Rev. Téc. Ing., Univ. Zulia
Vol. 5, Nº 2, 1982

REDUCIBLE H-FUNCTIONS OF TWO VARIABLES

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ABSTRACT

A reduction formula for certain H-functions of two variables to H-functions of one variable is obtained. A special case is the known reduction formula for Appell's functions F_2 to the Gauss function ${}_2F_1$.

RESUMEN

Se obtuvo una fórmula de reducción para ciertas funciones H de dos variables a funciones H de una sola variable. Un caso especial es la conocida fórmula de reducción para funciones F_2 de Appell a la función ${}_2F_1$ de Gauss.

The terminology reducible is taken in the sense used in {5} for hypergeometric functions of two variables; we mean simply that the H-functions of two variables can be expressed in terms of simpler functions for special cases of the parameters. Simple reduction formulas which occur because of matched parameters are well known.
Cases for which factorization can be made into a product of H-functions of one variable are known, for example, see {1}, {3} and {8}. Actually factorizations become more interesting in the case of N-variables, see {4}. Here we present a case of reducibility of an H-function of two variables to an H-function of one variable. This contains a known result involving Appell's function F_2. Reduction formulas for Appell's functions and their generalizations can be found, for example, in {5} and {7}.

The notations used follow {2} and {3}. We are assuming that α_k and β_k are complex; a_k and b_k, real but not zero.

AMS(MOS) subject classification (1980). 33A35

Key words and phrases. H-functions, reduction formulas.

The definition of the H-function of two variables in terms of a double Mellin-Barnes integral as in {2} and {3} allows us to express the reducible case as follows

$$H\{x,y\} = H\{x,y;\ (cp-q,c,1),(q,0,-1),(\alpha_k+pa_k,a_k,0)_m;$$

$$(cp,c,0),(\beta_k+pb_k,b_k,0)_n;\ L_0,L_0\}$$

$$= (2\pi i)^{-2}\int_{-i\infty}^{+i\infty}\int_{-i\infty}^{+i\infty}\Big(\Theta(s+p)\Gamma(cp-q+cs+t)\Gamma(q-t)/$$

$$\Gamma(cp+cs)\Big)\ x^s\ y^t\ dt\ ds$$

where

$$\Theta(s) = \prod_{k=1}^{m}\Gamma(\alpha_k + a_k s)\Big/\prod_{k=1}^{n}\Gamma(\beta_k + b_k s)$$

Next, by use of formula 7.3(15) of {6}, we can evaluate the integral with respect to t, which leaves an integral in the form for an H-function of one variable. Thus

$$H\{x,y\} = (2\pi i)^{-1}\int_{-i\infty}^{+i\infty}\Theta(s+p)\ y^q\ (1+y)^{-c(s+p)}x^s\ ds$$

$$= x^{-p}\ y^q\ H\{x(1+y)^{-c};\ (\alpha_k,a_k)_m;\ (\beta_k,b_k)_n\}.$$

The following conditions are sufficient for validity. Let

$$m^+ = \min_{a_k>0}\ (\mathrm{Re}(\alpha_k)/a_k),\ M^- = \max_{a_k<0}\ (\mathrm{Re}(\alpha_k)/a_k).$$

The choice of the imaginary axes for integrations are then justified if $p > 0$, $M^- < -p < m^+$, $0 < q < cp$. If $K_x > c$ where

$$K_x = \sum_{k=1}^{m}|a_k| - \sum_{k=1}^{n}|b_k|,$$

the convergence conditions in {2} guarantee absolute convergence of the double integral at least for

$$|\arg(x)| < \pi(K_x - c)/2, \quad |\arg(y)| < \pi/2 .$$

In the notation of {9} the result (with superscripts dropped as unnecesary for this special case) reads

$$H^{0,1:\,(\mu,\nu)\;;\;(1,0)}_{1,0:\,(B,D+1);\;(1,0)}\left(\begin{matrix} (q+1-cp:\ c,1):\{1-\alpha_k-pa_k:\ a_k\}, \\ -\quad :\{\alpha_k+pa_k:\ -a_k\}, \end{matrix}\right.$$

$$\left.\begin{matrix} \{\beta_k+pb_k:\ -b_k\}\quad ;\quad -\quad ; \\ \{1-\beta_k-pb_k:\ b_k\},(1-cp:\ c);\ (q:\ 1); \end{matrix}\ x,y\right) =$$

$$= x^{-p}y^{q}H^{\mu,\nu}_{B,D}\left[\ x(1+y)^{-c}\left|\begin{matrix}(1-\alpha_k,a_k),(\beta_k,-b_k)\\(\alpha_k,-a_k),(1-\beta_k,b_k)\end{matrix}\right.\right] ,$$

where we use the common notation for the H-function of one variable.

The corresponding reducible case for the G-function of two variables ($a_k = b_k = 1$) does not seem to be in the literature. By a judicious choice of parameters and some indentation of integration paths it can be shown that a special case of the result is the known reduction formula for the Appell function.

$$F_2(\alpha,\beta,\beta',\gamma,\beta';-x,-y) = (1+y)^{-\alpha}{}_2F_1(\alpha,\beta';\gamma';$$
$$-x(1+y)^{-1}),$$

see, for example, formula 5.10(2) of {5}. It is not clear as to whether or how the other known reducible cases for Appell's functions generalize. Generalizations which involve H-functions of more than two variables, as defined in {4}, can also be obtained.

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Recibido el 24 de octubre de 1980