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# Behaviour of hypergeometric function ${}_pF_{p-1}(z)$ in the vicinity of unity

ABSTRACT

The present paper deals with the behaviour of the generalized hypergeometric function ${}_pF_{p-1}(\alpha_1, \ldots, \alpha_p; \rho_1, \ldots \rho_{p-1}; z)$ in the neighbourhood of z=1. The fundamental solution systems for the corresponding generalized hypergeometric differential equation in the vicinity of singular points z=0, ∞ and z=1 are given for the ordinary, as well as for the logarithmic case.

RESUMEN

En este trabajo se estudia el comportamiento de la función hipergeométrica generalizada ${}_pF_{p-1}(\alpha_1, \ldots, \alpha_p; \rho_1, \ldots, \rho_{p-1}; z)$ en el entorno de z=1. Se dan los sistemas de soluciones fundamentales de la ecuación hipergeométrica generalizada correspondiente en el entorno de las singularidades z=0, ∞ y z=1, para el caso ordinario y también para el caso logarítmico.

In the present paper the type of the singularity of the generalized hypergeometric function [1] ${}_pF_{p-1}(\alpha_1, \ldots, \alpha_p; \rho_1, \ldots, \rho_{p-1}; z)$ at z=1 is studied and the fundamental solution systems for the corresponding generalized hypergeometric differential equation in the vicinity of singular points z=0, ∞ and z=1 are given both for the ordinary and for the logarithmic cases. The results presented here are obtained on the basis of N.E. Norlund's work [2] were the information is not presented separtely and different notations inconvenient for applications are used.

As is know, the generalized hypergeometric function ${}_pF_{p-1}(z) = {}_pF_{p-1}(\alpha_1, \ldots, \alpha_p; \rho_1, \ldots, \rho_{p-1}; z)$ in the z-plane is defined as the principal branch of the analytical function represented by the generalized hypergeometric series

$$ {}_pF_{p-1}(z) = {}_pF_{p-1}\begin{pmatrix}\alpha_1, \ldots, \alpha_p; z \\ \rho_1, \ldots, \rho_{p-1}\end{pmatrix} = $$

$$ \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ldots (\alpha_p)_k}{(\rho_1)_k \ldots (\rho_{p-1})_k} \frac{z^k}{k!}, \ |z| < 1, \qquad (1) $$

where

$$ (\alpha)_0 = 1, (\alpha)_k = \alpha(\alpha+1)\ldots(\alpha+k-1) = $$

$$ \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)} \equiv \Gamma\begin{bmatrix}\alpha + k \\ \alpha\end{bmatrix}, \qquad (2) $$

and is denoted by the same symbol as is used for the series itself.
Let in the following $\rho_j \neq 0, -1, -2, \ldots, j=1,2, \ldots, p-1$, otherwise the series (1) do not exist.

Series (1) is convergent at $|z|<1$. On the circle $|z|=1$ it converges when Re $\beta_p > 0$ where

$$ \beta_p = \sum_{j=1}^{p-1} (\rho_j - \alpha_j) - \alpha_p . \qquad (3) $$

If $-1 < \mathrm{Re}\beta_p \leq 0$, then series (1) is conventionally convergente at $|z|=1$, $z \neq 1$, and if Re $\beta_p \leq -1$, then over the whole circle $|z|=1$ it diverges.

The ray $[1, \infty)$ is a section of the principal branch of the analytical extension of series (1).

Function (1) is one of the solutions to the following ordinary linear homogeneous differential equation of the p-th order (the generalized hypergeometric equation)

$$ \left[\frac{d}{dz} \prod_{j=1}^{p-1} \left(z\frac{d}{dz} + \rho_j - 1\right) - \prod_{j=1}^{p} \left(z\frac{d}{dz} + \alpha_j\right)\right] U = 0. \qquad (4) $$

The above equation has three correct singular points, namely $z=0,\infty$ and $z=1$. Fundamental solution systems of equations (4) at $z=0,\infty$ are known [2]. If none of $\rho_k-\rho_j$, $j\neq k$ is an integer, then in the vicinity of $z=0$ $p$ generalized hypergeometric function of the form*

$$U_k^{\circ}(z) = z^{1-\rho_k}\ {}_pF_{p-1}\left(\begin{matrix}1+\alpha_1-\rho_k,\ldots,1+\alpha_p-\rho_k;z\\ 1+\rho_1-\rho_k,.*.,1+\rho_p-\rho_k\end{matrix}\right), \quad (5)$$

$$k=1,2,\ldots,p, \qquad \rho_p=1,$$

among which ${}_pF_{p-1}(z)$ is contained at $k=p$ : $u_p^{o}(z) = {}_pF_{p-1}(z)$, make the fundamental solution system. In the vicinity of $z=\infty$ provided that there are no integers among $\alpha_k-\alpha_j$, $j\neq k$, the fundamental solution system $u_k^{\infty}(z)$, $k=1,2,\ldots,p$, is obtained from the system $u_k^{o}(z)$, $k=1,2,\ldots,p$, if in the latter $z$ and $z^{-1}$, $\rho_k$ and $1-\alpha_k$, $k=1,2,\ldots,p$, interchanged with the subsequent assumption of $\rho_p=1$ :

$$U_k^{\infty}(z) = z^{-\alpha_k}\ {}_pF_{p-1}\left(\begin{matrix}1+\alpha_k-\rho_1,\ldots,1+\alpha_k-\rho_{p-1},\alpha_k;z^{-1}\\ 1+\alpha_k-\alpha_1,.*.,1+\alpha_k-\alpha_p\end{matrix}\right), \quad (6)$$

$$k=1,2,\ldots,p.$$

At non-integer $\alpha_k-\alpha_j$, $j\neq k$, the functions $u_p^{o}(z)$ and $u_k^{\infty}(z)$, $k=1,2,\ldots,p$, are related by the formula [3]

$${}_pF_{p-1}\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;z\\ \rho_1,\ldots\rho_{p-1}\end{matrix}\right) =$$

$$\Gamma\begin{bmatrix}\rho_1,\ldots,\rho_{p-1}\\ \alpha_1,\ldots,\alpha_p\end{bmatrix}\sum_{k=1}^{p}\Gamma\begin{bmatrix}\alpha_k,\alpha_1-\alpha_k,.*.,\alpha_p-\alpha_k\\ \alpha_1-\alpha_k,\ldots,\rho_{p-1}-\alpha_k\end{bmatrix}.$$

$$\cdot\ (z^{-1}e^{\pi i})^{\alpha_k}\ {}_pF_{p-1}\left(\begin{matrix}\alpha_k,1+\alpha_k-\rho_1,\ldots,1+\alpha_k-\rho_{p-1};z^{-1}\\ 1+\alpha_k-\alpha_1,.*.,1+\alpha_k-\alpha_p\end{matrix}\right), \quad (7)$$

$$0 < \arg z < 2\pi .$$

Here and in what follows $\Gamma[\,\equiv\,]$ denotes a ratio of the corresponding gamma-function products (see [3, 4]).

If there are integers among $\rho_k-\rho_j$, $j\neq k$, the respective solutions for $u_k^{o}(z)$ may coincide or they may not exist. In that case, to obtain the fundamental solution system of equation (4), a set of $\rho_j$ should be divided into groups so that every group would include all $\rho_j$, which differ from one another by integers. For example, let $\rho_1,\ldots,\rho_q$ compose one of such groups and $Re\rho_q \geq Re\rho_{q-1} \geq \ldots \geq Re\rho_1$. If $(A(\rho_k)\ {}_pF_{p-1})_{\rho_k}^{(m)}$ is the m-th order derivative with respect to the variable $\rho_k$ of the function ${}_pF_{p-1}$ in the right-hand side of (5) multiplied by an arbitrary constant $A(\rho_k)$, depending on $\rho_k$, we may compose the function

$$\tilde{U}_k^{o}(z) = z^{1-\rho_k}\ [\ (A(\rho_k)\ {}_pF_{p-1})_{\rho_k}^{(k-1)}\ -$$

$$\binom{k-1}{1}\ln z\,(A(\rho_k)\ {}_pF_{p-1})_{\rho_k}^{(k-2)}\ +$$

$$+\ \binom{k-2}{2}\ln^2 z\,(A(\rho_k)\ {}_pF_{p-1})_{\rho_k}^{(k-3)}\ -\ \ldots\ -$$

$$(-1)^k\ln^{k-1}z\ A(\rho_k)\ {}_pF_{p-1}\ ], \quad (8)$$

$$k = 1,\ 2,\ \ldots,\ q.$$

Since not all of the derivatives become zero, the functions form q linearly independent solutions, corresponding to the parameters $\rho_1,\ldots,\rho_q$.

Under certain conditions the terms, containing logarithms in $u_k^{\rho}(z)$ may be absent. To this end it is necessary but not sufficient that all $\rho_k$, $k = 1, 2,\ldots,q$, be different, i.e. $Re\rho_q > Re\rho_{q-1} > \ldots > Re\rho_1$. The following condition is one of the sufficient ones. If the following equality is fulfilled

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*) Here and in the following asterisk, .*., denotes that the component, containing $\rho_k - \rho_k$ (in the present case $1+\rho_k-\rho_k$) is omitted from the vector.

* Here and in the following the formula numbers of the type (5.40) denote the formula numbers from [2]

$$\prod_{j=o}^{\rho_k \rho_{k-1}-1} R(1-\rho_k+j) = 0 \quad , \quad k = 2,3, \ldots , q, \qquad (9)$$

where R(x) is defined by formula (20), then $u_1^o(z)$ and the sections of the generalized hypergeometric series

$$\overset{*}{U}{}_k^o(z) = z^{1-\rho_k} \sum_{m=o}^{\rho_k-\rho_{k-1}-1} z^m \prod_{j=1}^{p} \frac{(1+\alpha_j-\rho_k)_m}{(1+\rho_j-\rho_k)m} \, , \qquad (10)$$

$$k = 2,3, \ldots, q, \; \rho_p = 1,$$

*form q linearly independent solutions corresponding to the parameters $\rho_1,\ldots,\rho_q$ . Evidently, the solutions contain no logarithms.*

Near z=1 the fundamental solution system may be expressed in terms of hypergeometric functions only if p=2 (see Gauss's formulas 2.10 (1, 12 - 14) from [1]). As is shown in [2] with p > 2 the behaviour of $_pF_{p-1}(z)$ in the vicinity of z=1 is very complicated, not of a hypergeometric type, and is described by the following formulas.

1. If $\beta_p$ is non-integer, equality (5.40) is valid*

$$_pF_{p-1}\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;z\\ \rho_1,\ldots,\rho_{p-1}\end{matrix}\right) =$$

$$\Gamma\left[\begin{matrix}\rho_1,\ldots,\rho_{p-1},-\beta_p\\ \alpha_1,\ldots,\alpha_p\end{matrix}\right]\xi_p(z) + \zeta_p(z), \qquad (11)$$

$$|\arg(1-z)| < \pi ,$$

where $\xi_p(z)$ has a singularity at z=1 and $\zeta_p(z)$ is continuous :

$$\xi_p(z) = (1-z)^{\beta p} \sum_{k=o}^{\infty} \frac{C_{kp}}{(\beta_p+1)_k} (1-z)^k \, , \quad |1-z| < 1, \qquad (12)$$

$$\zeta_p(z) = \sum_{k=o}^{\infty} b_{kp}(z-1)^k \, , \quad |1-z| < 1, \qquad (13)$$

here always $c_{op} = 1$, $C_{k1} = 0$, k=1,2,3,..., and with $\beta_p > 0$

$$b_{op} = {}_pF_{p-1}\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;1\\ \rho_1,\ldots,\rho_{p-1}\end{matrix}\right) \qquad (14)$$

In particular, with p=2 the equalities

$$C_{k2} = \frac{(\rho_1-\alpha_1)_k(\rho_1-\alpha_2)_k}{k!} \, ,$$

$$b_{k2} = \Gamma\left[\begin{matrix}\rho_1,\beta_2\\ \rho_1-\alpha_1,\rho_1-\alpha_2\end{matrix}\right]\frac{(\alpha_1)_k(\alpha_2)_k}{(1-\beta_2)_k} \; \frac{(-1)^k}{k!} \qquad (15)$$

are valid and formula (11) becomes the above mentioned Gauss relation :

$$_2F_1(\alpha_1,\alpha_2;\rho_1;z) = \Gamma\left[\begin{matrix}\rho_1,\alpha_1+\alpha_2-\rho_1\\ \alpha_1,\alpha_2\end{matrix}\right](1-z)^{\rho_1-\alpha_1-\alpha_2} \cdot \qquad (16)$$

$$_2F_1\left(\begin{matrix}\rho_1-\alpha_1,\rho_1-\alpha_2;1-z\\ 1+\rho_1-\alpha_1-\alpha_2\end{matrix}\right) + \Gamma\left[\begin{matrix}\rho_1,\rho_2-\alpha_1-\alpha_2\\ \rho_1-\alpha_1,\rho_1-\alpha_2\end{matrix}\right]{}_2F_1\left(\begin{matrix}\alpha_1,\alpha_2;1-z\\ 1+\alpha_1+\alpha_2-\rho_1\end{matrix}\right)$$

In a general case the coefficients $c_{kp}$ are defined by (1.27), (1.28), which in the present notations become

$$c_{12} = R(\beta_2), \quad 2c_{22} = R(\beta_2+1)c_{12}, \ldots,$$

$$kc_{k2} = R(\beta_2+k-1)c_{k-1,2} \, ; \qquad (17)$$

$$c_{13} = \Delta R(\beta_3-1) - Q(\beta_3),$$

$$2c_{23} = \left[\Delta R(\beta_3) - Q(\beta_3+1)\right]c_{13} - R(\beta_3), \ldots,$$

$$kc_{k3} = \left[\Delta R(\beta_3+k-2) - Q(\beta_3+k-1)\right]c_{k-1,3} - \qquad (18)$$

$$- R(\beta_3+k-2)c_{k-2,3} \, ;$$

$$c_{1p} = \frac{\Delta^{p-2}R(\beta_p-p+2)}{(p-2)!} - \frac{\Delta^{p-3}\,Q(\beta_p-p+3)}{(p-3)!}$$

$$2c_{2p} = \left[\frac{\Delta^{p-2} R(\beta_p-p+3)}{(p-2)!} - \frac{\Delta^{p-3} Q(\beta_p-p+4)}{(p-3)!}\right]c_{1p} -$$

$$- \left[\frac{\Delta^{p-3} R(\beta_p-p+3)}{(p-3)!} - \frac{\Delta^{p-4} Q(\beta_p-p+4)}{(p-4)!}\right], \tag{19}$$

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$$kc_{kp} = \sum_{j=1}^{p-2} (-1)^{p-j} \left[\frac{\Delta^{j} R(\beta_p+k-p+1)}{j!} - \frac{\Delta^{j-1} Q(\beta_p+k-p+2)}{(j-1)!}\right]\cdot$$

$$\cdot c_{k-p+j+1,p} + (-1)^p R(\beta_p+k-p+1)c_{k-p+1,p},$$

where

$$R(x) = \prod_{j=}^{p} (x+\alpha_j), \quad Q(x) = x \prod_{j=1}^{p-1} (x+\rho_j-1), \tag{20}$$

and $\Delta$ is the difference operator defined by

$$\Delta^{\circ} f(x) = f(x), \quad \Delta f(x) = f(x+1)-f(x), \tag{21}$$

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k) \tag{22}$$

The coefficients $b_{kp}$ may be found from equalities (5.43)

$$b_{kp} = \Gamma\begin{bmatrix}\rho_1,\dots,\rho_{p-1},-\beta_p\\ \alpha_1,\dots,\alpha_p\end{bmatrix} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\xi_p(t)}{(t-1)^{k+1}}\, dt, \tag{23}$$

$$0<\gamma<1 .$$

The functions $\xi_p(z)$ and $\zeta_p(z)$ can be expressed in various forms, for example, as follows (see (2.8), (2.44), (5.44), (5.41)):

$$\xi_1(z) = (1-z)^{-\alpha_1}, \tag{24}$$

$$\xi_2(z) = (1-z)^{\beta_2}\, {}_2F_1(\rho_1-\alpha_1,\rho_1-\alpha_2;\beta_2+1;1-z), \tag{25}$$

$$\xi_j(z) = \Gamma\begin{bmatrix}\beta_j+1\\ \beta_{j-1}+1,\rho_{j-1}-\alpha_j\end{bmatrix}\cdot$$

$$z^{1-\rho_{j-1}} \int_z^1 t^{\alpha_{j-1}} (t-z)^{\rho_{j-1}-\alpha_j-1} \xi_{j-1}(t)\, dt, \tag{26}$$

$$\mathrm{Re}\beta_j > \mathrm{Re}\beta_{j-1} > -1,$$

$$\xi_p(z) = \Gamma(\beta_p+1) G^{p0}_{pp}\left(z\left|\begin{matrix}1-\alpha_1,\dots,1-\alpha_p\\ 0,1-\rho_1,\dots,1-\rho_{p-1}\end{matrix}\right.\right), \ |z|<1; \tag{27}$$

$$\zeta_1(z) = 0, \tag{28}$$

$$\zeta_2(z) = \Gamma\begin{bmatrix}\rho_1,\beta_2\\ \rho_1-\alpha_1,\rho_1-\alpha_2\end{bmatrix} {}_2F_1\binom{\alpha_1,\alpha_2;1-z}{1-\beta_2}, \tag{29}$$

$$\zeta_p(z) = \Gamma\begin{bmatrix}\rho_1,\dots,\rho_{p-1}\\ \alpha_1,\dots,\alpha_p\end{bmatrix} \frac{\pi}{\sin\beta_p\pi}\cdot$$

$$\{ G^{1,p}_{p+1,p+1}\left(z\left|\begin{matrix}1-\alpha_1,\dots,1-\alpha_p,\beta_p\\ 0,\beta_p,1-\rho_1,\dots,1-\rho_{p-1}\end{matrix}\right.\right) +$$

$$+ G^{p0}_{pp}\left(z\left|\begin{matrix}1-\alpha_1,\dots,1-\alpha_p\\ 0,1-\rho_1,\dots,1-\rho_{p-1}\end{matrix}\right.\right)\}, \tag{30}$$

$$\zeta_p(z) = \Gamma\begin{bmatrix}\rho_1,\dots,\rho_{p-1},-\beta_p\\ \alpha_1,\dots,\alpha_p\end{bmatrix} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\xi_p(t)}{t-z}\, dt, \tag{31}$$

where $0<\gamma<1$, and $z$ lies to the right of the integration path.

It follows from (27) that the property (2.23) is fulfilled

$$\xi_p(0) = \Gamma\begin{bmatrix}1+\beta_p,1-\rho_1,\dots,1-\rho_{p-1}\\ 1-\alpha_1,\dots,1-\alpha_p\end{bmatrix}, \quad \mathrm{Re}\rho_j<1, \tag{32}$$

$$\beta_p \neq -1,-2,\dots$$

The following expression of $\zeta_p(z)$ (5.45)* is important

$$\zeta_p(z) = \Gamma\begin{bmatrix}\rho_1,\dots,\rho_{p-1}\\ \alpha_1,\dots,\alpha_p\end{bmatrix}$$

$$\sum_{j=1}^{p-1} \cdot \frac{\prod_{k=1}^{p} \sin(\alpha_k-\rho_j)\pi}{\prod_{k=1}^{p-1}{}' \sin(\rho_k-\rho_j)\pi} R_j(z)\ , \qquad (33)$$

where $R_j(z)$ analytical in the circle $|z-1|<1$ may be expressed by any of the following formulas (see (5.1), (3.4), (1.13), (5.20), (5.21), (5.7)) :

$$R_j(z) = \frac{1}{\sin\beta_p\pi\sin\rho_j\pi} \Big\{ \Gamma\begin{bmatrix}\alpha_1,\dots,\alpha_p\\ \rho_1,\dots,\rho_{p-1}\end{bmatrix}. \qquad (34)$$

$${}_pF_{p-1}\begin{pmatrix}\alpha_1,\dots,\alpha_p;z\\ \rho_1,\dots,\rho_{p-1}\end{pmatrix} - \Gamma\begin{bmatrix}1+\alpha_1-\rho_j,\dots,1+\alpha_p-\rho_j\\ 2-\rho_j,1+\rho_1-\rho_j,\dots,1+\rho_{p-1}-\rho_j\end{bmatrix}.$$

$$z^{1-\rho_j}\ {}_pF_{p-1}\begin{pmatrix}1+\alpha_1-\rho_j,\dots,1+\alpha_p-\rho_j;z\\ 2-\rho_j,1+\rho_1-\rho_j,.*.,1+\rho_{p-1}-\rho_j\end{pmatrix}\Big\},$$

$$R_j(z) = \frac{\pi^{-1}}{\sin\beta_p\pi}\Gamma\begin{bmatrix}1+\alpha_1-\rho_j,1+\alpha_2-\rho_j,\alpha_1,\dots,\alpha_p,\rho_j\\ 1+\alpha_1+\alpha_2-\rho_j,\rho_1,\dots,\rho_{p-1}\end{bmatrix}.$$

$$\sum_{k=0}^{\infty}\frac{(1+\alpha_1-\rho_j)_k(1+\alpha_2-\rho_j)_k}{(1+\alpha_1+\alpha_2-\rho_j)_k\ k!}\ {}_pF_{p-1}\begin{pmatrix}-k,\alpha_3,\dots,\alpha_p,\rho_j;z\\ \rho_1,\dots,\rho_{p-1}\end{pmatrix},$$

(35)

$$\operatorname{Re}(\rho_j-\alpha_\ell) < 1, \quad \ell = 3,4,\dots,p,$$

$$R_j(z) = \frac{\pi^{-1}}{\sin\beta_p\pi}\ G^{2,p+1}_{p+1,p+1}\left(z\,\middle|\,\begin{matrix}1-\alpha_1,\dots,1-\alpha_p,1-\rho_j\\ 0,1-\rho_j,1-\rho_1,\dots,1-\rho_{p-1}\end{matrix}\right).$$

(36)

The functions $R_j(z)$, $j=1,2,\dots,p-1$, together with $\xi_p(z)$ compose the fundamental solution system $u_j^1(z)$, $z=1,2,\dots,p$, for equation (4) in the case of non-integral $\beta_p \neq 0,\pm1,\pm2,\dots$ Equalities (11), (33) reflect the important property of differential equations that any p+1 particular solutions of a linear differential equation of the p-th order are related by the linear equation with the concrete coefficients.

It should be noted that solution $\xi_p(z)$ is expressed in terms of $z^{1-\rho_j}\ {}_pF_{p-1}(z)$ by (3.44), which is the inverse of equalities (11), (33), (34) and is of a simpler form

$$\xi_p(z) = \sum_{j=1}^{p}\Gamma\begin{bmatrix}1+\beta_p,\rho_j-\rho_1,.*.,\rho_j-\rho_{p-1}\\ \rho_j-\alpha_1,\dots,\rho_j-\alpha_p\end{bmatrix} z^{1-\rho_j}\ .$$

(37)

$${}_pF_{p-1}\begin{pmatrix}1+\alpha_1-\rho_j,\dots,1+\alpha_p-\rho_j;z\\ 2-\rho_j,1+\rho_1-\rho_j,.*.,1+\rho_{p-1}-\rho_j\end{pmatrix},\rho_p=1\ .$$

We present here also the formula (1.21)

$${}_pF_{p-1}\begin{pmatrix}\alpha_1,\dots,\alpha_p;xz\\ \rho_1,\dots,\rho_{p-1}\end{pmatrix} =$$

$$(1-z)^{-\alpha_1}\sum_{k=0}^{\infty}\frac{(\alpha_1)_k}{k!}\ {}_pF_{p-1}\begin{pmatrix}-k,\alpha_2,\dots,\alpha_p;x\\ \rho_1,\dots,\rho_{p-1}\end{pmatrix}\left(\frac{z}{z-1}\right)^k,$$

(38)

which at p=2 and x=1 becomes the well-known equality

$${}_2F_1(\alpha_1,\alpha_2;\rho_1;z)=(1-z)^{-\alpha_1}\ {}_2F_1(\alpha_1,\rho_1-\alpha_2;\rho_1;\ \frac{z}{z-1})\ . \qquad (39)$$

2. Now let $\beta_p=m$, $m=0,1,2,\dots$ Then relation (6.2) is valid

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* Prime in the product $\prod_k{}' \sin(\rho_k-\rho_j)\pi$ denotes absence of $\sin(\rho_j-\rho_j)\pi$.

$$
{}_pF_{p-1}\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;z\\ \rho_1,\ldots,\rho_{p-1}\end{matrix}\right)=
$$

$$
\sum_{k=0}^{m-1} d_k(z-1)^k - \frac{(-1)^m}{m!}\,\Gamma\begin{bmatrix}\rho_1,\ldots,\rho_{p-1}\\ \alpha_1,\ldots,\alpha_p\end{bmatrix} \tag{40}
$$

$$
\cdot\left[\xi_p(z)\,\ell n(1-z)-\theta_p(z)\right], \qquad |\arg(1-z)|<\pi ,
$$

where $\xi_p(z)$ has been defined above and $d_k$ are expressed by the formulas

$$
d_k=\frac{(\alpha_1)_k\cdots(\alpha_p)_k}{(\rho_1)_k\cdots(\rho_{p-1})_k\,k!}\;{}_pF_{p-1}\left(\begin{matrix}\alpha_1+k,\ldots,\alpha_p+k;1\\ \rho_1+k,\ldots,\rho_{p-1}+k\end{matrix}\right), \tag{41}
$$

and the function $\theta_p(z)$ continuous in the vicinity of z=1 can be expressed in any of the following forms (see (6.3), (6.4)) :

$$
\theta_p(z)=-\frac{(1-z)^m}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\ell n(1-t)}{(1-t)^m}\,\frac{\xi_p(t)}{t-z}\,dt\ , \quad 0<\gamma<1, \tag{42}
$$

(z lies to the right of the integration path) or

$$
\theta_p(z)=(1-z)^m\sum_{k=0}^{\infty}e_k(1-z)^k\ , \qquad |1-z|<1, \tag{43}
$$

where

$$
e_k=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\ell n(1-t)\,\frac{\xi_p(t)}{(1-t)^{m+k+1}}\,dt\ , \ 0<\gamma<1. \tag{44}
$$

In order the the integrals (42), (44) converge, the conditions Re $\alpha_j>-m$, $\alpha_j\neq 0,-1,-2,\ldots, j=1,2,\ldots,p$, should be satisfied.

If the condition

$$
\prod_{j=1}^{m-p+1} R(j)=0\ , \tag{45}
$$

where R(x) is expressed by (20), is met, then in the equality (40) the second term in the right-hand side, which contains a logarithmic term, is absent and the sum, containing $d_k$, only remains.

In particular, at p=2 formula (40) becomes the following relation (see 2.10 (12-13) from [1]) :

$$
{}_2F_1(\alpha_1,\alpha_2;\alpha_1+\alpha_2+m;z)=\Gamma\begin{bmatrix}\alpha_1+\alpha_2+m\\ \alpha_1+m,\alpha_2+m\end{bmatrix}(m-1)!\ . \tag{46}
$$

$$
\cdot\sum_{k=0}^{m-1}\frac{(\alpha_1)_k(\alpha_2)_k}{(1-m)_k\,k!}(1-z)^k+(-1)^m(1-z)^m\,\Gamma\begin{bmatrix}\alpha_1+\alpha_2+m\\ \alpha_1,\alpha_2\end{bmatrix}\ .
$$

$$
\cdot\sum_{k=0}^{\infty}\frac{(\alpha_1+m)_k(\alpha_2+m)_k}{(k+m)!\,k!}\left[\psi(k+1)+\psi(k+m+1)-\psi(\alpha_1+k+m)\ -\right.
$$

$$
\left.-\psi(\alpha_2+k+m)-\ell n(1-z)\right](1-z)^k\ , \ |\arg(1-z)|<\pi\ .
$$

3. If $\beta_p=-m$, m=1,2,3, ..., then equality (6.11)

$$
{}_pF_{p-1}\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;z\\ \rho_1,\ldots,\rho_{p-1}\end{matrix}\right)=\Gamma\begin{bmatrix}\rho_1,\ldots,\rho_{p-1}\\ \alpha_1,\ldots,\alpha_p\end{bmatrix}\{(m-1)!\,(-z)^{-m}\ . \tag{47}
$$

$$
\sum_{k=0}^{m-1}\frac{c_{kp}}{(1-m)_k}(1-z)^k+(-1)^{m-1}\eta_p(z)\,\ell n(1-z)+X_p(z)\}\ ,
$$

$$
|\arg(1-z)|<\pi\ ,
$$

holds where the coefficients $c_{kp}$ are defined by (17) through (19) and $\eta_p(z)$, $X_p(z)$ continuous in the vicinity of z=1 can be expressed in any of the following forms (see(1.34), (2.40), (2.42), (6.12)-(6.14))

$$
\eta_p(z)=\sum_{k=0}^{\infty}\frac{c_{k+m,p}}{k!}(1-z)^k\ , \qquad |1-z|<1\ , \tag{48}
$$

$$\eta_p(z) = G^{p0}_{pp}\left(z \,\middle|\, \begin{matrix} 1-\alpha_1,\dots,1-\alpha_p \\ 0,1-\rho_1,\dots,1-\rho_{p-1} \end{matrix}\right) +$$

$$+(-1)^m G^{0p}_{pp}\left(z \,\middle|\, \begin{matrix} 1-\alpha_1,\dots,1-\alpha_p \\ 0,1-\rho_1,\dots,1-\rho_{p-1} \end{matrix}\right), \tag{49}$$

$$\eta_p(z) = \sum_{k=1}^{p} \Gamma\begin{bmatrix} \rho_k-\rho_1,\dots*\dots,\rho_k-\rho_p \\ \rho_k-\alpha_1,\dots,\rho_k-\alpha_p \end{bmatrix} z^{1-\rho_k} .$$

$$\cdot {}_pF_{p-1}\left(\begin{matrix} 1+\alpha_1-\rho_k,\dots,1+\alpha_p-\rho_k;\ z \\ 1+\rho_1-\rho_k,\dots*\dots,1+\rho_{p-1}-\rho_k,2-\rho_k \end{matrix}\right), |z|<1, \rho_p=1 \quad (50)$$

$$X_p(z) = \frac{(-1)^{m-1}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \ln(1-t) \frac{\eta_p(t)}{t-z}\, dt ,$$

$$0<\gamma<1, \quad (51)$$

(z lies to the right of the integration path) or

$$X_p(z) = (-1)^m \sum_{k=0}^{\infty} e'_k (1-z)^k , \quad |1-z|<1, \quad (52)$$

where

$$e'_k = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \ln(1-t) \frac{\eta_p(t)}{(1-t)^{k+1}}\, dt, \ 0<\gamma<1. \quad (53)$$

If follows from (50) that the equality

$$\eta_p(0) = \Gamma\begin{bmatrix} 1-\rho_1,\dots,1-\rho_{p-1} \\ 1-\alpha_1,\dots,1-\alpha_p \end{bmatrix}, \quad \mathrm{Re}\rho_k<1, \quad k=1,2,\dots,p-1, \quad (54)$$

is valid.

In the case when $\eta_p(z) \equiv 0$, series (47) contains no logarithmic terms and $X_p(z) \equiv 0$. In order that $\eta_p(z) \equiv 0$, the conditions

$$\prod_{j=1}^{m} R(1-\rho_k-j) = 0, \qquad \rho_p=1, \quad k=1,2,\dots,p \quad (55)$$

are necessary and sufficient.

In particular, with p=2 formula (47) becomes the following relation (see 2.10 (14-15) from [1])

$${}_2F_1(\alpha_1,\alpha_2;\alpha_1+\alpha_2-m;z) = \Gamma\begin{bmatrix} \alpha_1+\alpha_2-m \\ \alpha_1,\alpha_2 \end{bmatrix} (m-1)!(1-z)^{-m} .$$

$$\cdot \sum_{k=0}^{m-1} \frac{(\alpha_1-m)_k(\alpha_2-m)_k}{(1-m)_k k!}(1-z)^k + (-1)^m \Gamma\begin{bmatrix} \alpha_1+\alpha_2-m \\ \alpha_1-m,\alpha_2-m \end{bmatrix} . \quad (56)$$

$$\cdot \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k}{(k+m)!\,k!} \Big[\psi(k+1)+\psi(k+m+1)-\psi(\alpha_1+k)-\psi(\alpha_2+k)-$$

$$-\ln(1-z)\Big](1-z)^k , \qquad |\arg(1-z)|<\pi.$$

4. In the cases when $\beta_p=\pm m$, m=0.1,2,..., the fundamental solution system $u_j^1(z)$, j=1,2,...,p, for equation (4) in the vicinity of z=1 may be composed of the functions $R_j(z)$, j=1,2,...,p-1, regular at z=1 and the function $\tilde{R}_p(z)$, which has a logarithmic singularity at z=1. The function $u_p^1(z) = \tilde{R}_p(z)$ may be assumed equal to the right-hand sides of (40) or (47), and the functions $R_j(z)$ are defined by the equalities (34) through (36) at non-integral $\rho_j$ or from continuity from these equalities at integral $\rho_j$.

5. It follows from formulas (11), (40), (47) that at Re $\beta_p>0$ the relation

$${}_pF_{p-1}\left(\begin{matrix} \alpha_1,\dots,\alpha_p;z \\ \rho_1,\dots,\rho_{p-1} \end{matrix}\right) \xrightarrow[z\to1]{} {}_pF_{p-1}\left(\begin{matrix} \alpha_1,\dots,\alpha_p;1 \\ \rho_1,\dots,\rho_{p-1} \end{matrix}\right) |\arg(1-z)|<\pi, \quad (57)$$

holds. In the case of p=2, the limit value ${}_pF_{p-1}(1)$ is expressed in terms of gamma-functions by the formula (see (14)-(15)) :

$${}_2F_1(\alpha_1,\alpha_2;\rho_1;1) = \Gamma\begin{bmatrix} \rho_1,\rho_1-\alpha_1-\alpha_2 \\ \rho_1-\alpha_1,\rho_1-\alpha_2 \end{bmatrix} ,$$

$$\mathrm{Re}(\rho_1-\alpha_1-\alpha_2)>0. \quad (58)$$

With p>2 this value can be expressed in terms of gamma-functions only with special additional conditions for the parameters $\alpha_j$, $\rho_j$ and the order p (see Sec. 4.4 from [1]).

If $\beta_p=0$, then at $z\to 1$, $|\arg(1-z)| < \pi$, the relation

$$ {}_pF_{p-1}\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;z\\ \rho_1,\ldots,\rho_{p-1}\end{matrix}\right) \sim -\Gamma\left[\begin{matrix}\rho_1,\ldots,\rho_{p-1}\\ \alpha_1,\ldots,\alpha_p\end{matrix}\right] \ln(1-z)+O(1) \tag{59}$$

is valid, and if Re $\beta_p<0$, then at $z\to 1$, $|\arg(1-z)|<\pi$ the equality

$$ \lim_{z\to 1}\left\{ (1-z)^{-\beta_p}\, {}_pF_{p-1}\left(\begin{matrix}\alpha_1,\ldots,\alpha_p;z\\ \rho_1,\ldots,\rho_{p-1}\end{matrix}\right)\right\} = \Gamma\left[\begin{matrix}\rho_1,\ldots,\rho_{p-1},-\beta_p\\ \alpha_1,\ldots,\alpha_p\end{matrix}\right] \tag{60}$$

holds.

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