x)t^n \tag{3.9}$$

$$= (w)^{-a} F_A\left[a;(c_r);(-\beta);(-\alpha-\beta),(d_r);\frac{t}{w},\frac{(z_r)}{w}\right]$$

where $w = \left[1+\frac{1}{2}(x+1)t\right]$

In particular, by setting $r = 2$ in (3.9) we have

$$\sum_{n=o}^{\infty} \frac{(a)_n}{(-\alpha-\beta)_n} F_2(a+n;c_1,c_2;d_1,d_2;z_1,z_2) P_n^{(\alpha-n,\beta-n)}(x)t^n$$

$$= (w)^{-a} F_A\left[a,-\beta,c_1,c_2;-\alpha-\beta,d_1,d_2,\frac{t}{w},\frac{z_1}{w},\frac{z_2}{w}\right]$$

which was given earlier by Manocha ([4],p.457).

It is easy to see that for $A=2,B=C'=\ldots=C^{(r)}=0$, $D'=\ldots D^{(r)}=1,C=D-1=0,G=H-1=0,d_1=1+\beta,h_1=1+\alpha$, the result (1.3) will reduce to the generating function given by Saxena [(5),p.345] i.e.

$$\sum_{n=o}^{\infty} \frac{(\gamma)_n(\delta)_n}{(1+\alpha)_n(1+\beta)_n} F_C\,(\gamma+n,\delta+n;d_1\ldots d_r;z_1,\,.\,.\,,z_r) P_n^{(\alpha,\beta)}(x)t^n \tag{3.10}$$

$$=F_C^{(r+2)}\left[\gamma,\delta;d_1\ldots d_r,1+\alpha,1+\beta;z_1,\ .\ ,z_r\ \frac{1}{2}(x-1)t,\frac{1}{2}(x+1)t\right].$$

Finally, we remark that since ultrspherical polynomials,Gengenbauer polynomials, Legendre, Laguerre and various other polynomials are only either special cases or limiting forms of the Jacobi polynomials, generating functions of these related polynomials can be deduced easily from our general formula (1.3).

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