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ON A TEST STATISTIC CONNECTED WITH THE MULTIVARIATE BEHRENS - FISHER PROBLEM

ABSTRACT

The exact null distribution of a test statistic proposed earlier for testing equality to two mean vectors under the intraclass correlation models has been derived in terms of generalized hypergeometric function as well as in series form suitable for computation. Explicit expressions of the density are given for $p = 2, 3, 4$ and 5 .

RESUMEN

La distribución nula exacta de un estadístico de prueba propuesto con anterioridad para examinar la igualdad de dos vectores medios bajo los modelos de correlación de interclases, ha sido deducida en términos de la función hipergeométrica generalizada, así como en forma de series adecuadas para la computación. Se mencionan expresiones explícitos de la densidad para p = 2,3,4 y 5.

1. INTRODUCTION

Let X and Y be independent $p\times m$ and $p\times n$ random matrices whose columns are independently and identically distributed as

$N_p(\underset{\sim}{\mu}_1,\Sigma_1)$ and $N_p(\underset{\sim}{\mu}_2,\Sigma_2)$ respectively. Also, let

$$\Sigma_i = \Sigma(\rho_i,\sigma_i) = \sigma_i^2[(1-\rho_i)I_p + \rho_i \underset{\sim}{e}\underset{\sim}{e}'], \ i = 1,2,$$

where I_p is the identity matrix of order p and $\underset{\sim}{e} = (1,\ldots,1)'_{p\times 1}$.

Press (1967) combined Scheffé-Bennett procedure with the likelihood ratio test and derived a test statistic Λ for testing $H: \underset{\sim}{\mu}_1 = \underset{\sim}{\mu}_2$ and showed that h-th null moment of Λ is $(n>m)$

$$E(\Lambda^h) = \Big[\Gamma[(m-1)/2+h]\Gamma(m/2)/\{\Gamma(m/2+h)\Gamma((m-1)/2)\}\Big] \prod_{j=0}^{p-2}\Big[\Gamma[(m-1)/2+j/(p-1)+h]\cdot \Gamma[m/2+j/(p-1)]/\{\Gamma[(m-1)/2+j/(p-1)]\Gamma[m/2+j/(p-1)+h]\}\Big]. \quad (1.2)$$

This article deals with the distributional problem of Λ. For related work see Pillai and Gupta (1969), Gupta (1971) and references given there. In section 2, the density of Λ has been derived in terms of G-function, using inverse Mellin transform and the definition of G-function. In section 3, the series form of the density is given which has been obtained using residue theorem. Special cases of the density are given for p= 2, 3, 4 and 5.

2. DENSITY OF Λ

By using inverse Mellin transform and the expression (1.2), the density of Λ, denoted by $f(\lambda)$, is given by

$$f(\lambda) = K(m,p)(2\pi\omega)^{-1}\int_C \{\Gamma[(m-1)/2+h]/\Gamma[m/2+h]\}\prod_{j=0}^{p-2}\{\Gamma[(m-1)/2+j/(p-1)+h]/\Gamma[m/2+j/(p-1)+h]\}\lambda^{-1-h}dh. \quad 0<\lambda<1.$$

$$= K(m,p)(2\pi\omega)^{-1}\lambda^{(m-3)/2} \int_{C_1} [\Gamma(\alpha)/\Gamma(\alpha+1/2)] \prod_{j=0}^{p-2} \{\Gamma[\alpha+ j/(p-1)]/\Gamma[\alpha+1/2+j/(p-1)]\}\lambda^{-\alpha}d\alpha, \quad 0 < \lambda < 1 \tag{2.1}$$

where $\omega = (-1)^{1/2}$

$$K(m,p) = \{\Gamma[m/2]/\Gamma[(m-1)/2]\} \prod_{j=0}^{p-2} \{\Gamma[m/2+j/(p-1)]/ \Gamma[(m-1)/2 + j/(p-1)]\} \tag{2.2}$$

and C, C_1 are the suitable contours. From the theory of G-functions it is easy to see that C and C_1 can always be found and (2.1) can be put in terms of G-function [Erdélyi et al. (1953), p.207] as

$$f(\lambda) = K(m,p)\lambda^{(m-3)/2} G^{p,0}_{p,p}\left[\lambda \middle| \begin{matrix} 1/2, 1/2+j/(p-1), \\ 0, j/(p-1), \end{matrix} \right. \begin{matrix} j = 0,1,\ldots,p-2 \\ j = 0,1,\ldots,p-2 \end{matrix} \Bigg] \quad 0 < \lambda < 1. \tag{2.3}$$

Using the well known result on the G-function, namely,

$$G^{2,0}_{2,2}\left[z \middle| \begin{matrix} \alpha_1+\beta_1-1, \alpha_2+\beta_2-1 \\ \alpha_1-1 \quad , \alpha_2-1 \end{matrix}\right] = [z^{\alpha_2-1}(1-z)^{\beta_1+\beta_2-1}/\Gamma(\beta_1+\beta_2)] \; {}_2F_1(\alpha_2+\beta_2-\alpha_1, \beta_1; \beta_1+\beta_2; 1-z), \quad |z| < 1 \tag{2.4}$$

the density (2.3) for $p = 2$ is given by

$$f(\lambda) = K(m,2)\lambda^{(m-3)/2} \; {}_2F_1(1/2, 1/2; 1; 1-\lambda),$$

and for $p = 3$ by

$$f(\lambda) = K(m,3)\lambda^{(m-3)/2}\{(1-\lambda)^{1/2}/\Gamma(3/2)\} \; {}_2F_1(1, 1/2; 3/2; 1-\lambda). \tag{2.5}$$

On using (16) of Erdély et al. (1953, p.102), (2.5) becomes

$$f(\lambda) = 2K(m,3)\pi^{-1/2} \lambda^{(m-3)/2} \log\{(1 + \sqrt{1-\lambda})/\sqrt{\lambda}\}, \quad 0 < \lambda < 1. \tag{2.6}$$

3. DENSITY IN SERIES FORM

General series expansion of the density $f(\lambda)$ is obtained by using residue theorem on the right hand side of (2.1). Two cases, p-even and p-odd, are considered separately.

<u>Case I : p-odd.</u>

Cancelling out the common factors in the integrand of (2.1), the density $f(\lambda)$ is rewritten as

$$f(\lambda) = K(m,p)\lambda^{(m-3)/2} (2\pi\omega)^{-1} \int_{C_1} \{\Gamma(\alpha)/\Gamma(\alpha+1/2)\} \prod_{j=0}^{(p-3)/2} \left[\alpha+j(p-1)^{-1}\right]^{-1} \lambda^{-\alpha} d\alpha. \tag{3.1}$$

The integrand has poles of order unity at $\alpha=-i/(p-1)$, $i = 1,2,\ldots,(p-3)/2$; $\alpha= -i$, $i = 1,2,\ldots;$ and a pole of order 2 at $\alpha=0$. The residue at $\alpha =-i(p-1)^{-1}$ is

$$\Gamma[-i/(p-1)]\lambda^{i/(p-1)} / \{\Gamma[1/2 - i/(p-1)] \prod_{j=0}^{(p-3)/2} [(j-i)/(p-1)]\};$$

the residue at $\alpha = -i$ is

$$\lambda^{i} / [\Gamma[1/2 - i]\{ \prod_{\substack{j=0 \\ \neq i}}^{(p-3)/2} (-i+j/(p-1)) \prod_{j=0}^{i-1} (j-1)\}],$$

and the residue at $\alpha = 0$ is $[-\log\lambda + \psi(1) - \psi(1/2)$

$$- \sum_{j=1}^{(p-3)/2} \{j/(p-1)\}^{-1}\Big]/\{\Gamma(1/2)\sum_{j=1}^{(p-3)/2}(j/(p-1))\},$$

where $\psi(.)$ is the well known psi function [Erdélyi et al. (1953), p. 15]. Simplifying these with the help of conversion formula [Mathai(1982), eqn. 2.7], equations (6) and (7) of Erdélyi et al. (1953, p. 3); and equation (12) of Erdélyi et al. (1953, p. 16) and using the residue theorem one has the following result.

$$f(\lambda) = K(m,p)\lambda^{(m-3)/2}\Big[\Big[(p-1)^{(p-3)/2}/\{\Gamma(1/2)(\tfrac{p-3}{2})!\}\Big]\Big[-\log(\lambda/4) - \sum_{j=1}^{(p-3)/2}\{(p-1)j^{-1} + (-1)^{j}(p-1)j^{-1}\cot\{\pi j(p-1)^{-1}\}\}\Gamma[1/2+j/(p-1)]\lambda^{j/(p-1)}\Gamma(1/2)(\tfrac{p-3}{2})!\{\Gamma[j/(p-1)]\, j!(\tfrac{p-3}{2}-j)!\}\Big] + (-1)^{(p-1)/2}\sum_{j=1}^{\infty}\lambda^{j}\Gamma(1/2+j)/\{\Gamma^{2}(1/2)j!\prod_{k=0}^{(p-3)/2}(j-k/(p-1))\}\Big]\Big], \qquad (3.2)$$

$$0 < \lambda < 1.$$

For p=5, the above expresion reduces to

$$f(\lambda) = K(m,5)\lambda^{(m-3)/2}(4/\sqrt{\pi})\Big[-\log(\lambda/4) - 4 + 4\,\{\Gamma(3/4)\Gamma(1/2)/\Gamma(1/4)\}\,\lambda^{1/4} + \sum_{j=1}^{\infty}\lambda^{j}\Gamma(1/2+j)/\{\Gamma(1/2)j(j-1/4)j!\}\Big], \quad 0 < \lambda < 1.$$

Case II : p-even.

In this case the poles are of second order at $\alpha = -i$, $i = 0,1,\ldots$, and of order unity at $\alpha = (-i-k)/(p-1)$, $k = 1,2,\ldots, p-2$, $i = 0,1,\ldots$. The residue at $\alpha = -i$ is

$$\lambda^{i}\Big[-\log\lambda + 2\psi(1) - 2\sum_{j=0}^{i-1}(j-i)^{-1} - 2\psi(1/2-i) + \sum_{j=1}^{p-2}\{\psi(j/(p-1)-i) - \psi(j/(p-1)+1/2-i)\}\Big]\Big[\prod_{j=1}^{p-2}\{\Gamma[j/(p-1)-i]/\Gamma[1/2+j/(p-1)-i]\}\Big]/\Big[\Gamma^{2}(1/2-i)\prod_{j=0}^{i-1}(j-i)^{2}\Big].$$

The residue at $\alpha = -k/(p-1)-i$ is

$$\Gamma^{2}[-i-k/(p-1)]\prod_{\substack{j=1\\ \neq k}}^{p-2}\Gamma[(j-k)/(p-1)-i]/\{\Gamma^{2}[1/2-i-k/(p-1)]\prod_{j=0}^{i-1}(j-i)\prod_{j=1}^{p-2}\Gamma[1/2+(j-k)/(p-1)-i]\}.$$

Simplifying these with the help of the conversion formula [Mathai (1982), eqn.2.7] and 1.7.1 of Erdélyi et al. (1953, p. 16), and using the residue theorem the density given in (2.1) is obtained as

$$f(\lambda) = K(m,p)\lambda^{(m-3)/2}\sum_{i=0}^{\infty}\Big[\Big[-\log\lambda + 2\psi(i+1) - 2\psi(i+1/2) + \sum_{j=1}^{p-2}\{\psi(j/(p-1)) - \psi(1/2-j/(p-1)+i) - \pi\tan j\pi(p-1)^{-1}\}\Big].[\lambda^{i}\Gamma^{2}(1/2+i)/i!\pi]\prod_{j=1}^{p-2}\{\Gamma^{2}[j/(p-1)]\Gamma[1/2-j/(p-1)+i]/\{\Gamma[1/2+j/(p-1)]\,.\,\Gamma[i+j/(p-1)]\Gamma[1/2-j/(p-1)]\}\} + \sum_{k=1}^{p-2}\Big[\Gamma^{2}[-k/(p-1)]\,\Gamma^{2}[1+k/(p-1)].\Gamma^{2}[1/2+k/(p-1)+i]/\{\Gamma^{2}[1+i+k/(p-1)]\Gamma^{2}[1/2-k/(p-1)].\Gamma^{2}[1/2+k/(p-1)]i!\}\Big]\prod_{\substack{j=1\\ \neq k}}^{p-2}\Big[\Gamma[(j$$

$$-k)/(p-1)]\Gamma\left[1-(j-k)/(p-1)\right]/\Gamma\left[i+1-(j-k)/(p-1)\right]]\prod_{j=1}^{p-2}\left[\Gamma[1/2-(j-k)/(p-1)+i]/\{\Gamma[1/2-(j-k)/(p-1)]\,\Gamma[1/2+(j-k)/(p-1)]\}\right]\lambda^{i+k/(p-1)}],$$

$$0<\lambda<1.$$

As a special case, the density for p=4 is given by,

$$f(\lambda) = K(m,4)\lambda^{(m-3)/2}\sum_{i=0}^{\infty}\left[\left[-\log\lambda+2\psi(i+1)-2\psi(i+1/2)\right.\right.$$

$$+\psi(i+1/3)+\psi(i+2/3)-\psi(i+1/6)-\psi(i-1/6)]\lambda^{i}\,\Gamma^2(1/3)$$

$$\Gamma^2(2/3)\Gamma^2(i+1/2)\Gamma(1/6+i)\Gamma(-1/6+i)/\{\Gamma(5/6)\Gamma(7/6)$$

$$\Gamma(i+1/3)\pi\Gamma(i+2/3)\Gamma(1/6)\Gamma(-1/6)(i!)^2\}+\lambda^{1/3+i}$$

$$\Gamma^2(-1/3)\Gamma^2(4/3)\Gamma^2(5/6+i)\Gamma(1/3)\Gamma(2/3)\Gamma(1/2+i)$$

$$\Gamma(1/6+i)/\{\Gamma^2(4/3+i)(i!)\Gamma^3(1/6)\Gamma^3(5/6)\Gamma(2/3+i)$$

$$\Gamma^2(1/2)\}+\lambda^{2/3+i}\,\Gamma^2(-2/3)\Gamma^2(5/3)\Gamma^2(7/6+i)\Gamma(-1/3)$$

$$\Gamma(4/3)\Gamma(1/2+i)\Gamma(5/6+i)/\{\Gamma^2(5/3+i)(i!)\Gamma^2(-1/6)$$

$$\Gamma^2(7/6)\Gamma(4/3+i)\Gamma^2(1/2)\Gamma(1/6)\Gamma(5/6)\}],$$

$$0<\lambda<1.$$

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