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A NEW CLASS OF POLYNOMIALS AND THE POLYNOMIALS RELATED TO THEM

ABSTRACT

The present paper deals with the study of a general class of polynomials. We mention some particular cases and deduce some of their properties.

RESUMEN

En este trabajo se estudia una clase de polinomios generalizados. Se mencionan algunos casos particulares con sus propiedades.

1. INTRODUCTION

Recently Agrawal [1] introduced the polynomials defined by

$$(1.1)\qquad (1-pt^q)^{-c}\exp\left[\frac{-r^r\,xt}{(1-pt^q)^r}\right]=\sum_{n=0}^{\infty} f_n^c(x;p,q,r)\,t^n.$$

Also Panda [2] introduced the class of polynomials defined by

$$(1.2)\qquad (1-t)^{-c}\,G\left[\frac{xt^s}{(1-t)^r}\right]=\sum_{n=0}^{\infty} g_n^c(x,r,s)\,t^n.$$

where c is an arbitrary parameter, r is any integer, positive or negative, and s = 1,2,3,... .

In the present paper, we introduce the polynomials

$\{g_n^c(x,p,q,r)/n = 0,1,2,...\}$ defined by

$$(1.3)\qquad (1-pt^q)^{-c}\,G\left[\frac{xt}{(1-pt^q)^r}\right]=\sum_{n=0}^{\infty} g_n^c(x,p,q,r)\,t^n$$

where

$$(1.4)\qquad G(Z)=\sum_{n=0}^{\infty}\gamma_n Z^n\ ;\ (\gamma_0\neq 0)$$

q is any positive integer and other parameters are unrestricted in general.

The definitions (1.3) and (1.4) are motivated by the earlier work on (1.1) due to Agrawal [1], who considers the special case of (1.3) when $\gamma_n=\frac{(-1)^n r^{rn}}{n!}$, and also by the recent work on (1.2) due to Panda [2], who considers only the case when s = 1,2,3, ... and p=1, while in (1.3) we discuss the case when $s=\frac{1}{q}$; q = 1,2,3, ...

2. THE POLYNOMIALS $\{g_n^c(x,p,q,r)/n = 0,1,2,...\}$

Starting with the generating relation (1.3), we obtain the following results :

$$(2.1)\qquad g_n^c(x,p,q,r)=\sum_{k=0}^{[n/q]}\frac{(c+(n-qk)r)_k}{k!}\gamma_{n-qk}\,p^k x^{n-qk},$$

$$(2.2)\qquad x\,D_x\,g_n^c(x,p,q,r)-n\,g_n^c(x,p,q,r)=-p(cq+n)g_{n-q}^c(x,p,q,r)+(1-qr)px\,D_x\,g_{n-q}^c(x,p,q,r)$$

$$(2.3)\qquad x\,D_x\,g_k^c(x,p,q,r)-k\,g_k^c(x,p,q,r)=-\left[cq\sum_{n=0}^{[k/q]}g_{k-nq-q}^c(x,p,q,r)+qrx\sum_{n=0}^{[k/q]}D_x\,g_{k-(n+1)q}^c(x,p,q,r)\right]p^{k+1}$$

$$(2.4) \quad x D_x g_n^c(x,p,q,r) - n g_n^c(x,p,q,r)$$

$$= -\sum_{k=0}^{n-q} p^{\frac{n-k}{q}} (1-qr)^{(\frac{n-k}{q}-1)} \left[cq+qr(k+q)\right] g_k^c(x,p,q,r),$$

$$(2.5) \quad g_n^{c+1}(x,p,q,r) - p\, g_{n-q}^{c+1}(x,p,q,r) = g_n^c(x,p,q,r),$$

$$(2.6) \quad g_n^{b+c}(x,p,q,r) = \sum_{k=0}^{\left[\frac{n}{q}\right]} (b)_k p^k g_{n-qk}^c(x,p,q,r)$$

$$(2.7) \quad g_n^{c+n}(x,p,q,r) = g_n^c(x,p,q,r+1),$$

and

$$(2.8) \quad g_n^{c+nm}(x,p,q,r) = g_n^c(x,p,q,r+m)$$

3. AN INTERESTING SPECIAL CASE :

To discuss those properties which cannot be studied in general, we consider the following special case, when $\gamma_n = \frac{(b)_n}{n!}$

$$(3.1) \quad (1-pt^q)^{-c}\left[1-\frac{xt}{(1-pt^q)^r}\right]^{-b} = \sum_{n=0}^{\infty} g_n^{(b,c)}(x,p,q,r)\, t^n.$$

An appeal to the above result shows that

$$(3.2) \quad g_n^{(b+b',c+c')}(x,p,q,r) = \sum_{k=0}^{n} g_{n-k}^{(b,c)}(x,p,q,r)\, g_k^{(b',c')}(x,p,q,r),$$

$$(3.3) \quad g_n^{(b_1+\ldots+b_m,c_1+\ldots+c_m)}(x,p,q,r) = \sum_{i_1+i_2+\ldots+i_m=n} \prod_{j=1}^{m} Gi_j^{(b_j,\, c_j)}(x,p,q,r),$$

$$(3.4) \quad g_n^{(b+b',c)}(x,p,q,r) = \sum_{m=0}^{n} \frac{(b')_m}{m!} x^m g_{n-m}^{(b,c+rm)}(x,p,q,r),$$

$$(3.5) \quad g_n^{(b+1,c)}(x,p,q,r) - g_n^{(b,c)}(x,p,q,r) = x \sum_{k=0}^{\frac{n-1}{q}} \frac{(r)_k}{k!} p^k g_{n-qk-1}^{(b+1,c)}(x,p,q,r),$$

$$(3.6) \quad g_n^{(b,c-r)}(x,p,q,r) + x\, g_{n-1}^{(b+1,c)}(x,p,q,r) = \sum_{m=0}^{r} (-1)^m \binom{r}{m} p^m g_{n-qm}^{(b+1,c)}(x,p,q,r),$$

$$(3.7) \quad g_n^{(b,c)}(x,p,q,r) = \sum_{k=0}^{n} g_{n-k}^{(c,b)}(x,p,q,r)\, g_k^{(b-c,c-b)}(x,p,q,r),$$

$$(3.8) \quad D_x g_n^{(b,c)}(x,p,q,r) = b\, g_{n-1}^{(b+1,c+r)}(x,p,q,r),$$

$$(3.9) \quad D_x^m g_n^{(b,c)}(x,p,q,r) = (b)_m\, g_{n-m}^{(b+m,c+rm)}(x,p,q,r),$$

$$(3.10) \quad g_n^{(b,c)}(x+y,p,q,r) =$$

$$\sum_{m=0}^{\infty} \frac{(b)_m}{m!} y^m \, g_{n-m}^{(b+m,c+rm)}(x,p,q,r),$$

(3.11) $$g_n^{(b,c)}(x,p,q,r) =$$

$$\sum_{m=0}^{\infty} \frac{(b)_m}{m!} x^m \, g_{n-m}^{(b+m,c+rn)}(0,p,q,r),$$

and

(3.12) $$g_k^{(b,c)}(x,p,q,r) =$$

$$\sum_{k=0}^{[\frac{n}{q}]} \frac{(b)_{n-qk}}{(n-qk)!} x^{n-qk} \frac{(c+r(n-qk))_k}{k!} p^k .$$

4. THE POLYNOMIALS $\{A_n^{(b,c)}(x,p,q,r)/n = 0,1,2,\ldots\}$

Consider

(4.1) $$\sum_{k=0}^{[\frac{n}{q}]} A_k^{(b,c)}(x,p,q,r) \, g_{n-qk}^{(b,c+k)}(x,p,q,r) = 0,$$

$$n \geq 1$$

and

(4.2) $$A_o^{(b,c)}(x,p,q,r) = 1.$$

Therefore,

$$1 = \sum_{n=0}^{\infty} \sum_{k=0}^{[\frac{n}{q}]} A_k^{(b,c)}(x,p,q,r) \, g_{n-qk}^{(b,c+k)}(x,p,q,r) \, t^n$$

$$= \sum_{n,k=0}^{\infty} A_k^{(b,c)}(x,p,q,r) \, g_n^{(b,c+k)}(x,p,q,r) \, t^{n+qk}.$$

Thus

(4.3) $$\sum_{k=0}^{\infty} A_k^{(b,c)}(x,p,q,r) \, z^{qk} =$$

$$(1+pz^q)^{-c} \left[1-xz(1+pz^q)^{r-\frac{1}{q}}\right]^b$$

which further gives

(4.4) $$A_k^{(b,c)}(x,p,q,r) =$$

$$\sum_{k=0}^{k} \frac{(-b)_{kq-nq}}{(kq-nq)!} \frac{(c-(r-\frac{1}{q})(kq-nq))_n}{n!}$$

$$x(-p)^n x^{kq-nq}$$

Now (3.12) and (4.4) give the relation

(4.5) $$A_k^{(-b,c)}(x,-p,q,r+\frac{1}{q}) = g_{qk}^{(b,c)}(x,p,q,r).$$

An appeal to (4.3) shows that

(4.6) $$A_k^{(b+b',c+c')}(x,p,q,r)$$

$$= \sum_{m=0}^{k} A_{k-m}^{(b,c)}(x,p,q,r) \, A_m^{(b',c')}(x,p,q,r)$$

(4.7) $$A_k^{(b_1+\ldots+b_n,c_1+\ldots+c_n)}(x,p,q,r) =$$

$$\sum_{m_1+\ldots+m_n=k} \prod_{j=1}^{n} A_{m_j}^{(b_j,c_j)}(x,p,q,r)$$

and

$$(4.8)\quad A_k^{(b,c-1)}(x,p,q,r) = A_k^{(b,c)}(x,p,q,r) + p\,A_{k-1}^{(b,c)}(x,p,q,r).$$

5. THE POLYNOMIALS $\{R_n^{(a,b;c,d)}(x,p,q,r)/n=0,1,2,..\}$

Consider

$$(5.1)\quad R_n^{(a,b;c,d)}(x,p,q,r) = \sum_{k=0}^{nq} A_{n-[\frac{k}{q}]}^{(a,b+[\frac{k}{q}])} g_k^{(c,d)}(x,p,q,r).$$

Therefore

$$\sum_{n=0}^{\infty} R_n^{(a,b;c,d)}(x,p,q,r)z^{nq} =$$

$$\sum_{n,k=0}^{\infty} A_n^{(a,b+[\frac{k}{q}])}(x,p,q,r)g_k^{(c,d)}(x,p,q,r)\; z^{nq+k}$$

$$= (1+pz^q)^{-(b-d)}\left[1-xz(1+pz^q)^{r-\frac{1}{q}}\right]^{a-c}$$

$$= \sum_{n=0}^{\infty} A_n^{(a-c,b-d)}(x,p,q,r)\; z^{nq}.$$

Thus we obtain

$$(5.2)\quad R_n^{(a,b;c,d)}(x,p,q,r) = A_n^{(a-c,b-d)}(x,p,q,r),$$

$$(5.3)\quad \sum_{n=0}^{\infty} R_n^{(a,b;a,d)}(x,p,q,r)\; z^{nq} = (1+pz^q)^{-(b-d)},$$

$$(5.4)\quad \sum_{n=0}^{\infty} R_n^{(a,b;c,b)}(x,p,q,r)\; z^{nq} = \left[1-xz(1+pz^q)^{r-\frac{1}{q}}\right]^{a-c},$$

$$(5.5)\quad \sum_{n=0}^{\infty} R_n^{(a,b;a,b)}(x,p,q,r)\; z^{nq} = 1,$$

$$(5.6)\quad R_n^{(a,b;c,d)}(x,p,q,r) = \sum_{n=0}^{\infty} R_{n-k}^{(a',b';c,d)}(x,p,q,r)R_k^{(a,b;a',b')}(x,p,q,r),$$

$$(5.7)\quad R_n^{(a+a',b+b';c+c',d+d')}(x,p,q,r) = \sum_{k=0}^{n} R_{n-k}^{(a,b;c,d)}(x,p,q,r)R_k^{(a',b';c',d')}(x,p,q,r),$$

and

$$(5.8)\quad R_n^{(a_1+\ldots+a_m,b_1+\ldots+b_m;c_1+\ldots+c_m,d_1+\ldots+d_m)}(x,p,q,r) = \sum_{i+\ldots+i_m=n} \prod_{j=1}^{m} R_{i_j}^{(a_j,b_j;c_j,d_j)}(x,p,q,r).$$

From (5.2) and (4.5), we obtain

$$(5.9)\quad R_n^{(a,b;c,d)}(x,p,q,r)=g_{nq}^{(c-a,b-d)}(x,-p,q,r-\tfrac{1}{q}).$$

6. THE POLYNOMIALS $\{N_n^{(a,b;c,d)}(x,p,q,r)/n=0,1,2,\ldots\}$

Consider

$$(6.1)\quad N_n^{(a,b;c,d)}(x,p,q,r) = \sum_{k=0}^{[\frac{n}{q}]} A_k^{(a,b)}(x,p,q,r)g_{n-kq}^{(c,d+k)}(x,p,q,r).$$

Therefore

$$\sum_{n=0}^{\infty} N_n^{(a,b;c,d)}(x,p,q,r)z^n$$

$$= \sum_{n,k=0}^{\infty} A_k^{(a,b)}(x,p,q,r)g_n^{(c,d+k)}(x,p,q,r)z^{n+kq}$$

$$= (1-pz^q)^{-(d-b)}\left[1-\frac{xz}{(1-pz^q)^r}\right]^{-(c-a)}$$

$$= \sum_{n=0}^{\infty} g_n^{(c-a,d-b)}(x,p,q,r)\ z^n.$$

Thus we obtain

$$(6.2)\quad N_n^{(a,b;c,d)}(x,p,q,r) = g_n^{(c-a,d-b)}(x,p,q,r),$$

$$(6.3)\quad \sum_{n=0}^{\infty} N_n^{(a,b;a,d)}(x,p,q,r)\ z^n=(1-pz^q)^{b-d},$$

$$(6.4)\quad \sum_{n=0}^{\infty} N_n^{(a,b;c,b)}(x,p,q,r)\ z^n = \left[1-\frac{xz}{(1-pz^q)^r}\right]^{a-c},$$

$$(6.5)\quad \sum_{n=0}^{\infty} N_n^{(a,b;a,b)}(x,p,q,r)\ z^n = 1,$$

$$(6.6)\quad N_n^{(a,b;c,d)}(x,p,q,r) = \sum_{k=0}^{n} N_{n-k}^{(a',b';c,d)}(x,p,q,r)N_k^{(a,b;a',b')}(x,p,q,r),$$

$$(6.7)\quad N_n^{(a+a',b+b';c+c',d+d')}(x,p,q,r) = \sum_{k=0}^{n} N_k^{(a,b;c,d)}(x,p,q,r)N_{n-k}^{(a',b';c',d')}(x,p,q,r),$$

and

$$(6.8)\quad N_n^{(a_1+\ldots+a_m,b_1+\ldots+b_m;c_1+\ldots+c_m,d_1+\ldots+d_m)}(x,p,q,r) = \sum_{i_1+\ldots+i_m=n}\prod_{j=1}^{m} N_{i_j}^{(a_j,b_j;c_j,d_j)}(x,p,q,r).$$

From (6.2) and (3.11), we obtain

$$(6.9)\quad N_n^{(a,b;c,d)}(x,p,q,r) = \sum_{m=0}^{\infty}\frac{(c-a)_m}{m!}x^m\ g_{n-m}^{(c-a+m,d-b+rm)}(0,p,q,r).$$

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