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RECURRENCE RELATIONS FOR MODIFIED MOMENTS

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ABSTRACT

Recurrence relation for the modified moments

$$\int_{-1}^{1} f(x)C_k^{\lambda}(x)dx$$

of the function f with respect to the Gegenbauer polynomials C_k^{λ} of order $\lambda(\lambda>-1/2)$ is presented. The result is obtained under the assumption that f satisfies a linear differential equation with polynomial coefficients.

RESUMEN

Se presenta la relación de recurrencia para los momentos modificados

$$\int_{-1}^{1} f(x)C_k^{\lambda}(x)dx$$

de la función f con respecto a los polinomios de Gegenbauer C_k^{λ} de orden $\lambda(\lambda>-1/2)$. El resultado se obtiene bajo la suposición de que f satisface una ecuación diferencial lineal con coeficientes polinómicos.

1. INTRODUCTION

Let f be a function defined on the interval (-1,1) and such that

$$\int_{-1}^{1} f(x)x^k dx < \infty \qquad (k \geq 0).$$

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According to Gautschi [4], we call

$$m_k^{\lambda}[f] = \int_{-1}^{1} f(x)C_k^{\lambda}(x)dx \qquad (\lambda > -1/2) \quad (1.1)$$

modified moments of f with respect to the Gegenbauer polynomials C_k^{λ} (in short : the Gegenbauer moments). The case $\lambda = 0$ is of particular importance. We have

$$C_0^0 = T_0, \quad C_k^0 = \frac{2}{k} T_k \quad (k \geq 1),$$

where T_k are the Chebyshev polynomials of the first kind. We shall refer to the integrals

$$\tau_k[f] = \int_{-1}^{1} f(x)T_k(x)dx \qquad (1.2)$$

as the Chebyshev moments of f. Obviously,

$$\tau_0[f] = m_0^0[f], \quad \tau_k[f] = \frac{k}{2} m_k^0[f] \quad (k\geq 1). \qquad (1.3)$$

Some important computational problems in the field of numerical integration can be solved by procedures, which require the Gegenbauer or Chebyshev moments ([2], [4],[5],[9],[13],[14],[16], [17]). Sometimes these moments can be computed directly in integer form or in terms of special functions ([1],[3],[6],[7],[8],[18]). Most frequently, however, they are obtained from recurrence relations ([2],[9],[10],[11],[12],[13],[14],[19]).

In [10], some algorithms have been presented for obtaining recurrence relation for Gegenbauer moments of a function f which satisfies the linear differential equation

$$\sum_{m=0}^{n} p_m f^{(m)} = q \qquad (1.4)$$

of order n, where p_m are polynomials and the function q is such that the moments $m_k^{\lambda}[q]$ exist and are known. The first algorithm, which, however, is also the most complex, leads to a recurrence relation of the lowest possible order.

The aim here is to present a new algorithm, which has some remarkable features. First, the recurrence relation and its order are obtained in integer forms, being expressed in terms of the coefficients of equation (1.4). Second, the computational complexity is considerably reduced. Third, the algorithm can be easily implement in a computer language designed to perform symbolic manipulations of rational expressions. (Notice that Robertson [15] wrote an ALTRAN program which is an implementation of the first method of [10].)

Let us remark that recurrence relations for the Jacobi moments

$$\int_{-1}^{1} f(x)P_k^{(\alpha,\beta)}(x)\,dx \qquad (\alpha > -1,\ \beta > -1)$$

may be obtained using a similar technique.

2. DIFFERENCE OPERATORS

The results given in the following sections are expressed in terms of a certain type of linear operator. Consider the space S of all "doubly infinite" sequences of complex numbers (all functions $s : Z \to C$, if you will). Under the usual operations of pointwise addition and scalar multiplication, S becomes a linear space over C.

Let S* be the space of all linear mappings of S into itself. Given $T \in S^*$ and a sequence $\{z_k\} \in S$, let the k-th component of the sequence $T\{z_k\}$ be denoted by Tz_k, so that

$$T\{z_k\} = \{Tz_k\}.$$

Let the symbols I, Θ and E^m ($m \in Z$) denote the identity operator, the zero operator and the m-th shift operator on S, respectively. We then have

$$Iz_k = z_k, \quad \Theta z_k = 0, \quad E^m z_k = z_{k+m} \qquad (k \in Z).$$

Clearly $E^0 = I$.

Let L be the set of all operators from S* that can be expressed in the form

$$L = \sum_{j=0}^{r} \lambda_j(k)E^{u+j},$$

where $\lambda_0, \lambda_1, \ldots, \lambda_r$ are rational functions in S, $u, r \in Z$, $r \geq 0$. Every non-zero operator $L \in L$ can be expressed in the above form with $\lambda_0 \neq 0$ and $\lambda_r \neq 0$. $r = r(L)$ is referred to as the order of L, while $\lambda_0, \ldots, \lambda_r$ are called the coefficients of the operator L. The elements of the set L are known as difference operators. Let L_1, L_2 be two difference operators with

$$L_m = \sum_{j=0}^{r_m} \lambda_{mj}(k)E^{u_m+j} \qquad (m = 1, 2).$$

We define the product of L_1 and L_2 to be the operator

$$L_1L_2 = \sum_{i=0}^{1}\sum_{j=0}^{2} \lambda_{1i}(k)\lambda_{2j}(k+i+u_1)E^{u_1+u_2+i+j}$$

Under this multiplication and the addition defined in a natural manner, L forms a ring with unity I.

Let $L \in L$ and $\mu \in S$. Equation

$$Lz_k = \mu(k) \qquad (k \in Z)$$

is called the recurrence relation for the sequence $\{z_k\} \in S$. The order of the recurrence relation is the order of the difference operator L.

3. PROPERTIES OF GEGENBAUER MOMENTS

In the sequel it will be convenient to have defined Gegenbauer moments $m_k^{\lambda}[f]$ with negative indices. If 2λ is not an integer, we define

$$m^{\lambda}_{-k}[f] = 0 \qquad (k \geq 1).$$

When $2\lambda = h$, $h\epsilon Z$, we must define [10]

$$m^{\lambda}_{-k}[f] = 0 \qquad (k = 1, 2, \ldots, h-1),$$

$$= -m^{\lambda}_{k-h}[f] \quad (k = h, h+1, \ldots).$$

Let us define the difference operators D and X by

$$D = \frac{1}{2(k+\lambda)}(E^{-1}-E) \tag{3.1}$$

and

$$X = \frac{1}{2(k+\lambda)}\{(k+2\lambda-1)E^{-1}+(k+1)E\}. \tag{3.2}$$

From the following well-known properties of the Gegenbauer polynomials

$$2(k+\lambda)xC^{\lambda}_{k}(x) = (k+1)C^{\lambda}_{k+1}(x)+(k+2\lambda-1)C^{\lambda}_{k-1}(x),$$

$$2(k+\lambda)C^{\lambda}_{k}(x) = \frac{d}{dx}\left[C^{\lambda}_{k+1}(x)- C^{\lambda}_{k-1}(x)\right]$$

and from (1.1), it can be shown that [10]

$$m^{\lambda}_{k}[xf] = Xm^{\lambda}_{k}[f] \tag{3.3}$$

and

$$Dm^{\lambda}_{k}[f'] = m^{\lambda}_{k}[f]+D\phi^{\lambda}_{k}[f], \tag{3.4}$$

where

$$\phi^{\lambda}_{k}[f] = \left[f(x)C^{\lambda}_{k}(x)\right]^{x=1}_{x=-1} \tag{3.5}$$

It is easy to generalize (3.3) so that we can evaluate $m^{\lambda}_{k}[pf]$, where p is any polynomial. Namely, we have

$$m_{k}[pf] = p(X)m^{\lambda}_{k}[f]. \tag{3.6}$$

Similarly, we can replace (3.4) by

$$D^{i}m^{\lambda}_{k}[f^{(i)}] = m^{\lambda}_{k}[f]+\sum_{h=1}^{i} D^{h}\phi^{\lambda}_{k}[f^{(h-1)}] \quad (i\geq 0) \tag{3.7}$$

Let us recall that [10]

$$D^{i} = 2^{-i}(k+\lambda-i)^{-1}_{2i+1}\sum_{j=0}^{i} \rho_{ij}(k+\lambda)E^{2j-i}, \tag{3.8}$$

where

$$\rho_{ij}(k) = (-1)^{j}\binom{i}{j}(\kappa-i)_{j}(\kappa-i+2j)(\kappa+j+1)_{i-j}$$

$$(j = 0, 1, \ldots, i),$$

and that

$$X^{i} = 2^{-i}\sum_{j=0}^{i} \zeta_{ij}(k+\lambda)E^{2j-i}, \tag{3.9}$$

where ζ_{ij} are rational functions defined recursively by

$$\zeta_{00}(\kappa) = 1,$$

$$\zeta_{ij}(\kappa) = \kappa^{-1}\{(\kappa+\lambda-1)\zeta_{i-1,j}(\kappa-1)$$

$$+(\kappa-\lambda+1)\zeta_{i-1,j-1}(\kappa+1)\}$$

$(i \geq 1;\ j = 0, 1, \ldots, i;\ \xi_{i-1,-1} = \xi_{i-1,i} = 0);$

in particular,

$$X^i = 2^{-i}k^{-1}\left(\sum_{j=0}^{i} \binom{i}{j} E^{2j-i}\right)(KI) \qquad \text{for } \lambda = 0, \qquad (3.10)$$

$$= 2^{-i} \sum_{j=0}^{i} \binom{i}{j} E^{2j-i} \qquad \text{for } \lambda = 1.$$

Further properties of $m_k^{\lambda}[f]$ will be given in Lemmas 3.1 and 3.2. We first define the difference operators

$$A_i^{(\varepsilon)} = I - \varepsilon \frac{(2k+2\lambda+1)_2}{(2k+2\lambda+i+1)_2} \qquad (3.11)$$

and

$$Q_i^{(\varepsilon)} = (k+2\lambda-1)I + \varepsilon(k+i+2) \frac{(2k+2\lambda+1)_2}{(2k+2\lambda+i+1)_2} E \qquad (3.12)$$

for $i = 0, 1, \ldots$ and $\varepsilon \in [-1,1]$. It can be checked that

$$A_0^{(\varepsilon)}(X+\varepsilon I) = Q_0^{(\varepsilon)} D \qquad (3.13)$$

and that

$$A_i^{(\varepsilon)} Q_{i-1}^{(\varepsilon)} = Q_i^{(\varepsilon)} A_{i-1}^{(\varepsilon)} \qquad (i \geq 1). \qquad (3.14)$$

Let us introduce the notation

$$S_{ij}^{(\varepsilon)} = I \qquad \text{for } i < j, \qquad (3.15)$$

$$= A_i^{(\varepsilon)} S_{i-1,j}^{(\varepsilon)} \qquad \text{for } i \geq j \geq 0,$$

$$P_i^{(\varepsilon)} = S_{i-1,0}^{(\varepsilon)} \qquad \text{for } i \geq 0 \qquad (3.16)$$

and

$$R_h^{(\varepsilon)} = I \qquad \text{for } h = 0, \qquad (3.17)$$

$$= Q_{h-1}^{(\varepsilon)} R_{h-1}^{(\varepsilon)} \qquad \text{for } h \geq 1.$$

LEMMA 3.1. For any $i = 0, 1, \ldots$ and $\varepsilon \in [-1,1]$ we have

$$P_i^{(\varepsilon)} m_k^{\lambda}\left[\delta_\varepsilon^i f\right] = R_i^{(\varepsilon)} m_k^{\lambda}[f] + \sum_{h=1}^{i} S_{i-1,h}^{(\varepsilon)} R_h^{(\varepsilon)} D\phi_k^{\lambda}\left[\delta_\varepsilon^{i-h} f\right], \qquad (3.18)$$

where $\delta_\varepsilon = (x+\varepsilon)\dfrac{d}{dx}$.

Proof. For $i = 0$ equation 3.18 holds trivially, and for $i = 1$ it readily follows from (3.16), (3.6), (3.13), (3.4) and (3.17). Assuming that (3.18) is true for a certain $i (i \geq 1)$, we obtain from (3.15) - (3.17)

$$P_{i+1}^{(\varepsilon)} m_k^{\lambda}\left[\delta_\varepsilon^{i+1} f\right] = A_i^{(\varepsilon)} P_i^{(\varepsilon)} m_k^{\lambda}\left[\delta_\varepsilon^{i}(\delta_\varepsilon f)\right] \qquad (3.19)$$

$$= A_i^{(\varepsilon)} R_i^{(\varepsilon)} m_k^{\lambda}[\delta_\varepsilon f] + \sum_{h=1}^{i} S_{ih}^{(\varepsilon)} R_h^{(\varepsilon)} D\phi_k^{\lambda}\left[\delta_\varepsilon^{i-h+1} f\right].$$

From (3.14),

$$A_i^{(\varepsilon)} R_i^{(\varepsilon)} = Q_i^{(\varepsilon)} Q_{i-1}^{(\varepsilon)} \ldots Q_1^{(\varepsilon)} P_1^{(\varepsilon)}.$$

For i = 1, (3.18) reduces to

$$P_1^{(\varepsilon)} m_k^\lambda[\delta_\varepsilon f] = R_1^{(\varepsilon)} m_k^\lambda [f] + R_1^{(\varepsilon)} D\phi_k^\lambda[f].$$

Thus, the last expression in (3.19) can be written as

$$R_{i+1}^{(\varepsilon)} m_k^\lambda[f] + \sum_{h=1}^{i+1} S_{ih}^{(\varepsilon)} R_h^{(\varepsilon)} D\phi_k^\lambda\left[\delta_\varepsilon^{i+1-h}\right].$$

Define H, $H_i^{(\varepsilon)}$, $V_j^{(\varepsilon)} \in L$ by

$$H = \frac{1}{2(k+\lambda)} \{(k+2\lambda-2)_2 E^{-1} - (k+1)_2 E\}, \qquad (3.20)$$

$$H_i^{(\varepsilon)} = H + i(X+\varepsilon I) \qquad \text{for } i \geq 0 \text{ and } \varepsilon = \pm 1, \quad (3.21)$$

$$V_j^{(\varepsilon)} = I \qquad \text{for } j = 0 \text{ and } \varepsilon = \pm 1, \qquad (3.22)$$

$$= V_{j-1}^{(\varepsilon)} H_j^{(\varepsilon)} \qquad \text{for } j \geq 1 \text{ and } \varepsilon = \pm 1.$$

LEMMA 3.2. For any $j \geq 0$ and $\varepsilon \in \{-1,1\}$ we have the identity

$$m_k^\lambda\left[\delta_\varepsilon^j\{(x-\varepsilon)^j f(x)\}\right] = V_j^{(\varepsilon)} m_k^\lambda[f], \quad \delta_\varepsilon = (x+\varepsilon)\frac{d}{dx}. \qquad (3.23)$$

Proof. It can be shown that [10]

$$m_k^\lambda[(x^2-1)f'] = H m_k^\lambda[f]. \qquad (3.24)$$

We shall need the identity

$$\delta_\varepsilon\{(x-\varepsilon)^{j+1} f(x)\} = (x-\varepsilon)^j\{(x^2-1)f'(x) + (j+1)(x+\varepsilon)f(x)\} \quad (j \geq 0). \qquad (3.25)$$

Equation (3.23) holds trivially for j=0. Assuming that it is true for a certain j ($j \geq 0$) and using (3.25) and (3.24), we obtain

$$m_k^\lambda\left[\delta_\varepsilon^{j+1}\{(x-\varepsilon)^{j+1} f(x)\}\right] = V_j^{(\varepsilon)} m_k^\lambda\left[(x^2-1)f'(x) + (j+1)(x+\varepsilon)f(x)\right]$$

$$= V_j^{(\varepsilon)} \{H + (j+1)(X+\varepsilon I)\} m_k^\lambda[f] = V_{j+1}^{(\varepsilon)} m_k^\lambda[f].$$

4. RECURRENCE RELATION FOR GEGENBAUER MOMENTS

The main result of this paper is contained in Theorem 4.1 below. Let us observe first that the differential equation (1.4) may be written in the equivalent form

$$\sum_{i=0}^{n} (q_i f)^{(i)} = g, \qquad (4.1)$$

where

$$q_i = \sum_{j=i}^{n} (-1)^{j-i} \binom{j}{i} p_j^{(j-i)} \qquad (i = 0, 1, \ldots, n). \qquad (4.2)$$

We shall need the following result.

LEMMA 4.1. For every $i \geq 0$ and for $\varepsilon = \pm 1$ we have.

$$\frac{d^i}{dx^i}\{(x+\varepsilon)^i f(x)\} = \sum_{h=0}^{i} \beta_h^{(i)} \delta_\varepsilon^{i-h} f,$$

where $\delta_\varepsilon = (x+\varepsilon)\frac{d}{dx}$,

$$\beta_h^{(i)} = (-1)^h \binom{i}{h} B_h^{(i+1)}, \qquad (4.4)$$

and $B_s^{(a)}$ are the generalized Bernoulli numbers defined implicitly by

$$\left(\frac{t}{e^t-1}\right)^a = \sum_{s=0}^{\infty} \frac{t^s}{s!} B_s^{(a)}.$$

Proof. Combining the identity ([12], v.1, Eq. 2.8 (16))

$$(y+1)_i = \sum_{h=0}^{i} \beta_h^{(i)} y^{i-h}$$

with (ibid., Eq. 2.9 (4))

$$\frac{d^i}{dz^i}\{z^i q(z)\} = \prod_{j=1}^{i} (\delta+j) g(z) \qquad (\delta = z\frac{d}{dz}),$$

we obtain

$$\frac{d^i}{dz^i}\{z^i g(z)\} = \sum_{h=0}^{i} \beta_h^{(i)} \delta^{i-h} g(z).$$

For ε {-1,1}, let $z = x+\varepsilon$ and $f(x) = g(z)$. Obviously, $\delta_\varepsilon f(x) = \delta g(z)$ and (4.3) is simply a transcription of (4.5).

It should be remarked that the coefficients (4.4) can be calculated recursively using formulae (c.f. [12], v.1, Eq 2.8 (7))

$$\beta_h^{(i)} = \beta_h^{(i-1)} + i\beta_{h-1}^{(i-1)} \qquad (i \geq 1 \; ; \; h \geq 0),$$

$$\beta_0^{(i)} = 1, \quad \beta_{i+1}^{(i)} = 0 \qquad (i \geq 0). \qquad (4.6)$$

THEOREM 4.1. Let f be a real function defined on (-1,1) satisfying the differential equation (4.1), and let the Gegenbauer moments $m_k^\lambda[f^{(i)}]$ ($i = 0, 1, \ldots, n$) and $m_k^\lambda[q]$ exist ($\lambda > -1/2$). Let e_{hi} be non negative integers such that the coefficients q_i of equation (4.1) can be written in the form

$$q_i(x) = (x+1)^{e_{1i}}(x-1)^{e_{-1,i}} u_i(x) \qquad (4.7)$$

$$(i = 1,2,\ldots,n;\; q_i \not\equiv 0),$$

where u_i is a polynomial, $u_i(\pm 1) \neq 0$. Let

$$s_h = \max\{\max_{1\leq i\leq n,\, q_i\neq 0} (i-e_{hi}), 0\} \qquad (h = \pm 1) \qquad (4.8)$$

and let

$$\varepsilon = 1 \quad \text{for } s_1 \leq s_{-1}, \qquad (4.9)$$
$$\quad = -1 \quad \text{for } s_1 > s_{-1},$$

$$s = s_\varepsilon, \quad \sigma = s_{-\varepsilon}, \quad d = \sigma - s. \qquad (4.10)$$

Finally, define the polynomials

$$\mu_h(x) = \sum_{i=h}^{n-s} \beta_{i-h}^{(i)} (x+\varepsilon)^{-i} q_{i+s}(x) \quad \text{for } h=0,1,\ldots,n-s, \qquad (4.11)$$

and

$$\nu_j(x) = (x-\varepsilon)^{-j} \mu_{j+d}(x) \quad \text{for } j = 1,2,\ldots, n-\sigma. \qquad (4.12)$$

Then we have the recurrence relation

$$L m_k^{\lambda}[f] = \tau(k), \qquad (4.13)$$

where

$$L = P_d^{(\varepsilon)} \sum_{i=0}^{s-1} D^{s-i} q_i(X) + \sum_{h=0}^{d} S_{d-1,h}^{(\varepsilon)} R_h^{(\varepsilon)} \mu_h(X) + R_d^{(\varepsilon)} \sum_{h=1}^{n-\sigma} V_h^{(\varepsilon)} \nu_h(X) \qquad (4.14)$$

and

$$\tau(k) = P_d^{(\varepsilon)} D^s m_k^{\lambda}[q] - P_d^{(\varepsilon)} \sum_{j=1}^{s} D^{s-j+1} \sum_{h=0}^{n-j} \phi_k \left[(q_{h+j} f)^{(h)}\right] - \sum_{j=1}^{d} S_{d-1,j}^{(\varepsilon)} R_j^{(\varepsilon)} D \sum_{h=0}^{n-s-j} \phi_k^{\lambda}\left[\delta_\varepsilon^h \mu_{h+j} f)\right].$$

The order of (4.13) is expressed by

$$r(L) = s_{-1} + s_1 + 2 \max_{0 \le i \le n,\, q_i \ne 0} (\deg q_i - i). \qquad (4.16)$$

Proof. Equation (4.1) implies the identity

$$\sum_{i=0}^{n} m_k^{\lambda}\left[(q_i f)^{(i)}\right] = m_k^{\lambda}[q]. \qquad (4.17)$$

Let s be the integer defined in (4.10). Let the operator D^s act on both sides of (4.17) and use (3.6) and (3.7). We obtain, after some algebra,

$$\sum_{i=0}^{s-1} D^{s-i} q_i(X) m_k^{\lambda}[f] + \sum_{i=s}^{n} m_k^{\lambda}\left[(q_i f)^{(i-s)}\right] = \eta(k), \qquad (4.18)$$

where

$$\eta(k) = D^s m_k^{\lambda}[q] - \sum_{j=1}^{s} D^{s-j+1} \sum_{h=0}^{n-j} \phi_k^{\lambda}\left[(q_{h+j} f)^{(h)}\right]. \qquad (4.19)$$

Now, it readily follows from (4.8) - (4.10) that $e_{\varepsilon i} \ge i-s \ge i-\sigma$. Thus the formulae (4.11) and (4.12) actually define polynomials. Let us denote $\nu_i(x) = (x+\varepsilon)^{s-i} q_i(x)$ for $i \ge s$. Using Lemma 4.1 we transform the second sum on the l.h.s. of (4.18) :

$$\sum_{i=s}^{n} m_k^{\lambda}\left[(q_j f)^{(i-s)}\right] = \sum_{i=0}^{n-s} m_k^{\lambda}\left[\{(x+\varepsilon)^i \nu_{s+i}(x) f(x)\}^{(i)}\right]$$

$$= \sum_{i=0}^{n-s} \sum_{h=0}^{i} \beta_h^{(i)} m_k^{\lambda}\left[\delta_\varepsilon^{i-h}(\nu_{i+s} f)\right] = \sum_{h=0}^{n-s} m_k^{\lambda}\left[\delta_\varepsilon^{h}(\mu_h f)\right].$$

Here μ_h is the polynomial (4.11).

Let ε and d be the integers defined in (4.9) and (4.10), respectively. Apply the operator $P_d^{(\varepsilon)}$ (see (3.16)) to both sides of (4.18), then use Lemma 3.1 and (3.6). The result can be written in the form

$$\{P_d^{(\varepsilon)} \sum_{i=0}^{s-1} D^{s-i} q_i(X) + \sum_{h=0}^{d} S_{d-1,h}^{(\varepsilon)} R_h^{(\varepsilon)} \mu_h(X)\} m_k^{\lambda}[f] \qquad (4.20)$$

$$+ R_d^{(\varepsilon)} \sum_{h=1}^{n-\sigma} m_k^{\lambda}\left[\delta_\varepsilon^{h}(\mu_{h+d} f)\right] = \tau(k),$$

where

$$\tau(k) = P_d^{(\varepsilon)} \eta(k) - \sum_{j=1}^{d} S_{d-1,j}^{(\varepsilon)} R_j^{(\varepsilon)} D \sum_{h=0}^{n-s-j} \phi_k^{\lambda}\left[\delta_\varepsilon^{h}(\mu_{h+j} f)\right]. \qquad (4.21)$$

Using (4.12), Lemma 3.2 and (3.6), we have

$$\sum_{h=1}^{n-\sigma} m_k^{\lambda}\left[\delta_{\varepsilon}^{h}(\mu_{h+d}f)\right] = \sum_{h=1}^{n-\sigma} m_k^{\lambda}\left[\delta_{\varepsilon}^{h}\{(x-\varepsilon)^{h}\nu_h(x)f(x)\}\right]$$

$$= \sum_{h=1}^{n-\sigma} V_h^{(\varepsilon)} m_k^{\lambda}[\nu_h f] = \sum_{h=1}^{n-\sigma} V_h^{(\varepsilon)}\,{}_h(X)m_k^{\lambda}[f].$$

This together with (4.19) - (4.21) implies equation (4.13) in which the operator $L\in L$ and the function $\tau\in S$ are given by (4.14) and (4.15), respectively.

As we have remarked, equations (4.1) and (1.4) are equivelent. Now, we have seen that (4.13) is obtained with the help of the difference operator

$$P = P_d^{(\varepsilon)}D^s$$

such that

$$P\sum_{i=0}^{n} m_k^{\lambda}\left[p_i f^{(i)}\right] = Lm_k^{\lambda}[f]+Pm_k^{\lambda}[q]-\tau(k)$$

which implies [10]

$$r(L) = r(P)+2 \max_{0\le i\le n,\, q_i\ne 0} (\deg q_i - i).$$

Observing that $r(P) = r(P_d^{(\varepsilon)})+r(D^s) = d+2s = s+\sigma$, we obtain (4.16).

There are three special cases when (4.14) - (4.16) are significantly simplified.

First special case : $q_i(x) = (x^2-1)^i w_i(x)$ for a polynomial $w_i (i=0, 1, \ldots, n)$. Then

$$L = \sum_{h=0}^{n} V_h^{(1)}\nu_h(X), \tag{4.22}$$

$$\tau(k) = m_k^{\lambda}[q], \tag{4.23}$$

and

$$r(L) = 2 \max_{0\le i\le n;\, w_i\ne 0} (\deg w_i + i). \tag{4.24}$$

Here

$$\nu_h(x) = \sum_{j=h}^{n} \beta_{j-h}^{(j)}(x-1)^{j-h}w_j(x) \quad (h = 0,1, \ldots, n). \tag{4.25}$$

Second special case : $q_i(x) = (x+\varepsilon)^i\nu_i(x)$ for a polynomial $\nu_i (i = 0, 1, \ldots, n)$, and $q_n(\pm\varepsilon) \ne 0$, where $\varepsilon\in\{-1,1\}$. We then have

$$L = \sum_{h=0}^{n} S_{n-1,h}^{(\varepsilon)}R_h^{(\varepsilon)}\mu_h(X), \tag{4.26}$$

$$\tau(k) = P_n^{(\varepsilon)}m_k^{\lambda}[q] - \sum_{j=1}^{n} S_{n-1,j}^{(\varepsilon)}R_j^{(\varepsilon)}D\sum_{h=0}^{n-j}\phi_k^{\lambda}\left[\delta_{\varepsilon}^{h}(\mu_{j+h}f)\right], \tag{4.27}$$

where

$$\mu_h(x) = \sum_{i=h}^{n} \beta_{i-h}^{(h)}\nu_i(x) \quad (h = 0, 1, \ldots, n), \tag{4.28}$$

and

$$r(L) = n+2 \max_{0\le i\le n,\, \nu_i\ne 0} \deg \nu_i. \tag{4.29}$$

Third special case : $q_n(-1) \ne 0$ and $q_n(1) \ne 0$. Then

$$L = \sum_{i=0}^{n} D^{n-i} q_i(X), \qquad (4.30)$$

$$\tau(k) = D^n m_k^\lambda[q] - \sum_{j=1}^{n} D^{n-j+1} \sum_{h=0}^{n-j} \phi_k^\lambda\left[(q_{h+j} f)^{(h)}\right], \qquad (4.31)$$

and

$$r(L) = 2n+2 \max_{0\le i\le n, q_i\neq 0} (\deg q_i - i). \qquad (4.32)$$

A symmetry of the relation (4.13) has been discovered which seems to be useful for check purposes. We prove the following

THEOREM 4.2. Operator (4.14) has the form

$$L = (2k+2\lambda+d+1)_{d-1}^{-1} \sum_{j=0}^{r} \lambda_j(k) E^{j-u}$$

in which $r = r(L)$,

$$u = s + \max_{0\le i\le n, q_i\neq 0} (\deg q_i - i),$$

and $\lambda_0, \lambda_1, \ldots, \lambda_r$ are such that

$$\lambda_j(k) = -\varepsilon^d \lambda_{r-j}(-k-2\lambda-d) \qquad (k = 0, 1, \ldots, r).$$

Proof. We give a short sketch. Let P represent any of the operators

$$P_d^{(\varepsilon)}, \quad R_d^{(\varepsilon)},$$

$$S_{d-1,h}^{(\varepsilon)} R_h^{(\varepsilon)} \qquad (h = 0, 1, \ldots, d).$$

It can be shown by induction on d that

$$P = (2k+2\lambda+d+1)_{d-1}^{-1} \sum_{i=0}^{d} \pi_i(k) E^i,$$

in which

$$\pi_i(k) = -\varepsilon^d \pi_{d-i}(-k-2\lambda-d) \qquad (i = 0, 1, \ldots, d).$$

Further, let U stand for any of the following operators :

$$\sum_{i=0}^{s-1} D^{s-i} q_i(X), \qquad \sum_{h=1}^{n-\sigma} V_h^{(\varepsilon)} \nu_h(X),$$

$$\mu_h(X) \qquad (h = 0, 1, \ldots, d).$$

From (3.8), (3.9) and (3.22),

$$U = \sum_{j=0}^{2t} \psi_j(k) E^{j-t},$$

where

$$t = s + \max_{i\in J, q_i\neq 0} (\deg q_i - i)$$

and ψ_j are such that

$$\psi_j(k) = \psi_{2t-j}(-k-2\lambda) \qquad (j = 0, 1, \ldots, 2t).$$

Here J is $\{0,1,\ldots,s-1\}$, $\{\sigma+1,\sigma+2,\ldots,n\}$ or $\{s+h, s+h+1,\ldots,n\}$ $(h = 0, 1, \ldots, d)$, respectively.

Now, it can be checked that

$$PU = (2k+2\lambda+d+1)_{d-1}^{-1} \sum_{\iota=0}^{d+2t} X_\iota(k)E^{\iota-t}$$

and that

$$X_\iota(k) = -\varepsilon^d X_{d+2t-\iota}(-k-2\lambda-d) \qquad (\iota = 0, 1, \ldots, d+2t).$$

Using this in (4.14), the result follows.

5. EXAMPLE

Let us consider the evaluation of the integral

$$I = \int_0^1 (1-t)^\alpha t^\beta J_p(2at)g(t)dt, \tag{5.1}$$

where J_p is the Bessel function of the first kind and of order p, and where $\alpha>-1$, $\beta>-p-1$, $a>0$ are real numbers. We assume that g is a smooth function which can be approximated accurately by a polynomial of degree N, expressed in the form

$$p(t) = \sum_{k=0}^{N} a_k C_k^\lambda(2t-1) \qquad (0\le t\le 1). \tag{5.2}$$

Replacing g by p in (5.1), we obtain

$$I \approx \sum_{k=0}^{N} a_k \int_0^1 (1-t)^\alpha t^\beta J_p(2at)C_k^\lambda(2t-1)dt$$

$$= 2^{-\alpha-\beta-1} \sum_{k=0}^{N} a_k m_k^\lambda [w], \tag{5.3}$$

where

$$w(x) = (1-x)^\alpha(1+x)^\beta J_p(a(1+x)). \tag{5.4}$$

The success of this method of numerical integration, which is a natural generalization of the so-called modified Clenshaw-Curtis method due to Piessens and Branders [14], depends on the ability to compute the modified moments $m_0^\lambda[w], \ldots, m_N^\lambda[w]$ accurately. These moments can be obtained from a recurrence relation, judiciously employed, which is constructed using Theorem 4.1.

Function (5.4) satisfies

$$[(1-x^2)^2w]'' + [(1-x^2)(c_1x+c_2)w]'$$

$$+ [a^2(1-x^2)^2-p^2(1-x)^2+c_3x^2+c_4x+c_5]w = 0, \tag{5.5}$$

where

$$c_1 = 2\alpha+2\beta+7, \quad c_2 = 2\alpha-2\beta+1, \quad c_3 = (\alpha+\beta+3)^2,$$

$$c_4 = 2(\alpha+1)_2-2(\beta+1)^2, \quad c_5 = (\alpha-\beta+1)^2-2\beta-4.$$

Now, it is easy to observe that we have the first special case discussed at the end of Section 4. We obtain, therefore, the eighth order recurrence relation

$$Lm_k^\lambda[w] = 0, \tag{5.6}$$

where (cf. (4.22), (4.25))

$$L = \alpha^2(I-X^2)^2+(2-p^2)(I-X)^2+(c_3-c_1)X^2$$

$$+(c_1-c_2+c_4)X+(c_2+c_5)I \tag{5.7}$$

$$+H_1^{(1)}\left[(3-c_1)X-(c_2+3)I+H_2^{(1)}\right].$$

From (3.9) and (3.21), (5.6) is a seven-term relation of the form

$$K_0(k)m^{\lambda}_{k-4}[w] + \sum_{j=1}^{5} K_j(k)m^{\lambda}_{k-3+j}[w]$$

$$+ K_6(k)m^{\lambda}_{k+4}[w] = 0, \qquad (5.8)$$

in which

$$K_j(k) = K_{6-j}(-k-2\lambda) \quad (j = 0, 1, \ldots, 6).$$

In particular, the integrals

$$\tau_k[w] = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} J_p(a(1+x))T_k(x)dx \qquad (5.9)$$

obey

$$a^2\tau_{k-4}[w]+4[(k-\alpha-\beta-3)^2-p^2-a^2]\tau_{k-2}[w] \qquad (5.10)$$

$$+8[2(\alpha+1)_2-2(\beta+1)^2-(2\alpha-2\beta+1)k+2p^2]\tau_{k-1}[w]$$

$$-2[4k^2-12(\alpha-\beta+1)^2-16(\alpha+\beta+\alpha\beta)+12p^2-3a^2]\tau_k[w]$$

$$+8[2(\alpha+1)_2-2(\beta+1)^2+(2\alpha-2\beta+1)k+2p^2]\tau_{k+1}[w]$$

$$+4[(k+\alpha+\beta+3)^2-p^2-a^2]\tau_{k+2}[w]+a^2\tau_{k+4}[w] = 0.$$

Putting $\alpha = 0$ and $\beta = 0$, equation (5.12) reduces tc a recurrence relation obtained in [14] for the integrals

$$\int_{-1}^{1} J_p(a(1+x))T_k(x)dx. \qquad (5.11)$$

In the cited reference, the asymptotic forms for a fundamental set for the difference equation (5.10) were found and compared with the asymptotic behaviour of (5.11). The conclusion was that (5.11) can be computed using (5.10) in the forward direction.

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