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RESULTADOS CON POLINOMIOS DE JACOBI

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RESUMEN

Los polinomios ortogonales han tenido siempre una gran importancia en Matemática Aplicada. En el presente trabajo se obtienen algunos resultados sobre integrales que involucran polinomios de Jacobi, las cuales tienen aplicaciones en cuadraturas de Gauss y para derivar fórmulas de integración con singularidades. Se obtienen resultados generales y a partir de éstos, se deducen algunos casos particulares.

ABSTRACT

The orthogonal polynomials have always played a significant role in applied mathematics. In the present paper we obtain some results involving Jacobi polynomials. Such results are useful in Gauss' quadrature formulae and to derive integration formula with singularities. The results obtained here are general and some known results follow as special cases.

1. INTRODUCCION

Recientemente han aparecido varios trabajos [1,4,5,7,8,9] sobre integrales que involucran polinomios ortogonales, funciones algebráicas y logarítmicas, debido a sus aplicaciones en fórmulas de cuadraturas de Gauss y para derivar fórmulas de integración con singularidades algebráicas y logarítmicas.

Los artículos de Blue [1] y Gautschi [5], los cuales trabajaron con polinomios de Legendre, renovaron el interés en este campo. Kalla y Conde han tratado polinomios de Legendre y de Laguerre [7,8], mientras que Kalla, Conde y Luke [9] consideraron polinomios y funciones de Jacobi. Un artículo reciente de Gatteschi [4] considera polinomios de Laguerre y Jacobi. Kalla ha publicado un artículo de revisión [6] en el cual se incluyen los principales resultados obtenidos en los trabajos anteriormente mencionados. En el presente trabajo se sigue la misma línea para deducir algunas integrales con polinomios de Jacobi.

Los polinomios de Jacobi vienen dados por [3, 14]

$$P_n^{(\alpha,\beta)}(z) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left({-n,\ n+\lambda \atop \alpha+1}\Big|\,\frac{1-z}{2}\right) = \frac{(-1)^n(\beta+1)_n}{n!}\,{}_2F_1\left({-n,\ n+\lambda \atop \beta+1}\Big|\,\frac{1+z}{2}\right) \tag{1}$$

$$\lambda = \alpha+\beta+1 \quad ; \quad \alpha,\beta>-1$$

donde ${}_2F_1\left({a,b \atop c}\Big|\,z\right)$ es la función hipergeométrica de Gauss [2,10,13].

En un trabajo anterior, Kalla, Conde y Luke [9] consideraron la integral [11]

$$I_{n,\alpha,\beta}^{a,b} = \int_{-1}^{1} (1-x)^a(1+x)^b\,P_n^{(\alpha,\beta)}(x)\,dx \tag{2}$$

y sus derivadas parciales con respecto a los parámetros a y b. En este trabajo se obtienen nuevas integrales en donde intervienen los polinomios de Jacobi, algunas de las cuales generalizan los resultados dados en [9]

2. INTEGRALES QUE INVOLUCRAN POLINOMIOS DE JACOBI

i) Consideremos la integral

$$A_{n,\alpha,\beta}^{p,q,\gamma,\delta} = \int_{-p}^{q} (q-x)^{\gamma-1}(x+p)^{\delta-1}\,P_n^{(\alpha,\beta)}(x/p)\,dx \tag{3}$$

de la cual se obtiene (2) como caso particular, tomando $p=q=1$, $\gamma = a+1$ y $\delta = b+1$; esto es,

$$A^{1,1,a+1,b+1}_{n,\alpha,\beta} = I^{a,b}_{n,\alpha,\beta}$$

Se tiene que [12, p.581]

$$A^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\frac{(-1)^n}{n!}(\beta+1)_n B(\delta,\gamma)(q+p)^{\delta+\gamma-1}\, {}_3F_2\left(\begin{matrix}-n,n+\lambda,\delta\\ \delta+\gamma,\beta+1\end{matrix}\Big|\frac{p+q}{2p}\right) \quad (4)$$

$p>0$, $q>-p$; $Re(\gamma)$, $Re(\delta)>0$

donde, al igual que en (1), $\lambda = \alpha+\beta+1$.

${}_pF_q\left(\begin{matrix}a_1,a_2,\ldots,a_p\\ b_1,b_2,\ldots,b_q\end{matrix}\Big| z\right)$ es la función hipergeométrica generalizada [2,10,13]

De acuerdo con (3) y (4), se obtiene que

$$B^{p,q,\gamma,\delta}_{n,\alpha,\beta} = \frac{\partial}{\partial\gamma} A^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{q} [\ln(q-x)]\,(q-x)^{\gamma-1}(x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= [\ln(q+p) + \psi(\gamma)]A^{p,q,\gamma,\delta}_{n,\alpha,\beta}$$

$$- \frac{(-1)^n}{n!}(\beta+1)_n B(\delta,\gamma)(q+p)^{\delta+\gamma-1}\,\cdot$$

$$\cdot\sum_{k=o}^{n} \frac{(-n)_k (n+\lambda)_k (\delta)_k}{(\beta+1)_k\,(\delta+\gamma)_k\, k!}\,\psi(\delta+\gamma+k)\left(\frac{p+q}{2p}\right)^k \quad (5)$$

$p>o$, $q>-p$; $Re(\gamma)$, $Re(\delta)>0$

donde $\psi(z)$ es la derivada logarítmica de la función Gamma [2,10].

También se deduce que

$$C^{p,q,\gamma,\delta}_{n,\alpha,\beta} = \frac{\partial}{\partial\delta} A^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{q} [\ln(x+p)]\,(q-x)^{\gamma-1}(x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= [\ln(q+p)]A^{p,q,\gamma,\delta}_{n,\alpha,\beta} + \frac{(-1)^n}{n!}(\beta+1)_n B(\delta,\gamma)(q+p)^{\delta+\gamma-1}\,.$$

$$\cdot\sum_{k=o}^{n} \frac{(-n)_k (n+\lambda)_k (\delta)_k}{(\beta+1)_k\,(\delta+\gamma)_k\, k!}[\psi(\delta+k) - \psi(\delta+\gamma+k)]\left(\frac{p+q}{2p}\right)^k \quad (6)$$

$p>0$, $q>-p$; $Re(\gamma)$, $Re(\delta)>0$

Teniendo en cuenta (5) y (6), se tiene que

$$D^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{q} [\ln\,(q-x)(x+p)]\,(q-x)^{\gamma-1}(x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= B^{p,q,\gamma,\delta}_{n,\alpha,\beta} + C^{p,q,\gamma,\delta}_{n,\alpha,\beta} \quad (7)$$

y

$$E^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{q} \left[\ln\left(\frac{q-x}{x+p}\right)\right](q-x)^{\gamma-1}(x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= B^{p,q,\gamma,\delta}_{n,\alpha,\beta} - C^{p,q,\gamma,\delta}_{n,\alpha,\beta} \quad (8)$$

Tomando $p=q=1$, $\gamma=a+1$ y $\delta=b+1$, las fórmulas (5) a (8), se reducen a las correspondientes dadas en [9].

Si en (4) tomamos $p=q$, tenemos que

$$A^{p,p,\gamma,\delta}_{n,\alpha,\beta} = \int_{-p}^{p} (p-x)^{\gamma-1}(x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= \frac{(-1)^n}{n!} (\beta+1)_n B(\delta,\gamma)(2p)^{\delta+\gamma-1} {}_3F_2 \left({-n,\ n+\lambda,\ \delta \atop \delta+\gamma,\ \beta+1} \Big| 1 \right) \tag{9}$$

$p,\ Re\langle\gamma\rangle,\ Re\langle\delta\rangle > 0$

Es fácil ver que

$$A^{p,p,\gamma,\delta}_{n,\alpha,\beta} = (-1)^n A^{p,p,\delta,\gamma}_{n,\beta,\alpha} \tag{10}$$

Según (5), se tiene que

$$B^{p,p,\gamma,\delta}_{n,\alpha,\beta} = \frac{\partial}{\partial\gamma} A^{p,p,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{p} [\ln(p-x)] (p-x)^{\gamma-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\, dx$$

$$= [\ln 2p + \psi(\gamma)] A^{p,p,\gamma,\delta}_{n,\alpha,\beta}$$

$$- \frac{(-1)^n}{n!} (\beta+1)_n B(\delta,\gamma) (2p)^{\delta+\gamma-1} \cdot$$

$$\cdot \sum_{k=o}^{n} \frac{(-n)_k (n+\lambda)_k (\delta)_k}{(\beta+1)_k (\delta+\gamma)_k k!} \psi(\delta+\gamma+k) \tag{11}$$

$p, Re(\gamma),\ Re(\delta)>0$

y entonces

$$C^{p,p,\gamma,\delta}_{n,\alpha,\beta} = \frac{\partial}{\partial\delta} A^{p,p,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{p} [\ln(x+p)] (p-x)^{\gamma-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\, dx$$

$$= \frac{\partial}{\partial\delta} \left[(-1)^n A^{p,p,\delta,\gamma}_{n,\beta,\alpha} \right]$$

$$= (-1)^n B^{p,p,\delta,\gamma}_{n,\beta,\alpha} \tag{12}$$

Luego queda que

$$D^{p,p,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{p} [\ln(p^2 - x^2)] (p-x)^{\gamma-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\, dx$$

$$= B^{p,p,\gamma,\delta}_{n,\alpha,\beta} + (-1)^n B^{p,p,\delta,\gamma}_{n,\beta,\alpha} \tag{13}$$

y

$$E^{p,p,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{p} \left[\ln(\tfrac{p-x}{x+p})\right] (p-x)^{\gamma-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/p)\, dx$$

$$= B^{p,p,\gamma,\delta}_{n,\alpha,\beta} - (-1)^n B^{p,p,\delta,\gamma}_{n,\beta,\alpha} \tag{14}$$

Si ahora tomamos $\delta = \gamma$ y $\beta = \alpha$ en (13) y (14), obtenemos que

$$D^{p,p,\gamma,\gamma}_{n,\alpha,\alpha} =$$

$$\int_{-p}^{p} [\ln(p^2 - x^2)] (p^2 - x^2)^{\gamma-1} P_n^{(\alpha,\alpha)}(x/p)\, dx$$

$$= \begin{cases} 0, & n \text{ impar} \\ 2\, B^{p,p,\gamma,\gamma}_{n,\alpha,\alpha}, & n \text{ par} \end{cases} \tag{15}$$

y

$$E^{p,p,\gamma,\gamma}_{n,\alpha,\alpha} = \int_{-p}^{p} \left[\ln\left(\frac{p-x}{x+p}\right)\right] (p^2 - x^2)^{\gamma-1} P_n^{(\alpha,\alpha)}(x/p)\, dx$$

$$= \begin{cases} 0, & n \text{ par} \\ 2\, B^{p,p,\gamma,\gamma}_{n,\alpha,\alpha}, & n \text{ impar} \end{cases} \qquad (16)$$

Los resultados (14) a (18) obtenidos por Kalla, Conde y Luke [9] resultan como casos particulares de (12) a (16).

La integral

$$F^{p,q,\gamma,\delta}_{n,\alpha,\beta} = \int_{-p}^{q} (q-x)^{\gamma-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/q)\, dx \qquad (17)$$

puede expresarse en términos de (3), haciendo $x/q = -u/p$ y usando

$$P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z)$$

resultando que

$$F^{p,q,\gamma,\delta}_{n,\alpha,\beta} = (-1)^n \left(\frac{q}{p}\right)^{\gamma+\delta-1} A^{p,p^2/q,\delta,\gamma}_{n,\beta,\alpha} \qquad (18)$$

Luego, usando (18) y (6), obtenemos

$$G^{p,q,\gamma,\delta}_{n,\alpha,\beta} = \frac{\partial}{\partial\gamma} F^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{q} \left[\ln (q-x)\right] (q-x)^{\gamma-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/q)\, dx$$

$$= (-1)^n \left(\frac{q}{p}\right)^{\gamma+\delta-1} \left[\left[\ln\left(\frac{q}{p}\right)\right] A^{p,p^2/q,\delta,\gamma}_{n,\beta,\alpha} + C^{p,p^2/q,\delta,\gamma}_{n,\beta,\alpha} \right] \qquad (19)$$

Análogamente, de (18) y (5), resulta

$$H^{p,q,\gamma,\delta}_{n,\alpha,\beta} = \frac{\partial}{\partial\delta} F^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{q} \left[\ln (x+p)\right] (q-x)^{\delta-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/q)\, dx$$

$$= (-1)^n \left(\frac{q}{p}\right)^{\gamma+\delta-1} \left[\left[\ln\left(\frac{q}{p}\right)\right] A^{p,p^2/q,\delta,\gamma}_{n,\beta,\alpha} + B^{p,p^2/q,\delta,\gamma}_{n,\beta,\alpha} \right] \qquad (20)$$

Entonces

$$U^{p,q,\gamma,\delta}_{n,\alpha,\beta} = \int_{-p}^{q} \left[\ln (q-x)(x+p)\right] (q-x)^{\gamma-1} (x+p)^{\delta-1} P_n^{(\alpha,\beta)}(x/q)\, dx$$

$$= G^{p,q,\gamma,\delta}_{n,\alpha,\beta} + H^{p,q,\gamma,\delta}_{n,\alpha,\beta} \qquad (21)$$

y

$$V^{p,q,\gamma,\delta}_{n,\alpha,\beta} =$$

$$\int_{-p}^{q}\left[\ln\left(\frac{q-x}{x+p}\right)\right](q-x)^{\gamma-1}(x+p)^{\delta-1}P_n^{(\alpha,\beta)}(x/q)\,dx$$

$$= G_{n,\alpha,\beta}^{p,q,\gamma,\delta} - H_{n,\alpha,\beta}^{p,q,\gamma,\delta} \qquad (22)$$

Tomando $q=p$ en (17) a (22) se obtienen de nuevo las fórmulas (9) a (14).

Podemos obtener otros casos particulares interesantes en términos de los polinomios de Legendre $P_n(z)$ y de Tchebysheff $T_n(z)$, sabiendo que

$$P_n^{(o,o)}(z) = P_n(z) \quad y \quad \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2},-\frac{1}{2})}(z) = T_n(z)$$

ii) Consideremos la integral [12, p.587]

$$P_{n,\alpha,\beta}^{p,\delta,\theta} = \int_{-p}^{p}(p-x)^{\alpha}(x+p)^{\delta-1}(z\pm x)^{-\theta}P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= \frac{(-1)^n}{n!}(\beta-\delta+1)_n\,B(\delta,\alpha+n+1)\,(2p)^{\delta+\alpha}(z\mp p)^{-\theta}.$$

$$\cdot\, {}_3F_2\left(\begin{matrix}\delta,\theta,\delta-\beta\\ \delta-\beta-n,\delta+\alpha+n+1\end{matrix}\;\middle|\;\frac{2p}{p\mp z}\right) \qquad (23)$$

p, $Re(\delta)>0$; $\delta-\beta \neq m \leq n$, $m=o,1,2\ \ldots;$

$|\arg(z^2-p^2)|<\pi$

Tomando $\theta = 0$ en (23), nos queda

$$P_{n,\alpha,\beta}^{p,\delta,0} = \int_{-p}^{p}(p-x)^{\alpha}(x+p)^{\delta-1}P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= \frac{(-1)^n}{n!}(\beta-\delta+1)_n\,B(\delta,\alpha+n+1)(2p)^{\delta+\alpha} \qquad (24)$$

p, $Re(\delta)>0$; $|\arg(z^2-p^2)|<\pi$

La fórmula (24) puede obtenerse como caso particular de (4), tomando $q = p$ y $\gamma = \alpha+1$. Para obtener el resultado basta aplicar el teorema de Saalschutz

$${}_3F_2\left(\begin{matrix}a,b,-n\\ c,1+a+b-c-n\end{matrix}\;\middle|\;1\right) = \frac{(c-a)_n\,(c-b)_n}{(c)_n\,(c-a-b)_n}$$

y la relación

$$(-z)_n = \frac{(-1)^n\,\Gamma(z+1)}{\Gamma(z-n+1)}$$

De (23) se deduce que

$$Q_{n,\alpha,\beta}^{p,\delta,\theta} = \frac{\partial}{\partial\delta}P_{n,\alpha,\beta}^{p,\delta,\theta} =$$

$$\int_{-p}^{p}[\ln(x+p)]\,(p-x)^{\alpha}(x+p)^{\delta-1}(z\pm x)^{-\theta}P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$= [\ln 2p - \psi(\beta-\delta+n+1)+\psi(\delta-\beta-n)$$

$$+\psi(\beta-\delta+1)-\psi(\delta-\beta)]\,P_{n,\alpha,\beta}^{p,\delta,\theta} +$$

$$+\frac{(-1)^n}{n!}(\beta-\delta+1)_n\,B(\delta,\alpha+n+1)(2p)^{\delta+\alpha}(z\mp p)^{-\theta}\,\cdot$$

$$\cdot\sum_{k=o}^{\infty}\frac{(\delta)_k\,(\theta)_k\,(\delta-\beta)_k}{(\delta-\beta-n)_k\,(\delta+\alpha+n+1)_k\,k!}\,.$$

$$\cdot\left[\psi(\delta+k)+\psi(\delta-\beta+k)-\psi(\delta-\beta-n+k)\right.$$

$$\left.-(\delta+\alpha+n+k+1)\right]\left(\frac{2p}{p\mp z}\right)^k \qquad (25)$$

$p,\mathrm{Re}(\delta)>0;\ \delta-\beta\neq m\leq n,\ m=0,1,2,\ldots;\ |\arg(z^2-p^2)|<\pi$

Tenemos que

$$\psi(1-z)-\psi(z)=\pi\operatorname{ctg}\pi z$$

Luego, (25) también puede escribirse como

$$Q^{p,\delta,\theta}_{n,\alpha,\beta}=\left[\ln 2p+\pi\operatorname{ctg}\pi(\delta-\beta)\right]-$$

$$-\pi\operatorname{ctg}\pi(\delta-\beta-n)\Big]P^{p,\delta,\theta}_{n,\alpha,\beta}$$

$$+\frac{(-1)^n}{n!}(\beta-\delta+1)_n B(\delta,\alpha+n+1)(2p)^{\delta+\alpha}(z\mp p)^{-\theta}\cdot$$

$$\cdot\sum_{k=o}^{\infty}\frac{(\delta)_k(\theta)_k(\delta-\beta)_k}{(\delta-\beta-n)_k(\delta+\alpha+n+1)_k\,k!}\cdot$$

$$\cdot\left[\psi(\delta+k)+\psi(\delta-\beta+k)-\psi(\delta-\beta-n+k)-\right.$$

$$\left.-\psi(\delta+\alpha+n+k+1)\right]\left(\frac{2p}{p\mp z}\right)^k$$

(26)

$p,\mathrm{Re}(\delta)>0;\ \delta-\beta\neq m\leq n,\ m=0,1,2,\ldots;|\arg(z^2-p^2)|<\pi$

Tomando los valores apropiados de los parámetros en (23) a (26), se obtienen las correspondientes fórmulas para los polinomios de Legendre y de Tchebysheff.

Usando (24), vemos que para $\theta=o$ la expresión (25) se reduce a

$$Q^{p,\delta,0}_{n,\alpha,\beta}=\int_{-p}^{p}\left[\ln(x+p)\right](p-x)^{\alpha}(x+p)^{\delta-1}P_n^{(\alpha,\beta)}(x/p)\,dx$$

$$=\left[\ln 2p-\psi(\beta-\delta+n+1)+\psi(\beta-\delta+1)+\psi(\delta)-\right.$$

$$\left.-\psi(\delta+\alpha+n+1)\right]P^{p,\delta,o}_{n,\alpha,\beta}$$

(27)

$p,\ \mathrm{Re}(\delta)>0;\ |\arg(z^2-p^2)|<\pi$

Por otro lado, tenemos de (6) con $q=p$ y $\gamma=\alpha+1$, que

$$Q^{p,\delta,0}_{n,\alpha,\beta}=(\ln 2p)\,P^{p,\delta,0}_{n,\alpha,\beta}+$$

$$+\frac{(-1)^n}{n!}(\beta+1)_n B(\delta,\alpha+1)(2p)^{\delta+\alpha}\cdot$$

$$\cdot\sum_{k=o}^{n}\frac{(-n)_k(\alpha+\beta+n+1)_k(\delta)_k}{(\beta+1)_k(\delta+\alpha+1)_k\,k!}\left[\psi(\delta+k)-\psi(\delta+\alpha+k+1)\right]$$

(28)

Igualando los resultados dados por (27) y (28), se deduce que

$$\sum_{k=o}^{n}\frac{(-n)_k(\alpha+\beta+n+1)_k(\delta)_k}{(\beta+1)_k(\delta+\alpha+1)_k\,k!}\left[\psi(\delta+k)-\psi(\delta+\alpha+k+1)\right]=$$

$$=\frac{(\beta-\delta+1)_n B(\delta,\alpha+n+1)}{(\beta+1)_n B(\delta,\alpha+1)}\left[\psi(\delta)+\psi(\delta-\beta+1)-\right.$$

$$\left.\psi(\beta-\delta+n+1)-\psi(\delta+\alpha+n+1)\right]$$

$\mathrm{Re}(\delta)>0;\ \alpha,\beta>-1$

(29)

iii) Consideremos la integral [12, p.614].

$$X^{a,b,\alpha,m,n}_{\lambda,\nu,\rho,\sigma} =$$

$$\int_{-a}^{a} (x+a)^{\alpha-1} (a-x)^{\rho} (1-ab-bx)^{\lambda} P_m^{(\lambda,\nu)}(2bx+2ab-1) P_n^{(\rho,\sigma)}(x/a)dx$$

$$= \frac{(-1)^{m+n}}{m!\, n!} (1+\nu)_m (1+\sigma-\alpha)_n B(\rho+n+1,\alpha) (2a)^{\alpha+\rho} .$$

$$. {}_4F_3 \left(\begin{matrix} -m-\lambda, \nu+m+1, \alpha-\sigma, \alpha \\ \alpha+\rho+n+1, \nu+1, \alpha-\sigma-n \end{matrix} \middle| \, 2ab \right)$$

a,b, $Re(\alpha)>0$; $2ab<1$; $Re(\rho)>-1$; $\alpha-\sigma \neq i \leq n$, $i = 0,1,2,..$

(30)

Diferenciando ambos miembros de (30) respecto a α, resulta que

$$Y^{a,b,\alpha,m,n,}_{\lambda,\nu,\rho,\sigma} = \frac{\partial}{\partial\alpha} X^{a,b,\alpha,m,n}_{\lambda,\nu,\rho,\sigma}$$

$$= \int_{-a}^{a} [\ln(x+a)] (x+a)^{\alpha-1} (a-x)^{\rho} (1-ab-bx)^{\lambda} .$$

$$. P_m^{(\lambda,\nu)} (2bx+2ab-1) P_n^{(\rho,\sigma)} (x/a) \; dx$$

$$= [\ln 2a + \pi ctg\pi(\alpha-\sigma) - \pi ctg\pi(\alpha-\sigma-n)] .$$

$$. X^{a,b,\alpha,m,n}_{\lambda,\nu,\rho,\sigma} + \frac{(-1)^{m+n}}{m!\, n!} (1+\nu)_m (1+\sigma-\alpha)_n .$$

$$. B(\rho+n+1,\alpha)(2a)^{\alpha+\beta} \sum_{k=o}^{\infty} \frac{(-m-\lambda)_k}{(\alpha+\rho+n+1)_k} .$$

$$. \frac{(\nu+m+1)_k (\alpha-\sigma)_k (\alpha)_k}{(\nu+1)_k (\alpha-\sigma-n)_k k!} [\psi(\alpha-\sigma+k) +$$

$$+ \psi(\alpha+k) - \psi(\alpha+\rho+n+k+1) - \psi(\alpha-\sigma-n+k)] (2ab)^k$$

(31)

a,b, $Re(\alpha)>0$; $2ab<1$; $Re(\rho)>-1$; $\alpha-\sigma \neq i \leq n$, $i=0,1,2,...$

De nuevo, tomando convenientemente los parámetros en (30) y (31), pueden obtenerse resultados particulares donde estén involucrados los polinomios de Legendre y/o de Tchebysheff.

3. INTEGRALES QUE INVOLUCRAN POLINOMIOS DE JACOBI Y OTRAS FUNCIONES ESPECIALES

Las expresiones dadas a continuación son bastante generales y, partiendo de éstas, es posible obtener varios casos particulares, asignándole los valores apropiados a los parámetros.

Consideremos las siguientes integrales [12, p. 600, 606, 607]

$$\Delta = \int_{-a}^{a} (x+a)^{\alpha-1} (a-x)^{\rho} e^{-b^2x} H_{2m+\varepsilon} (b \sqrt{x+a}) .$$

$$P_n^{(\rho,\sigma)} (x/a) \; dx \; =$$

$$= \frac{(-1)^{m+n}}{n!} (\sigma-\alpha-\frac{\varepsilon}{2}+1)_n (\varepsilon+\frac{1}{2})_m \; 2^{\alpha+\frac{3\varepsilon}{2}+\rho+2m}$$

$$. B(\alpha+\frac{\varepsilon}{2}, \rho+n+1) \; 2^{\alpha+\rho+\frac{\varepsilon}{2}} b^{\varepsilon} e^{ab^2} .$$

$$. {}_3F_3 \left(\begin{matrix} m+\varepsilon+\frac{1}{2}, \alpha-\sigma+\frac{\varepsilon}{2}, \alpha+\frac{\varepsilon}{2} \\ \alpha+\rho+\frac{\varepsilon}{2}+n+1, \alpha-\sigma+\frac{\varepsilon}{2}-n, \varepsilon+\frac{1}{2} \end{matrix} \middle| -2ab^2 \right)$$

(32)

$\varepsilon = 0,1$; $a>0$; $Re(\rho)>-1$; $Re(\alpha)>-\frac{\varepsilon}{2}$; $\alpha-\sigma+\frac{\varepsilon}{2} \neq i \leq n$, $i = 0,1,2,...$

$$\Lambda = \int_{-a}^{a} (x+a)^{\gamma+\sigma-1/2} (a-x)^{\beta-1} C_m^{\gamma}(x/a)\, P_n^{(\rho,\sigma)}(x/a)\, dx =$$

$$= \frac{(2\gamma)_m}{m!\; n!} (1+\rho-\beta)_n\, B(\beta,\sigma+n+1)\, (2a)^{\beta+\sigma+\gamma-1/2}$$

$${}_4F_3\left(\begin{matrix}\frac{1}{2}-\gamma-m,\ \frac{1}{2}+\gamma+m,\beta-\rho,\beta\\ \beta+\sigma+n+1,\ \frac{1}{2}+\gamma,\beta-\rho-n\end{matrix}\ \Big|\ 1\right) \qquad (33)$$

a, $Re(\beta)>0$; $Re(\gamma+\sigma)>-\frac{1}{2}$; $\beta-\rho \neq i \leq n$, $i = 0,1,2,..$

$$\phi = \int_a^{\infty} (x-a)^{\gamma+\rho-1/2} (x+a)^{-\tau} C_m^{\gamma}(x/a)\, P_n^{(\rho,\sigma)}(x/a)\, dx =$$

$$= \frac{2^{\gamma+\rho-\tau+2m+1/2}(\gamma)_m}{m!\quad n!} \left(\frac{1}{2}+\tau+\sigma-\gamma-m\right)_n \cdot$$

$$B(\rho+n+1,\tau-\rho-\gamma-m-n-\tfrac{1}{2})\, a^{\gamma+\rho-\tau+1/2}\ .$$

$${}_4F_3\left(\begin{matrix}\frac{1}{2}-\gamma-m,\ 1-2\gamma-m,\ \frac{1}{2}+\tau+\sigma-\gamma+n-m,\tau-\rho-\gamma-m-n-1/2\\ 1-2\gamma-2m,\ \frac{1}{2}+\tau+\sigma-\gamma-m,\frac{1}{2}+\tau-\gamma-m\end{matrix}\ \Big|1\right)$$

$a>0$;$-\frac{1}{2}<Re(\gamma+\rho)<Re(\tau-m-n-\frac{1}{2})$; $1-2\gamma\neq i\leq 2m$;$\frac{1}{2}+\tau+\sigma-\gamma \neq j\leq m$;

$\frac{1}{2}+\tau-\gamma\neq l\leq m$; $i,j,l = 0,1,2,...$ (34)

$$\Omega = \int_{\pm a}^{\infty} (x\mp a)^{\alpha-1} J_\nu(b\sqrt{x\mp a})\, P_n^{(\rho,\sigma)}(x/a)\, dx$$

$$= \frac{(\pm 1)^n (\rho+\sigma+n+1)_n\, 2^{2\alpha+n}}{n!\, a^n\, b^{2\alpha+2n}} \frac{\Gamma(\frac{\nu}{2}+\alpha+n)}{\Gamma(\frac{\nu}{2}-\alpha-n+1)}\ .$$

$$\cdot\ {}_2F_3\left(-\rho-\sigma-2n, 1-\alpha-\frac{\nu}{2}-n,\ \frac{\nu}{2}-\alpha-n+1\ \Big|\ \frac{ab^2}{2}\right) \qquad (35)$$

$a,b,Re(2\alpha+\nu)>o$; $Re(\alpha)<\frac{3}{4}-n$;$\omega=\{^{\rho}_{\sigma}\}$; $-\rho-\sigma\neq i\leq 2n$;

$1-\alpha-\frac{\nu}{2}\neq j\leq n$; $\frac{\nu}{2}-\alpha+1\neq \ell\leq n$; $i,j,\ell = 0,1,2,\ ...$

Los símbolos $H_n(z)$ y $C_m^{\gamma}(z)$ corresponden a los polinomios de Hermite y Gegenbauer [3,14], respectivamente, y $J_\nu(z)$ es la función de Bessel de primera clase de orden [3,10,15].

Derivando ambos miembros de las ecs. (32)a(35) respecto a los parámetros que aparecen como subíndices en la notación, obtenemos los siguienes resultados :

$$\Delta_\alpha = \int_{-a}^{a} [\ln(x+a)]\, (x+a)^{\alpha-1} (a-x)^{\rho}\, e^{-b^2x}\ ,$$

$$H_{2m+\varepsilon}(b\sqrt{x+a})\, P_n^{(\rho,\sigma)}(x/a)\, dx = [\ln 2a+\pi ctg\pi(\alpha-\sigma+\tfrac{\varepsilon}{2})-$$

$$\pi\, ctg\, \pi(\alpha-\sigma+\tfrac{\varepsilon}{2}-n)]\ \Delta + \frac{(-1)^{m+n}}{n!}(\sigma-\alpha-\tfrac{\varepsilon}{2}+1)_n\ .$$

$$.\ (\varepsilon+\tfrac{1}{2})_m\, 2^{\alpha+\frac{3\varepsilon}{2}+\rho+2m}\ B(\alpha+\tfrac{\varepsilon}{2},\rho+n+1)$$

$$a^{\alpha+\rho+\frac{\varepsilon}{2}}\, b^{\varepsilon} \cdot e^{ab^2}\ .$$

$$\sum_{k=o}^{\infty} \frac{(m+\varepsilon+\frac{1}{2})_k\ (\alpha-\sigma+\frac{\varepsilon}{2})_k\ (\alpha+\frac{\varepsilon}{2})_k}{(\alpha+\rho+\frac{\varepsilon}{2}+n+1)_k\ (\alpha-\sigma+\frac{\varepsilon}{2}-n)_k\ (\varepsilon+\frac{1}{2})_k\ k!}\ .$$

$$[\psi(\alpha-\sigma+\tfrac{\varepsilon}{2}+k) + \psi(\alpha+\tfrac{\varepsilon}{2}+k) - \psi(\alpha+\rho+\tfrac{\varepsilon}{2}+n+k+1) -$$

$$\psi(\alpha-\sigma+\tfrac{\varepsilon}{2}-n+k)]\ (-2ab)^k \qquad (36)$$

$\varepsilon = 0,1$;$a>0$; $Re(\rho)>-1$; $Re(\alpha)>-\frac{\varepsilon}{2}$;$\alpha-\sigma+\frac{\varepsilon}{2}\neq i\leq n$,$i = 0,1,2,...$

$$\Lambda_\beta = \int_{-a}^{a} [\ln(a-x)]\,(x+a)^{\gamma+\sigma-\frac{1}{2}}\,(a-x)^{\beta-1}\,C_m^{\gamma}(x/a)\,\cdot$$

$$P_n^{(\rho,\sigma)}(x/a)\,dx = [\ln 2a + \pi\,\mathrm{ctg}\,\pi(\beta-\rho) - \pi\,\mathrm{ctg}\,\pi(\beta-\rho+n)]\Lambda +$$

$$+ \frac{(2\gamma)_m}{m!\,n!}(1+\rho-\beta)_n\;B(\beta,\sigma+n+1)\,(2a)^{\beta+\sigma+\gamma-\frac{1}{2}}\;\cdot$$

$$\sum_{k=o}^{\infty}\frac{(\frac{1}{2}-\gamma-m)_k\,(\frac{1}{2}+\gamma+m)_k\,(\beta-\rho)_k\,(\beta)_k}{(\beta+\sigma+n+1)_k\,(\frac{1}{2}+\gamma)_k\,(\beta-\rho-n)_k\,k!}\;\cdot$$

$$[\psi(\beta-\rho+k)+\psi(\beta+k)-\psi(\beta+\sigma+n+k+1)-\psi(\beta-\rho-n+k)] \qquad (37)$$

a, $Re(\beta)>0$; $Re(\gamma+\sigma)>-\frac{1}{2}$; $\beta-\rho \neq i \leq n$, $i = 0,1,2,\ldots$

$$\emptyset_\tau = \int_a^{\infty}[\ln(x+a)]\,(x-a)^{\gamma+\rho-\frac{1}{2}}\,(x+a)^{-\tau}\,C_m^{\gamma}(x/a)\;\cdot$$

$$\cdot P_n^{(\rho,\sigma)}(x/a)\,dx = (\ln 2a)\emptyset + \frac{2^{\gamma+\rho-\tau+2m+\frac{1}{2}}\,(\gamma)_m}{m!\;n!}\cdot$$

$$(\tfrac{1}{2}+\tau+\sigma-\gamma-m)_n\;B(\rho+n+1,\tau-\rho-\gamma-m-n-\tfrac{1}{2})\;\cdot$$

$$a^{\gamma+\rho-\tau+\frac{1}{2}}\;\cdot$$

$$\sum_{k=o}^{\infty}\frac{(\frac{1}{2}+\gamma-m)_k(1-2\gamma-m)_k(\frac{1}{2}+\tau+\sigma-\gamma+n-m)_k(\tau-\rho-\gamma-m-n-\frac{1}{2})_k}{(1-2\gamma-2m)_m\,(\frac{1}{2}+\tau+\sigma-\gamma-m)_k\,(\frac{1}{2}+\tau-\gamma-m)_k\,k!}\cdot$$

$$[\psi(\tfrac{1}{2}+\tau+\sigma-\gamma-m+k)+\psi(\tfrac{1}{2}+\tau-\gamma-m+k)-\psi(\tfrac{1}{2}+\tau+\sigma-\gamma+n-m+k)-$$

$$-\psi(\tau-\rho-\gamma-m-n+k-\tfrac{1}{2})] \qquad (38)$$

$a>0$; $-\frac{1}{2}<Re(\gamma+\rho)<Re(\tau-m-n-\frac{1}{2})$; $1-2\gamma \neq i \leq 2m$; $\frac{1}{2}+\tau+\sigma-\gamma \neq j \leq m$; $\frac{1}{2}+\tau-\gamma \neq \ell \leq m$; $i,j,\ell = 0,1,2,\ldots$

$$\Omega_\alpha = \int_{\pm a}^{\infty}[\ln(x\mp a)]\,(x\mp a)^{\alpha-1}\,J_\nu(b\sqrt{x\mp a})\,P_n^{(\rho,\sigma)}(x/a)\,dx =$$

$$=[2\ln(\tfrac{2}{b})-\pi\,\mathrm{ctg}\,\pi(\tfrac{\nu}{2}+\alpha+n)]\Omega + (\pm 1)^n\cdot$$

$$\frac{(\rho+\sigma+n+1)_n\,2^{2\alpha+n}}{n!\,a^n\,b^{2\alpha+2n}}\;\frac{\Gamma(\frac{\nu}{2}+\alpha+n)}{\Gamma(\frac{\nu}{2}-\alpha-n+1)}\;\cdot$$

$$\cdot\sum_{k=o}^{n}\frac{(-n)_k\,(-\omega-n)_k}{(-\rho-\sigma-2n)_k(1-\alpha-\frac{\nu}{2}-n)_k\,(\frac{\nu}{2}-\alpha-n+1)_k\,k!}\cdot$$

$$[\psi(1-\alpha-\tfrac{\nu}{2}-n+k)+\psi(\tfrac{\nu}{2}-\alpha-n+k+1)]\,\left(\pm\frac{ab^2}{2}\right)^k \qquad (39)$$

$a,b,Re(2\alpha+\nu)>0$; $Re(\alpha)<\frac{3}{4}-n$; $-\rho-\sigma \neq i \leq 2n$; $1-\alpha-\frac{\nu}{2} \neq j \leq n$; $\frac{\nu}{2}-\alpha+1 \neq 1 \leq n$; $i,j,1 = 0,1,2,\ldots$

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