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O.I. Marichev, Vu Kim Tuan
Byelorussian State University
Minsk, U.S.S.R.

THE PROBLEMS OF DEFINITIONS AND SYMBOLS OF G- AND H-FUNCTIONS OF SEVERAL VARIABLES

*(Al Libertador Simón Bolívar,
en el bicentenario de su nacimiento)*

ABSTRACT

In the present paper some problems concerned with the notations used in the theory of G- and H-functions of several variables are considered. The G- and H-functions of several variables are defined and the definitions include all the known ones of the type. For designation of these functions some symbols are suggested, which are very simple or complète for description of the function dependence on a large number of parameters. In one of the symbols ternary calculus is used. Different particular cases of G- and H-functions are examined, sufficient conditions of their existence and some of their properties are presented, and for G-functions the appropriate partial differential equations and a certain set of its solutions are given.

RESUMEN

Este trabajo se trata del problema de notaciones en la teoría de las funciones G y H. Se define una función H de n - variables. Se sugiere algunos símbolos para la descripción completa de estas funciones. Se menciona algunos casos pariculares, las condiciones suficientes para su existencia y ciertas propiedades de estas funciones. Se da la ecuación diferencial en dirivadas parciales para la función G.

1. INTRODUCTION

The most important and urgent problem in the theory of general hypergeometric functions is the development of the theory for H-functions of many variables. For the case of Fox's H-function of one variable, defined in [8], such a theory is successfully developed and presented in detail in [1,5,12, 13]. In those monographs many applications of the H-function and its particular case, Meijer's G-function, to physics, statistics and the integration theory are given. In particular the integration algorithm, developed in [1], may be used for calculating more than a half of the single integrals from [9] in terms of hypergeometric series. It was employed for preparation of Handbook [2] for computing the integrals of different complexities. If the theory of the H-function of many variables were developed, it would be possible to extend this algorithm to multiple integrals of the products of functions of hypergeometric type.

Up till now some works have been published where analogues of Fox's H-functions are composed for two [4, 14, 15, 21], three [11, 16, 17], four [7] and n variables [3, 10, 18 -21]. Those definitions are not however general and have no common notations, which would be most suitable for the structure of the H-fuction of many variables. So, in particular, the symbols for H-function of two variables, given in [4,14,15], are not the most general since 23 from 34 double series of Horn's list [5] cannot be designated by those notations. But the series reflect Pochhammer's symbols $(a)_n$ and are therefore of hypergeometric type. The functions of several variables [6] $C_n^{(k)}(b_1,\ldots, b_n, a, a'; x_1,\ldots,x_n)$, $D_{(n)}^{p,q}(a, b_1,\ldots, b_n; c, c'; x_1,\ldots, x_n)$ are also of the hypergeometric type but they cannot be expressed in terms of the symbols of H-function of n variables from [3, 18 - 21].

The principal difficulty in the development of the most general and convenient symbols for designation of the H-function of many variables is to find the optimal description in the defining symbol the dependence of this function on a large number of parameters, which are grouped in a certain way.

For example, Meijer's G-function, introduced by Meijer in 1941, is defined in terms of the following Mellin-Barnes integral over the special contour

$$\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)} z^s ds \qquad \ldots(1)$$

It will be denoted by

$$G_{pq}^{mn}\left(z \,\middle|\, {a_1,\ldots, a_p \atop b_1,\ldots, b_q}\right),\ G_{pq}^{mn}\left(z \,\middle|\, {a_r \atop b_s}\right),\ G_{pq}^{mn}(z) \text{ or } G(z) \qquad \ldots(2)$$

The symbols more or less completely describe the dependence of the function on four parameter groups.

It seems more simple and convenient to use Slater's designation of the G-function in terms of the integral

$$I_L(z) = \frac{1}{2\pi i}\int_L \Gamma\left[{(a) + s,\ (b) - s \atop (c) + s,\ (d) - s}\right] z^{-s} ds \qquad (3)$$

Here and in the following symbol $\Gamma(\lambda)$ denotes the ratio of the products of the appropriate gamma-functions, whose arguments in the symbol are separated by commas

$$\Gamma\begin{bmatrix}(a)+s,\ (b)-s\\(c)+s,\ (d)-s\end{bmatrix}=\frac{\prod_{i=1}^{A}\Gamma(a_i+s)\prod_{i=1}^{B}\Gamma(b_i-s)}{\prod_{i=1}^{C}\Gamma(c_i+s)\prod_{i=1}^{D}\Gamma(d_i-s)}\qquad\ldots(4)$$

In accordance with the following the integral $I_L(z)$ could be denoted by any of the following symbols, which are more illustrative

$$\begin{matrix}A & B\\ & G\\ C & D\end{matrix}[z]=\begin{matrix}A & B\\ & G\\ C & D\end{matrix}\left[z\left|\begin{matrix}(a_A)\ ;\ (b_B)\\(c_C)\ ;\ (d_D)\end{matrix}\right.\right]=$$

$$=\begin{matrix}A & B\\ & G\\ C & D\end{matrix}\left[z\left|\begin{matrix}(a);(b)\\(c);(d)\end{matrix}\right.\right]=$$

$$=\begin{matrix}A & B\\ & G\\ C & D\end{matrix}\left[z\left|\begin{matrix}a_1,\ldots,a_A\ ;\ b_1,\ldots,b_B\\c_1,\ldots,c_C\ ;\ d_1,\ldots,d_D\end{matrix}\right.\right]\ldots(5)$$

It is evident that

$$G_{pq}^{mn}\left(z\left|\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}\right.\right)=$$

$$\begin{matrix}m & & n\\ & G & \\ p-n & & q-m\end{matrix}\left[z\left|\begin{matrix}b_1,\ldots,bm;1-a_1,\ldots,1-a_n\\a_{n+1},\ldots,a_p;\ 1-b_{m+1},\ldots,1-b_q\end{matrix}\right.\right]\qquad\ldots(6)$$

But the traditional notations could hardly be changed and the changes seem unreasonable.

Fox's function, which is more general, differs from G-function (1) by the non-negative coefficients α_i, β_j at s and may be defined in a similar way using a symbol, which is also not very convenient

$$\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}z^s ds=H(z)=$$

$$=H_{pq}^{mn}(z)=H_{pq}^{mn}\left(z\left|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right.\right)$$

$$=H_{pq}^{mn}\left(z\left|\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right.\right)\qquad\ldots(7)$$

In this paper two approaches to the H-function of n variables and some symbols for designation of the function are suggested. The symbols may be used in certain cases and are most simple or complete for description of the dependence on the parameters. The notation from Section 2 gives a simple form of the convergence conditions for Mellin-Barnes' corresponding multiple integral and a system of partial differential equations, which is satisfied by the G-function, a particular case of the H-function. The notation from Sect. 4 is very specific and complete for description of the dependence on the parameters and is therefore bulky and requires careful treatment. But it could hardly be simplified because of extreme complexity and generality of the subject. Such a notation is useful for description of particular cases and properties of the H-function. A combination of the notations suggested may eliminate many difficulties in applications of the H-function.

The H-function defined here is the most general. With n=2 it contains all 34 series from Horn's list [5] and the H-functions of two variables defined in [4,14,15]. The functions from [6,7, 16-20] are also particular cases of this H-function, this fact is demonstrated in Sect. 3 by using the function from [18]. Other particular cases are considered here as an example (Sect. 3,5) and for one of them, G-function, the appropiate differential equations and some of their solutions are given. In Sect. 2 sufficient conditions for the H-function to exist are also written.

2. DEFINITION OF THE H-FUNCTION $\overset{n}{H}[z]$ AND G-FUNCTION $\overset{n}{G}[z]$

Definition 1. Any integral of the form

$$\overset{n}{H}[z]\equiv\frac{1}{(2\pi i)^n}\int_L\Gamma\begin{bmatrix}a+\alpha s\\b+\beta s\end{bmatrix}z^s ds\equiv\frac{1}{(2\pi i)^n}\int_{L_1}\overset{n}{\ldots}\int_{L_n}\frac{\prod_{j=1}^{A}\Gamma(a^j+\alpha^j s)}{\prod_{j=1}^{B}\Gamma(b^j+\beta^j s)}z^s ds,\qquad\ldots(8)$$

where $a^1,a^2,\ldots,a^A;b^1,b^2,\ldots,b^B$ are the arbitrary complex pairs (at B=0 the empty product from the denominator is substituted by unity); $\alpha^j=(\alpha_1^j,\ldots,\alpha_n^j)$, $j=\overline{1,A}$, $\beta^j=(\beta_1^j,\ldots,\beta_n^j)$, $j=\overline{1,B}$ are non-zero real vectors; $s=(s_1,\ldots,s_n)$, $z=(z_1,\ldots,z_n)$, $z^s=z_1^{s_1}\ldots z_n^{s_n}$, $ds=ds_1\ldots ds_n$; $\alpha^j s=\alpha_1^j s_1+\ldots+\alpha_n^j s_n$, $\beta^j s=\beta_1^j s_1+\ldots+\beta_n^j s_n$ are the scalar products of the vectors α^j and β^j by s, will be called the H-function of n variables. The arbitrary contours $L_1,\ldots,L_n$, extending infinitely at both sides, are supposed to meet the two following conditions: 1. The contours should be such ones that at certain constraints for the parameters and variables $z_1,\ldots,z_n$ integral (8) would converge. 2. The "separability condition" of the poles of the integrand function should be satisfied for every of the contours L_i to separate all the poles of $\prod_{\alpha_i^j<0}\Gamma(a^j+\alpha^j s)/\prod_{j=1}^{B}\Gamma(b^j+\beta^j s)$ from those of $\prod_{\alpha_i^j>0}\Gamma(a^j+\alpha^j s)/\prod_{j=1}^{B}\Gamma(b^j+\beta^j s)$ with the assumption that $s_k\in L_k$, $k\neq i$ ($\prod_{\alpha_i^j<0}$ denotes the product over all $j=\overline{1,A}$, for which $\alpha_i^j<0$).

Remark. The problem of finding all types of the conditions for the contours and parameters, at which integral (8) converges, is very complicated and has not been completely studied. It can be demonstrated that for (8) to converge it is sufficient that the curves between $-i$ and $+i$ over s_k are taken as contours L_k and the conditions

$$\sum_{j=1}^{A}|\alpha_i^j| > \sum_{j=1}^{B}|\beta_i^j|, \quad i=1,\dots,n, \qquad \dots(9)$$

$$|\arg z_i| < \frac{\pi}{2}\left(\sum_{j=1}^{A}|\alpha_i^j| - \sum_{j=1}^{B}|\beta_i^j|\right), \quad i=1,\dots,n, \qquad \dots(10)$$

are satisfied for the parameters. Then at (10) the function $\overset{n}{H}[z]$ will be analytical and, generally speaking, polyvalent.

Definition 2. The particular case of the H-function (8), when all the components of α^j, $j=\overline{1,A}$; β^j, $j=1,B$, could be 0, 1 or -1, will be referred to as G-function of n variables and designated by $\overset{n}{G}[z]$.

It must be noted that the G-function defined here generalizes all the earlier G-function from the other works.

It may be found that the G-function $\overset{n}{G}[z]=U(z)$ is one of the solutions of the following system of n equations

$$\Big[(-1)^{B_i} z_i \prod_{\alpha_i^j=1}(a^j+\alpha^j\delta)\prod_{\beta_i^j=1}(1-b^j-\beta^j\delta) - \prod_{\alpha_i^j=-1}(a^j+\alpha^j\delta)\prod_{\beta_i^j=1}(1-b^j-\beta^j\delta)\Big]U(z)=0, \quad i=\overline{1,n}, \qquad \dots(11)$$

where $B_i=\sum_{\beta_i^j\neq 0}1$, $\quad \delta=\left(x_1\frac{\partial}{\partial x_1},\dots,x_n\frac{\partial}{\partial x_n}\right)$,

which is of the order

$$\max_{1\le i\le n}\ \max\left(\sum_{\alpha_i^j=1}1+\sum_{\beta_i^j=-1}1,\ \sum_{\alpha_i^j=-1}+\sum_{\beta_i^j=1}1\right)$$

The functions of the form

$$U_C(z)=\frac{1}{(2\pi i)^n}\int\overset{n}{\dots}\int\frac{\prod_{j\in C}\Gamma(a^j+\alpha^j s)}{\prod_{j\in(A+B)/C}\Gamma(1-a^j-\alpha^j s)}z^s ds, \qquad \dots(12)$$

where $a^{j+A}=1-b^j$, $\alpha^{j+A}=-\beta^j$, $j=\overline{1,B}$, and C is an arbitrary non-empty subset from the set $(A+B)=\{1,2,\dots,A+B\}$, $(A+B)/C$ is its supplement to $(A+B)$, may be other solutions to the above system. In order that the functions (12) could be solutions of equations (11), it is necessary and sufficient that two conditions more should be satisfied: the sums $\sum_{j\in(A+B)/C,\ \alpha_i^j\neq0}1$ and B_i have the same evenness and the inequality

$$\sum_{j\in C}|\alpha_i^j| - \sum_{j\in(A+B)/C}|\alpha_i^j| > 0, \quad i=\overline{1,n}, \qquad \dots(13)$$

be valid, which provides for analiticity domain of the type (10) for $U_c(z)$ to exist. It can easily be seen that the number of the functions of the type (12) is within 2^{A+B}, the number of functions of the type (12) but with the condition (13) could not exceed 2^{A+B-1}. It is possible that the solutions (12) do not form a complete system of solutions of (11). Moreover, apart from equations (11), other independent equations can exist, which are satisfied by the function $\overset{n}{G}[z]$. For example, Appell's function F_1 from Horn's list [5] satisfies not only two equations of the type (11) but also the third independent equation, which is shown in the wellknown book by Appell and Kampé de Fériet (1926).

3. THE FUNCTION $\overset{n}{H_1}[z]$ AND ITS EXPRESSION BY SERIES

In a general case the problem of expressing the H-function in terms of the residue sum in the intergrand function poles, which is very important for the theory and applications, involves great difficulties, which cannot be overcome as yet. Here, we only consider the particular case of the H-function, denoted by $\overset{n}{H_1}[z]$. For this case such a problem in the regular (nonlogarithmic) case is solvable. Let

$$\overset{n}{H_1}[z]=\frac{1}{(2\pi i)^n}\int_{L_1}\overset{n}{\dots}\int_{L_n}\frac{\prod_{j=1}^{A}\Gamma(a^j+\alpha^j s)\prod_{i=1}^{n}\prod_{j=1}^{C_i}\Gamma(c_i^j-C_i^j s_i)}{\prod_{j=1}^{B}\Gamma(b^j+\beta^j s)}z^s ds, \qquad \dots(14)$$

where α_i^j, $j=\overline{1,A}$; C_i^j, $j=\overline{1,C_i}$; $i=\overline{1,n}$ are non-negative real numbers (the situation of non-positive α_i^j, C_i^j can be reduced to this case by substitution of z^{-1} for z). In accordance with Definition 1 each of the contours L_i is assumed to separate all the poles of $\prod_{j=1}^{A} \Gamma(a^j + \alpha^j s) / \prod_{j=1}^{B} \Gamma(b^j + \beta^j s)$ from those of $\prod_{j=1}^{C_i}{}' \Gamma(c_i^j - C_i^j s_i) / \prod_{j=1}^{B} \Gamma(b^j + \beta^j s)$. For the integral (14) convergence conditions (9), (10) become

$$\sum_{j=1}^{A} \alpha_i^j + \sum_{j=1}^{C_i} C_i^j > \sum_{j=1}^{B} |\beta_i^j|, \quad i = \overline{1,n}, \qquad \ldots(15)$$

$$|\arg z_i| < \frac{\pi}{2}\left(\sum_{j=1}^{A} \alpha_i^j + \sum_{j=1}^{C_i} C_i^j - \sum_{j=1}^{B} |\beta_i^j| \right), \quad i = \overline{1,n} \quad (16)$$

At (16) $\overset{n}{H}_1[z]$ is an analytical function.

Let the condition

$$\sum_{j=1}^{A} \alpha_i^j < \sum_{j=1}^{C_i} C_i^j + \sum_{j=1}^{B} \beta_i^j, \quad i = \overline{1,n} \qquad \ldots(17)$$

which is an analogue of A + D < B + C (from [2], Sect. 4), is satisfied. Then, integral (14) should be computed in terms of the residue sum at the poles

$$S_i = \frac{c_i^{k_i} + m_i}{C_i^{k_i}}, \quad 1 \le k_i \le C_i, \; m_i = 0,1,\ldots, \quad i = \overline{1,n}. \qquad \ldots(18)$$

Let us assume that these poles do not contain multiple ones, that is the following equalities cannot be satisfied simultaneously

$$c_i^j - C_i^j s_i = -m_i, \; c_i^k - C_i^k s_i = -n_i, \quad j \neq k$$

As is known, at poles (18) the following formula may be used to find the residues of the corresponding gamma-function

$$\underset{S_i = \frac{c_i^{k_i} + m_i}{C_i^{k_i}}}{\operatorname{Res}} (\Gamma(c_i^{k_i} - C_i^{k_i} s_i)) = \frac{(-1)^{m_i}}{C_i^{k_i} m_i!}, \quad m_i = 0,1,2,\ldots \qquad \ldots(19)$$

In order to express integral (14) in terms of the sum of residues at poles (18), we substitute s_i from (18) into the integrand from (14) and change the sign $\frac{1}{(2\pi i)^n} \int_{L_1} \cdots \int_{L_n} ds$ by the sign of the sum over all the poles $\sum_{k=1}^{C} \sum_{m=0}^{\infty}$ where

$$\sum_{m=0}^{\infty} = \sum_{m_n=0}^{\infty} \cdots \sum_{m_1=0}^{\infty}, \quad \sum_{k=1}^{C} = \sum_{k_n=1}^{C_n} \cdots \sum_{k_1=1}^{C_1},$$

and replace $\Gamma(-m_i)$ by the residue (19) but with the inverse sign (since integration over L_i is clockwise for the region of s_i). As the result, we find the desired relation

$$\overset{n}{H}_1[z] = \sum_{k=1}^{C} \frac{1}{\prod_{k=1}^{n} C_i^{k_i}} \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{A} \Gamma\left(a^j + \frac{c^k}{C^k}\alpha^j + \frac{\alpha^j}{C^k} m\right)}{\prod_{j=1}^{B} \Gamma\left(b^j + \frac{c^k}{C^k}\beta^j + \frac{\beta^j}{C^k} m\right)} \cdot$$

$$\prod_{i=1}^{n} \prod_{j=1}^{C_i}{}' \Gamma\left(c_i^j - \frac{c_i^{k_i} C_i^j}{C_i^{k_i}} - \frac{C_i^j m_i}{C_i^{k_i}}\right) \frac{(-1)^{|m|} z^{\frac{m+c^k}{C^k}}}{m!},$$

where

$$(-1)^{|m|} = (-1)^{m_1 + \ldots + m_n},$$

$$\frac{c^k}{C^k} = c^k \cdot \frac{1}{C^k} = \frac{c_1^{k_1}}{C_1^{k_1}} + \ldots + \frac{c_n^{k_n}}{C_n^{k_n}},$$

$$\frac{\alpha^j}{C^k} m = \frac{\alpha_1^j}{C_1^{k_1}} m_1 + \ldots + \frac{\alpha_n^j}{C_n^{k_n}} m_n, \qquad \ldots(21)$$

a prime in the product $\prod_{j=1}^{C_i}$ denotes absence of the factor with index $j = k_i$.

Particularly, if $C_i^j = 1$, $j = \overline{1,C_i}$, $i = \overline{1,n}$ and 0 or ± 1 are components of α^j, β^k, the H-function $\overset{n}{H}_1[z]$ becomes the G-function, which will be denoted by $\overset{n}{G}_1[z]$. Under the same conditions $\overset{n}{G}_1[z]$ may be expressed in terms of generalized multiple hypergeometric series as

$$\overset{n}{G}_1[z] =$$

$$= \sum_{k=1}^{C} \Gamma\left[\begin{matrix}(a^A + \alpha^A c^k), (c_1^{C_1} - c_1^{k_1})', \ldots, (c_n^{C_n} - c_n^{k_n})' \\ (b^B + \beta^B c^k)\end{matrix}\right] z^{c^k}$$

$$\cdot \sum_{m=0}^{\infty} \frac{\prod_{j=1}^{A} (a^j + \alpha^j c^k)_{\alpha^j m} ((-1)^C z)^m}{\prod_{j=1}^{B} (b^j + \beta^j c^k)_{\beta^j m} \prod_{i=1}^{n} \prod_{j=1}^{C_i}{}' (1 - c_i^j + c_i^{k_i})_{m_i} \, m!}, \qquad \ldots(22)$$

where

$$(a^A + \alpha^A c^k) = a^1 + \alpha^1 c^k, \ldots, a^A + \alpha^A c^k;$$
$$(b^B + \beta^B c^k) = b^1 + \beta^1 c^k, \ldots, b^B + \beta^B c^k; \qquad \ldots(23)$$

$$(a)_{\alpha m} = \frac{\Gamma(a + \alpha m)}{\Gamma(a)} = \frac{\Gamma(a + \alpha_1 m_1 + \ldots + \alpha_n m_n)}{\Gamma(a)} \qquad \ldots(24)$$

Function $\overset{n}{G}_1[z]$ is more general than

$$S^{A:B';\dots;B^{(n)}}_{C:D';\dots;D^{(n)}}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \sum_{m_1=0}^{\infty}\dots\sum_{m_n=0}^{\infty}\frac{\prod_{j=1}^{A}\Gamma\left[a_j+\sum_{i=1}^{n} m_i\vartheta_j^{(i)}\right]}{\prod_{j=1}^{C}\Gamma\left[c_j+\sum_{i=1}^{n} m_i\psi_j^{(i)}\right]}$$

$$\frac{\prod_{j=1}^{B'}\Gamma\left[b'_j+m_1\phi'_j\right]\dots\prod_{j=1}^{B^{(n)}}\Gamma\left[b_j^{(n)}+m_n\phi_j^{(n)}\right]}{\prod_{j=1}^{D'}\Gamma\left[d'_j+m_1\delta'_j\right]\dots\prod_{j=1}^{D^{(n)}}\Gamma\left[d_j^{(n)}+m_n\delta_j^{(n)}\right]}\ \frac{x_1^{m_1}\dots x_n^{m_n}}{m_1!\dots m_n!}$$

...(25)

from [18], which is obtained form (22) at particular values of α^k and β^k, when $C_i^j = 1$.

4. OTHER NOTATIONS FOR THE H-FUNCTIONS $\overset{n}{H}[z]$

For complete description of the dependence of the H-function upon all its parameters divided into several groups, it seems convenient to use more concrete notations, which will be given in this section. Some auxiliary notions and symbols will be preliminary introduced.

In accordance with the theory of Meijer's G-function of one variable, for calculation of the H-function of one and many variables distribution of signs at α_i^j and β_i^j at s_i from (8) is of great importance. It is therefore convenient to substitute α^js and β^js by the following

$$\bar{i}\alpha_i^j s = i_1\alpha_{i,1}^j s_1+\dots+i_n\alpha_{i,n}^j s_n,$$

$$\bar{i}\beta_i^j s = i_1\beta_{i,1}^j s_1+\dots+i_n\beta_{i,n}^j s_n, \qquad \dots(26)$$

where

$\alpha_{i,k}^j \geq 0$, $j=\overline{1,A}$; $\beta_{i,k}^j \geq 0$, $j=\overline{1,B}$; $i=\overline{1,3^n-1}$,

$k=\overline{1,n}$,

and the vector sign $\bar{i}$ consists of three components, which admit only 0 or ± 1:

$$\bar{i} = (i_1,\dots,i_n),\ i_j = 0 \text{ or } 1 \text{ or } -1,\ j=\overline{1,n} \qquad (27)$$

To find particular distribution of 0 and ± 1 in every $\bar{i}$, we use the ternary calculus with 0, 1 and 2. To this end, each -1 in $\bar{i}$ will be substituted by 2 and commas will be ommitted. The resultant $i_1\dots i_n$ will be transformed from ternary to decimal system where its value will be denoted by i, i.e. $(i_1 i_2\dots i_n)_3 = (i)_{10} = i$. It is evident that with fixed n the number of all possible different vectors $\bar{i}$ does not exceed 3^n-1 (here $\bar{0} = (0,\dots,0)$ is eliminated).

In many cases instead of definiton 1 the following equivalent definition of the H-function may be used.

Definition 1'. The integral

$$\frac{1}{(2\pi i)^n}\int_{L_1}\overset{n}{\dots}\int_{L_n}\frac{\prod_{i=1}^{3^n-1}\prod_{j=1}^{A_i}\Gamma(a_i^j+\bar{i}\alpha_i^j s)}{\prod_{i=1}^{3^n-1}\prod_{j=1}^{B_i}\Gamma(b_i^j+\bar{i}\beta_i^j s)}z^s ds = \overset{n}{H}[z_1,\dots,z_n] =$$

$$= \overset{n}{H}\,{}^{1,\dots,\ i,\dots,\ 3^n-1}_{A_1,\dots,\ A_i,\dots,\ A_{3^n-1}}\ {}^{1,\dots,\ i,\dots,\ 3^n-1}_{B_1,\dots,\ B_i,\dots,\ B_{3^n-1}}[z_1,\dots,z_n],$$

...(28)

will be referred to as the H-function of n variables. Here $A_1,\dots,A_i,\dots,A_{3n-1}$; $B_1,\dots,B_{3n-1}$ are non-negative integers, and a_i^j, $j=\overline{1,A}$; b_i^j, $j=\overline{1,B}$, $i=\overline{1,n}$ are arbitrary complex parameters. The other notations and conditions for L_k, $k=\overline{1,n}$, are given in Definition 1 and in (26), (27).

In the last of the symbols from (28) indices A_i, B_i show the number of gamma-functions from the numerator and the denominator, for which $\bar{i}$ is the same. They are assumed to be associated with the number i of the vector, given in the upper index series. At $A_i>0$ the pairs$\binom{i}{A_i}$should necessarily be indicated and at $A_i = 0$ the pair to be ommitted from the set of indices. The number i of each of the pairs$\binom{i}{A_i}$must be located strictly over A_i. All the indices and numbers are to be separated by commas, permutation of pairs$\binom{i}{A_i}$ to the left of $\overset{n}{H}$ being allowed.The same holds for pairs $\binom{i}{B_i}$

For example, any of the symbols

$$\overset{3}{H}\,{}^{9,\ 7}_{1,11}[z] \equiv \overset{3}{H}\,{}^{7,\ 9}_{11,\ 1}[z_1,z_2,z_3]$$

denotes the following integral

$$\frac{1}{(2\pi i)^3}\int_{L_1}\int_{L_2}\int_{L_3}\Gamma(a_g^1+\alpha_{g,1}^1 s_1)\prod_{j=1}^{11}\Gamma(a_7^j+0.\alpha_{7,1}^j s_1-$$

$$\alpha_{7,2}^j s_2+\alpha_{7,3}^j s_3)\, z_1^{s_1} z_2^{s_2} z_3^{s_3}\, ds_1 ds_2 ds_3.$$

...(29)

In order to find its $\bar{i}$, the above transformation $(i)_{10} = (7)_{10} = (021)_3 \leftrightarrow (0,-1,1) = \bar{i}$ and formulas (26) to (28) have been used. It can be noted that if $\binom{g}{1}$, i.e. $\Gamma(a_g^1 + \alpha_{g,1}^1 s_1)$ is removed, the integral corresponding to $\overset{7\;3}{\underset{11}{H}}[z]$ will be divergent.

For the most complete description of the dependence of the H-function of n variables on all its parameters symbols (28) can be substituted by more concrete and therefore more bulky

$$\overset{n}{H}\begin{matrix} 1,\ldots,\, i,\ldots,\, 3^n-1 \\ A_1,\ldots,\, A_i,\ldots,\, A_{3^n-1} \\ B_1,\ldots,\, B_i,\ldots,\, B_{3^n-1}\end{matrix}\Big[z_1,\ldots,z_n\Big|$$

$$([1: a_1^{A_1};\, 0,\ldots,0,\alpha_{1,n}^{A_1}]);\ldots;([i: a_i^{A_1};\, \overline{i\alpha_i}^{A_i}]);\ldots$$

$$([1: b_1^{B_1};\, 0,\ldots,0,\beta_{1,n}^{B_1}]);\ldots;([i: b_i^{B_1};\, \overline{i\beta_i}^{B_i}]);\ldots$$

$$\left.\begin{matrix}([3^n-1: a_{3^n-1}^{A_{3^n-1}};\, -\alpha_{3^n-1,1}^{A_{3^n-1}},\ldots,\, -\alpha_{3^n-1,n}^{A_{3^n-1}}]) \\ (3^n-1: b_{3^n-1}^{B_{3^n-1}};\, -\beta_{3^n-1,1}^{B_{3^n-1}},\ldots,\, -\beta_{3^n-1,n}^{B_{3^n-1}})\end{matrix}\right],$$

$$\ldots(30)$$

where the following vector notations are used

$$\overline{i\alpha_i}^{A_i} = {}_{i_1}\alpha_{i,1}^{A_i},\ldots,\, {}_{i_n}\alpha_{i,n}^{A_i},$$

$$([i: a_i^{A_i};\, \overline{i\alpha_i}^{A_i}]) = [\,i: a_i^1;\, i_1\alpha_{i,1}^1,\ldots,i_n\alpha_{i,n}^1],\ldots,$$

$$[i: a_i^{A_i};\, i\alpha_{i,1}^{A_i},\ldots,\, i_n\alpha_{i,n}^{A_i}] \qquad \ldots(31)$$

For example, function (29) can be written in terms of the above as

$$\overset{3}{H}\begin{matrix}9,\ 7\\ 1,\ 11\end{matrix}\Big[z_1,z_2,z_3\Big|\ [9: a_9^1;\, 0,0,\, \alpha_{9,3}^1];$$

$$([7: a_7^{11};\, 0,\, -\alpha_{7,2}^{11},\, \alpha_{7,3}^{11}])\Big] \qquad \ldots(32)$$

(the coefficients $\alpha_{9,1}^1$, $\alpha_{9,2}^1$, $\alpha_{7,1}^i$, $i = \overline{1,11}$ for this function should not necessarily be specified since they are multipled by zero components of the vectors (0,0,1), (0,-1,1) and the function is independent of them).

5. SOME PARTICULAR CASES AND PROPERTIES OF THE H-FUNCTION

We will give here some particular cases and properties of the H-function of n variables. So, Fox's H-function (7) may be written in terms of (30) as

$$H_{p\,q}^{m\,n}\left(z\Big|\begin{matrix}(a_p,\alpha_p)\\(b_q,\beta_q)\end{matrix}\right) = \overset{1}{H}\begin{matrix}1,\ 2 & 1 & 2\\ n,\ m & q-m, & p-n\end{matrix}$$

$$\left[z\Big|\begin{matrix}([1: 1-a_n;\alpha_n]);\ ([2: b_m;-\beta_m])\\ [1: 1-b_{m+1};\, \beta_{m+1}],\ldots,\, [1: 1-b_q;\, \beta_q];\end{matrix}\right.$$

$$\left.[2: a_{n+1};\, -\alpha_{n+1}],\ldots,\, [2: a_p;\, -\alpha_p]\right], \qquad \ldots(33)$$

and $\overset{n}{H}_1[z]$ (14) can be expressed as

$$\overset{n}{H}_1[z] = \overset{n}{H}\begin{matrix}2,\ 6,\ldots,\, 2\cdot3^i,\ldots,2\cdot3^{n-1},\, \frac{3^n-1}{2}\\ C_n, C_{n-1},\ldots,\, C_{n-i},\ldots,\, C_1,\ A\end{matrix}\Big[z_1,\ldots,z_n\Big|$$

$$([2: c_n^{C_n};\, 0,\ldots,0,\, -C_n^{C_n}]);\ldots;$$

$$([2\cdot3^{n-1}: c_1^{C_1};\, -C_1^{C_1},\, 0,\ldots,0]);$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$([\tfrac{3^n-1}{2}: a^A;\, \alpha_1^A,\ldots,\alpha_n^A])\Big]$$

$$\cdots\cdots\cdots\cdots\cdots, \qquad \ldots(34)$$

where dots are given because there is no order in the signs of the parameters in the denominator of (14).

The "degenerated" case of the H-function (28) is of interest when the n-multiple integral is divided into the product of n simple integrals. Then the H-function of n variables is the product of n Fox's H-funtions (7)

$$\overset{k}{H}\begin{matrix}1,\ 2,\ 3,\ 6,\ldots,\, 3^{k-1},\, 2\cdot3^{k-1} & 1, & 2, & \ldots,\\ m_k,\, n_k,\, m_{k-1},\, n_{k-1},\ldots,\, m_1,\, n_1 & q_k-n_k, & p_k-m_k & \ldots,\\ 3^{k-1},\, 2\cdot3^{k-1} & & & \\ q_1-n_1,\, p_1-m_1 & & & \end{matrix}\left[\frac{1}{z_1},\ldots,\frac{1}{z_k}\Bigg|\begin{matrix}([1: b_{m_k}^k;0,\ldots,0,\beta_{m_k}^k]);\\ ([1: a_j^k;0,\ldots,0,\alpha_j^k])n_k < j \le p_k;\end{matrix}\right.$$

$$([2: 1-a_{n_k}^k;\, 0,\ldots,\, 0,\, -\alpha_{n_k}^k]);\ldots;$$

$$([3^{k-1}: b_{m_1}^1;\, \beta_{m_1}^1,\, 0,\ldots,0]);$$

$$([2:\ 1-b_j^k;\ 0,\dots,0,\ -\beta_j^k])_{m_k<j\le q_k}\ ;\dots;$$

$$([3^{k-1}:\ a_j^1;\ \alpha_j^1,\ 0,\dots,0])_{n_1<j\le p_1}\ ;$$

$$\left.\begin{matrix}([2\cdot 3^{k-1}:\ 1-a_{n_1}^1;\ -\alpha_{n_1},\ 0,\dots,0]) \\ ([2\cdot 3^{k-1}:\ 1-b_j^1\ ;\ -\beta_j^1,\ 0,\dots,0])_{m_1<j\le q_1}\end{matrix}\right] =$$

$$= \prod_{i=1}^{k} H^{m_i\, n_i}_{p_i\, q_i}\left(z \middle| \begin{matrix}(a^i_{p_i},\ \alpha^i_{p_i}) \\ (b^i_{q_i},\ \beta^i_{q_i})\end{matrix}\right), \qquad \dots(35)$$

where

$$([f(j)])_{n<j\le p} = [f(n+1)]\ ,\dots,\ [f(p)]\ .$$

To conclude the paper, the property of the shift, generalizing the corresponding formula (5.3.1) from [5] will be given:

$$z_1^{\sigma 1}\cdots z_n^{\sigma n}\ \overset{n}{H}[z] = \begin{matrix}1\ ,\dots,\ i\ ,\dots,\ 3^n-1 \\ A_1\ ,\dots,\ A_i\ ,\dots,\ A_{3^n-1}\end{matrix}$$

$$\overset{n}{H}\begin{matrix}1\ ,\dots,\ i\ ,\dots,\ 3^n-1 \\ B_1,\dots,\ B_i\ ,\dots,\ B_{3^n-1}\end{matrix}$$

$$\left[z_1,\dots,z_n\ \middle|\ \begin{matrix}([1:a_1^{A_1}-\alpha_{1,n}^{A_1}\sigma_n;\ 0,\dots,0,\ \alpha_{1,n}^{A_1}])\ ;\dots; \\ ([1:b_1^{B_1}-\beta_{1,n}^{B_1}\sigma_n;\ 0,\dots,0,\ \beta_{1,n}^{B_1}])\ ;\dots;\end{matrix}\right.$$

$$([i:\ a_i^{A_i}-\overline{i\sigma}\,\alpha_i^{A_i};\ \overline{i\alpha_i}^{A_i}])\ ;\dots;$$

$$([3^n-1:\ a_{3^n_{-1}}^{A_{3^n-1}} + \sum_{j=1}^{n}\alpha_{3^n-1,j}^{A_{3^n-1}}\sigma_j\ ;$$

$$([i:\ b_i^{B_i}-\overline{i\sigma}\,\beta_i^{B_i};\ \overline{i\beta_i}^{B_i}])\ ;\dots;$$

$$([3^n-1:\ b_{3^n-1}^{B_{3^n-1}} + \sum_{j=1}^{n}\beta_{3^n-1}^{B_{3^n-1}}\sigma_j\ ;$$

$$\left.\begin{matrix}-\alpha_{3^n-1,1}^{A_{3^n-1}},\dots,-\alpha_{3^n-1,n}^{A_{3^n-1}}]) \\ -\beta_{3^n-1,1}^{B_{3^n-1}},\dots,-\beta_{3^n-1,n}^{B_{3^n-1}}])\end{matrix}\right] \qquad \dots(36)$$

REFERENCES

1) MARICHEV, O.I.: *"Handbook of Integrals of Higher Transcendental Functions: Theory and Algorithmic Tables"*. Ellis Horwood Ltd. Publishers, Chichester, 1983.

2) PRUDNIKOV, A.P., BRYCHKOV, Yu.A. and MARICHEV, O.I.: *"Integrals and Series. Elementary Functions", 1981; "Integrals and Series. Special Functions"*, 1983; Nauka Publishers, Moscow.

3) CHANDEL, R.C.S.: *"On Some Multiple Hypergeometric Functions Related to Lauricella Functions"*. Iñānabha, Sect. A, No.3, 1973, pp. 119-136.

4) CHATURVEDI, K.K. and GOYAL, A.N.:*"A*-function"*. (I). Indian J. Pure and Appl. Math., Vol.3, No. 3, 1972, pp. 357-360.

5) ERDELYI, A. et al.:*"Higher Transcendental Functions"*. Vol.1, McGraw-Hill, New York, 1953.

6) EXTON, H.: *"Multiple Hypergeometric Functions and Applications"*. Ellis Horwood Ltd. Publishers, Chichester, 1976.

7) EXTON, H.: *"Certain Hypergeometric Function of Four Variables"*. Bull. Soc. Math. Greece N.S., 13, 1972, pp. 104-113.

8) FOX, C.: *"The G and H Functions as Symmetrical Fourier Kernels"*. Trans. Amer. Math. Soc., Vol. 98, 1961, pp. 395-429.

9) GRADSHTEIN, I.S. and RUZHIK, I.M.: *"Table of Integrals, Series, and Products"*. Academic Press, New York and London, 1980.

10) KARLSSON, P.W.: *"Reduction of Certain Generalized Kampé de Fériet Functions"*. Math. Scand., Vol. 32, 1973, pp. 265-268.

11) KARLSSON, P.W.: *"Reduction of Certain Hypergeometric Functions of Three Variables"*. Glasnik Math., Ser.III, 8(28), 1973, pp. 199-204.

12) MATHAI, A.M. and SAXENA, R.K.: *"Generalized Hypergeometric Functions With Applications in Statistics and Physical Sciences"*. Springer-Verlag, Lect. Notes Math., Ser., No. 348, Heidelberg and New York, 1973.

13) MATHAI, A.M. and SAXENA, R.K.: *"The H-function with Applications in Statistics and Other Disciplines"*. A Halsted Press Book, New York and London, 1978.

14) MITTAL, P.K. and GUPTA, K.C.: *"An Integral Involving Generalized Function of Two Variables"*. Proc. Indian Acad. Sci., Vol. 75A, 1972, No. 3, pp. 117-123.

15) MUNOT, P.C. and KALLA, S.L.: *"On an Extension of a Generalized Function of Two Variables"*.

Univ. Nac. Tucumán, Rev., Ser. A21, 1971, pp. 67-84.

16) SARAN, S.: "*Hypergeometric Functions of Three Variables*". Ganita, Vol. 5, 1954, pp. 77-91.

17) SRIVASTAVA, H.M.: "*Hypergeometric Functions of Three Variables*". Ganita, Vol. 15, 1964, pp. 97 -108.

18) SRIVASTAVA, H.M. and DAOUST, M.C.: "*Certain Generalized Neumann Expansions Associated with the Kempé de Fériet Function*". Nederl. Akad. Wetensch. Proc., A72, 1969, pp. 449-457.

19) SRIVASTAVA, H.M. and PANDA, R.: "*Some Bilateral Generating Functions for a Class of Generalized Hypergeometric Polynomials*". J. Reine Angew. Math., 283/284, 1976, pp. 265-274.

20) SRIVASTAVA, H.M. and PANDA, R.: "*Expansion Theorems for the H-function of Several Complex Variables*". J. Reine Angew. Math., Vol. 288, 1976, pp. 129-145.

21) SRIVASTAVA, H.M., GUPTA, K.C. and GOYAL, S.P.: "*The H-Functions of One and Two Variables with Applications*". South Asian Publishers Pvt Ltd, New Delhi, 1982.

Addendum : M. Santana de Galindo and S. L. Kalla have also studied a generalized H-function of two variables [sobre una extensión de la función generalizada de dos variables. Univ. Nac. Tucumán Rev. Ser. A 25(1975), 221-229]. The notations used by these authors are the same as [15] cited above. Recently R. G. Buschman has studied H – function of two variables in a series of papers [Indian J. Math 20 (1978), 139-153, Jñānabha, 7 (1977), 105 - 116, Pure Appl. Math. Sci. 9 (1979), 13-18]. The H-function of n variables has been trated by Buschman in [Ranchi Univ. Math. J. 10 (1979), 81-88]. A reducible case of H-functions of two variables is considered in this Journal, Vol. 5, (ii) (1982), 1-2. In this series of papers, Buschman explores conditions sufficient for the convergence of the double Mellin-Barnes integrals, which define the H-function of two variables. He also introduces a set of notations to avoid confusion in the use of symbols.

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