Rev. Téc. Ing., Univ. Zulia

NOTA TECNICA

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INTEGRATION OVER A HYPERCUBE

*(Al Libertador Simón Bolívar,
en el bicentenario de su nacimiento)*

The major difficulty one usually encounters in evaluating iterated integrals is the determination of the limits of integration.

$$\text{Evaluate} \int\limits_{\substack{0 \le x_1x_2 + x_3x_4 \le 1, \\ 0 \le x_k \le 1,\ k= 1,2,3,4}} x_1x_2^2x_3^2x_4^2 \; dx_1 dx_2 dx_3 dx_4$$

In order to get an idea on how to determine just what portions of the hypercube $0 \le x_k \le 1$, k=1, ,2,3,4 constitute the region of integration, consider an analogous problem in ordinary three-space using x_1, x_2, x_3 for x, y, z. Then we will do the four-space problem by analogy. So, as a preliminary problem evaluate

$$\iiint_T x_1x_2^2x_3^2 \; dV$$

where T denotes the region defined by

$$T: \begin{cases} 0 \le x_1x_2 + x_1x_3 \le 1 \\ 0 \le x_k \le 1,\ k= 1,2,3 \end{cases}$$

The region T is some portion (or portions) of the cube C: $0 \le x_k \le 1$, k= 1,2,3. To find out what portion or portions of C make up the region T, let us first put x_1 at its maximum possible value of 1 Thus the intersection of the upper plane of the cube C by the surface S: $x_1x_2 + x_1x_3= 1$ gives the line L: $x_1= 1$, $x_2 + x_3= 1$ as in Fig. I.

The line L is obtained by taking x_1 at its maximum possible value. We see, therefore, that the entire half of the cube lying on the same side of the plane $x_2 + x_3= 1$ as the origin makes up part of the region of integration T, namely, all those points in space for which $0 \le x_1 \le 1$, $0 \le x_2+x_3 \le 1$.

There remains the question: What part of the other half of cube C belongs to T? To obtain the answer to this question one must observe that in this half of cube C we must have $x_2 + x_3 > 1$. This means that x_1 must now be sufficiently less than unity as will satisfy the requirement $x_1x_2+x_1x_3 \le 1$, wich in turn means that we can admit only those points of the cube C that lie on or below the surface S: $x_1x_2 + x_1x_3= 1$ as shown in Fig. II.

Thus, the region of integration T is made up of the prism resting on the triangle POQ in Fig. I

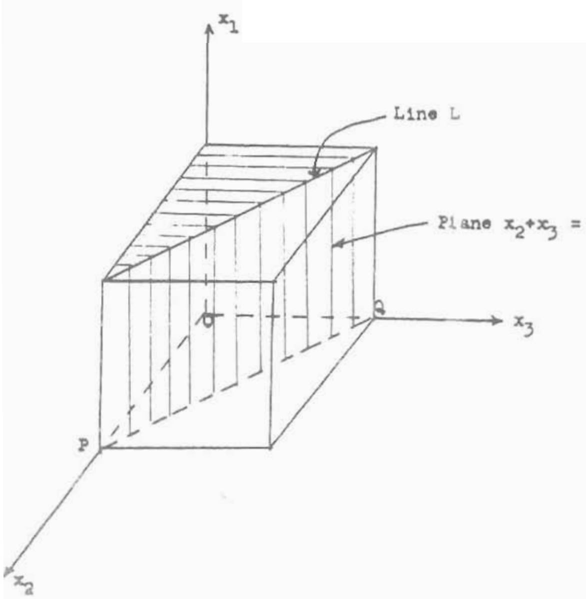

Fig. I

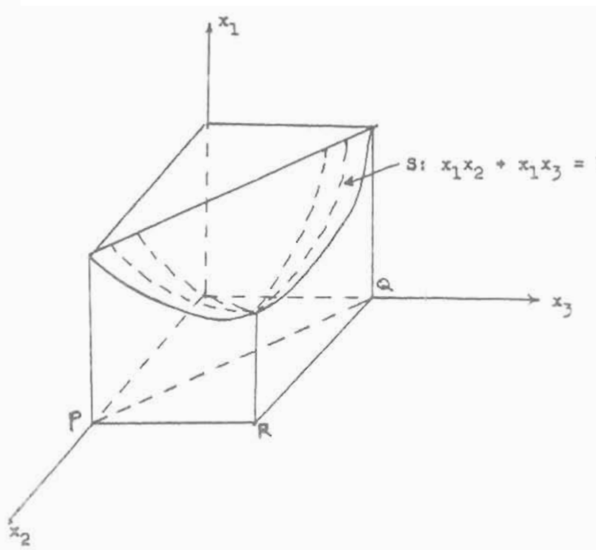

Fig. II

together with that much of the other prism wich lies below the surface S and wich rests upon triangle PRQ. And we have

$$\iiint_T x_1 x_2^2 x_3^2 \, dV = \int_0^1 x_3^2 \, dx_3 \int_0^{1-x_3} x_2^2 \, dx_2 \int_0^1 x_1 \, dx_1$$
$$+ \int_0^1 x_3^2 \, dx_3 \int_{1-x_3}^1 x_2^2 \, dx_2 \int_0^{1/(x_2+x_3)} x_1 \, dx_1 \qquad \text{...Eq. 1}$$

Now using the preceding example as an analogy, we seek to evaluate

$$\iiiint_R x_1 x_2^2 x_3^2 x_4^2 \, dW$$

where dW denotes the rectangular element in Cartesion 4-space and where R denotes the region of the 4-dimensional hyperspace defined by

$$R: \begin{cases} 0 \le x_1 x_2 + x_3 x_4 \le 1 \\ 0 \le x_k \le 1, \; k = 1,2,3,4 \end{cases}$$

For convenience let us denote the hypercube $0 \le x_k \le 1$ by H. And now, guided by the 3-dimensional example, we proceed to determine what portions of H constitute the region R.

The hypersurface $x_2 + x_3 x_4 = 1$, obtained by putting $x_1 = 1$, is analogous to the plane $x_2 + x_3 = 1$ which divided the cube C in the 3-space problem into two prisms. Similarly, the hypersurface $x_2 + x_3 x_4 = 1$ divides the hypercube H into two parts. Let us see how much of each part belongs to R. Incidentally, we can actually picture $x_2 + x_3 x_4 = 1$ in a 3-space diagram as in Fig. III.

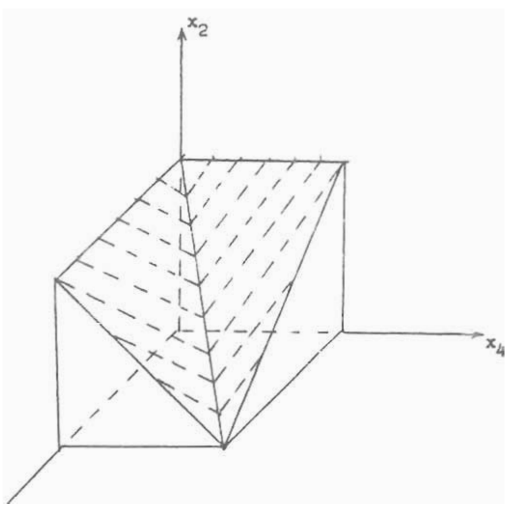

Fig. III

working with arguments similar to those to obtain the prism in Fig. I as part of region T, we conclude that one portion of R is made up of those points of H for wich $0 \le x_2 + x_3 x_4 \le 1$, $0 \le x_1 \le 1$.

Next for the points of H where we have $x_2 + x_3 x_4$ greater than unity, we must take x_1 sufficiently less than unity as will satisfy $x_1 x_2 + x_3 x_4 \le 1$. The question is: How much less than unity? The immediate answer is x_1 can have any value for which $0 \le x_1 \le (1 - x_3 x_4)/x_2$.

In the preceding 3-space problem the last integral to be evaluated (farthest to the left) in each of the two iterated integrals in Eq. 1 had zero to unity as its limits of integration. Similarly, the last two integrals in the problem before us and also the next to last will have both zero to unity as limits of integration. And so we have

$$\iiiint_R x_1 x_2^2 x_3^2 x_4^2 \, dW = \int_0^1 x_4^2 \, dx_4 \int_0^1 x_3^2 \, dx_3 \int_0^{1-x_3x_4} x_2^2 \, dx_2 \int_0^1 x_1 \, dx_1 + \int_0^1 x_4^2 \, dx_4$$
$$\int_0^1 x_3^2 \, dx_3 \int_{1-x_3x_4}^1 x_2^2 \, dx_2 \int_0^{(1-x_3x_4)/x_2} x_1 \, dx_1 \qquad \text{...Eq. 2}$$

When evaluating $\iiiint x_1 x_2^2 x_3^2 x_4^2 \, dx_1 dx_2 dx_3 dx_4$ the inner integral is evaluated first. Thus after integrating with respect to x_1 in the first of the two iterated integrals on the right hand side of Eq. 2 we get

$$\iiint \frac{1}{2} x_2^2 x_3^2 x_4^2 \, dx_2 dx_3 dx_4$$

The second integration with respect to x_2 yields

$$\iint \frac{1}{2} \cdot \frac{1}{3} (1 - x_3 x_4)^3 x_3^2 x_4^2 \, dx_3 dx_4$$

After removing the parenthesis and integrating with respect to x_3 and x_4 with limits zero and unity of integration in each case, we obtain (1/2) (1/3)(1/9 - 3/16 + 3/25 - 1/36)= 19/7200.

The second set of iterated integrals in Eq. 2 yields 37/7200. Finally, (19 + 37) / 7200= 7/900 = =.007777... . It seems as if there is no way of verifying the above result by classical analysis. An excellent project for numerical analyst to work on is the verification of the result by a computer program.

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