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TRANSFORMATION FORMULAE FOR DOUBLE HYPERGEOMETRIC SERIES

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ABSTRACT

In this paper we obtain two transformation formulae for hypergeometric series of two variables.

RESUMEN

En este trabajo obtenemos dos fórmulas de transformación por las series hipergeométricas de dos variables.

I. INTRODUCTION

Sharma [4], Sharma and Abiodun [3] and Singal [5] have obtained transformation formulae for double series. In this paper we add some interesting results to the Literature dealing with the hypergeometric series of two variables.

The following notation due to Burchnall and Chaundy [2] is used to represent the hypergeometric series of higher order and of two variables.

$$F\left[\begin{matrix}(a_p):(c_q);(c_r);x,y\\(d_s):(e_h);(f_1);\end{matrix}\right] = \sum_{m,n=0}^{\infty}\frac{[(a_p)]_{m+n}\,[(b_p)]_m\,[(c_r)]_n\,x^m\,y^n}{[(d_s)]_{m+n}\,[(e_h)]_m\,[(f_1)]_n\,m!\,n!}, \quad \ldots \quad (1)$$

where (a_p) and $[(a_p)]_{m+n}$ will mean $a_1, \ldots, a_p$ and $(a_1)_{m+n} \cdots (a_p)_{m+n}$.

The method in proving the results is straightforward and is based on series rearrangement.

The following formulae are required in the proof : - Bailey [1, p.30 (4.5) (1)]

$${}_4F_3\left[\begin{matrix}a, b, c, -m; 1\\1+a-b, 1+a-c, w\end{matrix}\right] = \frac{(w-a)_m}{(w)_m}$$

$${}_5F_4\left[\begin{matrix}1+a-w, \tfrac{1}{2}a, \tfrac{1}{2}(1+a), 1+a-b-c, -m: 1\\1+a-b, 1+a-c, \tfrac{1}{2}(1+a-w-m), 1+\tfrac{1}{2}(a-w-m);\end{matrix}\right]. \quad (2)$$

Bailey [1, p.21 (3.8) (1)]

$${}_3F_2\left[\begin{matrix}a, b, c; 1\\e, f;\end{matrix}\right] = \frac{\Gamma(e)\Gamma(e-a-b)}{\Gamma(e-a)\Gamma(e-b)}$$

$${}_3F_2\left[\begin{matrix}a, b, f-c; 1.\\a+b-e+1, f;\end{matrix}\right] +$$

$$+\frac{\Gamma(e)\Gamma(f)\Gamma(a+b-e)\Gamma(e+f-a-b-c)}{\Gamma(a)\Gamma(b)\Gamma(f-c)\,(\Gamma(e+f-a-b)}$$

$${}_3F_2\left[\begin{matrix}e-a, e-b, e+f-a-b-c; 1\\e-a-b, c+f-a-b;\end{matrix}\right]. \quad \ldots \quad \ldots \quad (3)$$

2. The first formula to be proved is

$$F\left[\begin{matrix}-n: a_1, b_1, c_1; a_2, b_2, c_2; 1, 1\\a: 1+a_1-b_1, 1+a_1-c_1; 1+a_2-b_2, 1+a_2-c_2;\end{matrix}\right]$$

$$= \frac{(a-a_1-a_2)_n}{(a)_n}$$

$$F\left[\begin{matrix} -n, 1+a_1+a_2-\alpha : \frac{1}{2}a_1, \frac{1}{2}(1+a_1), 1+a_1-b_1-c_1; \frac{1}{2}a_2, \\ \frac{1}{2}(1+a_1+a_2-\alpha-n), \frac{1}{2}(3+a_1+a_2-\alpha-n) : 1+a_1-b_1, 1+a_1-c_1; \end{matrix}\right.$$

$$\left.\begin{matrix} \frac{1}{2}(a_2+1), 1+a_2-b_2-c_2; 1, 1 \\ 1+a_2-b_2, 1+a_2-c_2; \end{matrix}\right]. \qquad \ldots \quad \ldots \quad (4)$$

PROOF : - To prove (4), we start with the L.H.S. of (4).

$$F\left[\begin{matrix} -n: a_1, b_1, c_1; a_2, b_2, c_2; 1, 1 \\ \alpha; 1+a_1-b_1, 1+a_1-c_1 : 1+a_2-b_2, 1+a_2-c_2; \end{matrix}\right]$$

$$= \sum_{r=0}^{n} \frac{(-n)_r (a_2)_r (b_2)_r (c_2)_r}{(\alpha)_r (1+a_2-b_2)_r (1+a_2-c_2)_r \, r!}$$

$${}_4F_3\left[-(n-r), a_1, b_1, c_1; \alpha+r, 1+a_1-b_1, 1+a_1-c_1; 1\right]$$

$$= \sum_{r=0}^{n} \frac{(-n)_r (a_1)_r (b_1)_r (c_1)_r (\alpha+r-a_1)_{n-r}}{(\alpha)_r (1+a_1-b_1)_r (1+a_1-c_1)_r (\alpha+r)_{n-r} \, r!}$$

$${}_5F_4\left[\begin{matrix} 1+a_1-\alpha-r, {}^1/_2\, a_1, {}^1/_2(1+a_1), 1+a_1-b_1-c_1, -n+r; 1 \\ 1+a_1-b_1, 1+a_1-c_1, {}^1/_2(1+a_1-\alpha-n), {}^1/_2(2+a_1-\alpha-n); \end{matrix}\right]$$

by (2)

$$= \frac{(\alpha-a_1)_n}{(\alpha)_n}$$

$$\sum_{s=0}^{n} \frac{(-n)_s ({}^1/_2 a_1)_s ({}^1/_2+{}^1/_2 a_1)_s (1+a_1-b_1-c_1)_s (1+a_1-\alpha)_s}{s!\,(1+a_1-b_1)_s (1+a_1-c_1)_s (1+a_1-\alpha-n)_{2s} \, 2^{-2s}}$$

$${}_4F_3\left[-(n-s), a_2, b_2, c_2; \alpha-a_1-s, 1+a_2-b_2, 1+a_2-c_2; 1\right]$$

$$= \frac{(\alpha-a_1)_n}{(\alpha)_n}$$

$$\sum_{s=0}^{n} \frac{(-n)_s ({}^1/_2 a_1)_s ({}^1/_2+{}^1/_2\, a_1)_s (1+a_1-b_1-c_1)_s (1+a_1-\alpha)_s (\alpha-a_1-s-a_2)_{n-s}}{s!\,(1+a_1-b_1)_s (1+a_1-c_1)_s (1+a_1-\alpha-n)_{2s} (\alpha-a_1-s)_{n-s} \, 2^{-2s}}$$

$${}_5F_4\left[\begin{matrix} 1+a_1+a_2-\alpha+s, {}^1/_2 a_2, {}^1/_2+{}^1/_2\, a_2, 1+a_2-b_2-c_2, -n+s; 1 \\ 1+a_2-b_2, 1+a_2-c_2, {}^1/_2(1+a_2+a_1-\alpha-n+2s), {}^1/_2(3+a_1+a_2-\alpha-n+2s); \end{matrix}\right]$$

by (2)

$$= \frac{(\alpha-a_1-a_2)_n}{(\alpha)_n} F\left[\begin{matrix} -n, 1+a_1+a_2-\alpha : {}^1/_2\, a_1, {}^1/_2+{}^1/_2 a_1, \\ {}^1/_2\,(1+a_1+a_2-\alpha-n), {}^1/_2(3+a_2+a_2-\alpha-n) : \end{matrix}\right.$$

$$\left.\begin{matrix} 1+a_1-b_1-c_1; {}^1/_2\, a_2, {}^1/_2+{}^1/_2\, a_2, 1+a_2-b_2-c_2; 1, 1 \\ 1+a_1-b_1, 1+a_1-c_1; 1+a_2-b_2, 1+a_2-c_2; \end{matrix}\right]$$

by (1)

This completes the proof of (4)

3. The second formula to be proved is

$$F\left[\begin{matrix} c: a_1, b_1; a_2, b_2; 1, 1 \\ f: c_1; c_2; \end{matrix}\right]$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1-a_1-b_1)}{\Gamma(c_1-a_1)\Gamma(c_2-a_2)\Gamma(c_1-b_1)}$$

$$\frac{\Gamma(c_2-a_2-b_2)}{\Gamma(c_2-b_2)} F\left[\begin{matrix} f-c: a_1, b_1; a_2, b_2; 1, 1 \\ f: 1+a_1+b_1-c_1; 1+a_2+b_2-c_2; \end{matrix}\right] +$$

$$\frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_2-a_2-b_2)\Gamma(f)\Gamma(a_1-b_1-c_1)\Gamma(c_1+f-a_1-b_1-c)}{\Gamma(c_2-a_2)\Gamma(c_2-b_2)\Gamma(a_1)\Gamma(b_1)\Gamma(f-c)\Gamma(f+c_1-a_1-b_1)}$$

$$F\left[\begin{matrix} f-a_1-b_1+c_1-c : c_1-a_1, c_1-b_1; a_2, b_2; 1, 1 \\ f+c_1-a_1-b_1 : c_1-a_1-b_1+1; 1+a_2+b_2-c_2; \end{matrix}\right]$$

$$+ \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_2)\Gamma(b_2)} \frac{\Gamma(f)\Gamma(c_1-a_1-b_1)\Gamma(a_3+b_2-c_2)\Gamma(c_2+f-a_2-b_2-c)}{\Gamma(f-c)\Gamma(c_1-a_1)\Gamma(c_1-b_1)\Gamma(c_2+f-a_2-b_2)}$$

$$F\left[\begin{matrix} c_2+f-a_2-b_2-c\ a_1, b_1; c_2-a_2, c_2-b_2; 1, 1 \\ c_2+f-a_2-b_2 : 1+a_1+b_1-c_1; 1+c_2-a_2-b_2; \end{matrix}\right]$$

$$+ \frac{\Gamma(f)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)} \frac{\Gamma(c_2)\Gamma(a_1+b_1-c_1)\Gamma(a_2+b_2-c_2)\Gamma(c_1+c_2+f-a_1-a_2-b_1-b_2-c)}{\Gamma(b_1)\Gamma(b_2)\Gamma(f-c)\Gamma(c_1+c_2+f-a_1-a_2-b_1-b_2)}$$

$$F\left[\begin{matrix} c_1+c_2+f-a_1-b_1-a_2-b_2-c : c_1-a_1, c_2-b_1; c_2-a_2, c_2-b_2; 1, 1 \\ c_1+c_2+f-a_1-a_2-b_1-b_2 : 1+c_1-a_1-b_1; 1+c_2-a_2-b_2; \end{matrix}\right] \quad \ldots\ \ldots\ \ldots \quad (5)$$

PROOF : (5) can be proved in a similar manner as (4) by using (3) instead of (2).

In particular, if we put $c=f+m$ in (5), it reduces to the following interesting formula.

$$F\left[\begin{matrix} f+m : a_1, b_1; a_2, b_2; 1, 1 \\ f; c_1 : c_2; \end{matrix}\right] = \frac{\Gamma(c_1)\Gamma(c_2)\Gamma(c_1-a_1-b_1)}{\Gamma(c_1-a_1)\Gamma(c_2-a_2)\Gamma(c_1-b_1)} \frac{\Gamma(c_2-a_2-b_2)}{\Gamma(c_2-b_2)} F\left[\begin{matrix} -m : a_1, b_1; a_2, b_2; 1, 1 \\ f : 1+a_1+b_1-c_1; 1+a_2+b_2-c_2; \end{matrix}\right]. \quad \ldots (6)$$

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