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ON THE WIENER-LAGUERRE TRANSFORMATION

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ABSTRACT

In this paper we investigate a discrete integral transformation on the real line, which, with respect to the convolution structure, is better adapted than the discrete Hermite transformation, introduced by Debnath. The kernels are complex-valued rational functions which are the Fourier transforms of the Laguerre polynomials and which were introduced by N. Wiener. Some remarks about a real version, following a paper of Christov are added

ABSTRACT

En este trabajo se investiga una transformación integral discreta sobre la línea real, la cual, considerando la estructura de la convolución, se adapta mejor que la transformación discreta de Hermite introducida por Debnath. Los núcleos son funciones racionales de valores complejos, las cuales son transformadas de Fourier de los polinomios de Laguerre introducidos por N. Wiener. Se incluyen algunas observaciones sobre la versión real, siguiendo un trabajo de Christov.

1. INTRODUCTION

In [4], [5] Debnath has investigated the discrete Hermite transforms, that are the Fourier-coefficients with respect to the complete orthonormal system (CON) of the Hermite functions on the real line. As was pointed out in [5], the convolution theorem exists only for odd arguments (see also [6], [7]), a gap, which can be closed considering the Hermite transformation of generalized functions (see [8]). The reason for the non-existence of a general convolution theorem (In the classical case) is the non-existence of a linearization formula for the product of two Hermite polynomials.

Therefore another CON on the real line, which consists of complex-valued rational functions, and which was introduced in [12], 1.03 by Wiener, seems to be better adapted for the study of an integral transformation on the real line. There exist linearization formulas with respect to the argument as well as with respect to the index. The members of the CON are essentially the Fourier transforms of the Laguerre functions. In [3] Christov has given a real-valued CON, which is connected with the complex-valued CON above analogous to the functions cos x and sin x with the function exp ix.

In section 2 we repeat some results on the kernel of the integral transform under consideration, using [12], 1.03, [9], 2.6.4. and [3]. Also we give a linearization of the product of two kernels with the same index and different arguments (see (2.13)). In section 3 we define the transformation $\mathcal{T}$ in certain original spaces. An inversion theorem is proved and a connexion between $\mathcal{T}$ and the z - and Stieltjes- transformations is derived. Section 4 deals with the operational calculus of the associated integral transformation including convolution theorems in the original- and in the image-domain, which are missing in the case of the Hermite transformation. In section 5 we will remark on a real version of the kernel and the transformation, following Christov [3].

It must be mentioned, that in [3] Christov has explained, that the CON of the Wiener-Laguerre -functions is very well adapted for the determination of solutions of interesting non-linear differential equations of mathematical physics which are square integrable on the real line because of the asymptotic behaviour of the functions of the CON at $\pm\infty$ (like x^{-1}) and the existence of linearization formulas.

NOTATIONS. In this paper N denotes the set of natural numbers, $N_0 = N \cup \{0\}$, Z the set of integers, R the field of real numbers, R^+, R^- the subsets of positive and negative real numbers respectively. With $\mathbb{C}$ we denote the field of complex numbers and a^* is the complex conjugate of a.

2. THE WIENER-LAGUERRE FUNCTIONS

It is well known, that the Laguerre functions (of order zero)

$$l_n(t) = e^{-t/2} L_n(t) , \quad n \in N_o \qquad (2.1)$$

where L_n are the Laguerre polynomials

$$L_n(t) = (n!)^{-1} e^{t/2} D^n(e^{-t}t^n) , \quad n \in N_o \qquad (2.1')$$

form a complete orthonormal system in $L_2(R^+)$. Let $\mathcal{F}$ be the (unitary) Fourier transformation defined by

$$\mathcal{F}[f](x) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(t)e^{-ixt}\, dt \qquad (2.2)$$

and η be the Heaviside function

$$\eta(t) = \begin{cases} 1 , & t > 0 , \\ 0 , & t < 0 . \end{cases} \qquad (2.3)$$

From the fact that the sequence $\{\sqrt{2}\eta(t)l_n(2t)\}_{n=0}^{\infty}$ forms a CON in $L_2(R^+)$ too, we conclude that the sequence $\{\rho_n\}_{n=0}$ with

$$\rho_n(x) = \mathcal{F}[\sqrt{2}\,\eta(t)l_n(2t)](x),\ x \in R ,\ n \in N_o \qquad (2.4)$$

forms a CON in that subspace L_2^+ of $L_2(R)$, for which the inverse Fourier transforms vanish on the negative real line R^-. Similarly by straight forward calculation we have that the sequence

$$\{-\sqrt{2}\eta(-t)l_n(-2t)\}_{n=0}^{\infty} = \{-\sqrt{2}\,\eta(-t)l_{n-1}(-2t)\}_{n=1}^{\infty}$$

forms a CON in $L_2(R^-)$. Therefore, the sequence $\{\rho_{-n}\}_{n=1}^{\infty}$ with

$$\rho_{-n}(x) = \mathcal{F}[-\sqrt{2}\eta(-t)l_{n-1}(-2t)](x),\ n \in N \qquad (2.5)$$

forms a CON in that subspace L_2^- of $L_2(R)$ for which the inverse Fourier transforms vanish on the positive real line R^+. Because of $L_2(R)=L_2^- + L_2^+$ we conclude, that the set $\{\rho_n\}_{n=-\infty}^{\infty}$ forms a CON in $L_2(R)$. If we take the Laplace transform

$$L(f) = \int_0^{\infty} e^{-zt} f(t)dt, \qquad (2.2')$$

then from (2.4) we get

$$\rho_n(x) = \frac{1}{2\sqrt{\pi}} L[l_n](\tfrac{ix}{2}),\ n \in N_o , \qquad (2.4')$$

and from (2.5) we have

$$\rho_{-n}(x) = \frac{-1}{2\sqrt{\pi}} L[l_n](-\tfrac{ix}{2}),\ n \in N \qquad (2.5')$$

Using [2], 4.11., (31) after a simple calculation from (2.4), (2.5) we have

$$\rho_n(x) = \frac{1}{\sqrt{\pi}} \frac{(ix-1)^n}{(ix+1)^{n+1}} ,\ n \in Z ,\ x \in R . \qquad (2.6)$$

For details we refer to [9], section 2.6.4.

In the following we let $n,m,k \in Z$ and $t,x,y \in R$. Now let us look at some properties of the functions ρ_n, which are proved by straightforward calculations. From (2.6) we derive

$$\rho_n^* = -\rho_{-n-1}, \qquad (2.6')$$

$$\rho_n(-x) = \rho_n^*(x), \qquad (2.7)$$

and

$$|\rho_n| \le 1/\sqrt{\pi} . \qquad (2.8)$$

The asymptotic as x tends to $\pm\infty$ is given by

$$\rho_n(x) \sim -i/\sqrt{\pi}\,x,\ x \to \pm\infty. \qquad (2.9)$$

By means of the first of the formulas

$$\rho_{n-1}(x) - \rho_n(x) = \frac{2}{ix+1}\ \rho_{n-1}(x), \qquad (2.10)$$

and

$$\rho_{n-1}(x) + \rho_n(x) = 2ix\ \rho_{n-1}(x)/(ix+1) \qquad (2.10')$$

we can derive the recurrences

$$\rho_n' = \frac{i}{2}\left[n\ \rho_{n-1} - (2n+1)\ \rho_n + (n+1)\rho_{n+1}\right], \qquad (2.11)$$

and

$$2x\,\rho_n'(x) = n\,\rho_{n-1}(x) - \rho_n(x) - (n+1)\,\rho_{n+1}(x) \qquad (2.11')$$

From the recurrence for the Laguerre polynomials, see [1], 10,12, (8), and (2.11'), (2.4'), (2.5'), after a short calculation, we obtain

$$(x-i)\ \rho_n'(x) = n\,\rho_{n-1}(x) - (n+1)\,\rho_n(x), \qquad (2.12)$$

and

$$(x+i)\rho_n'(x) = n\rho_n(x) - (n+1)\rho_{n+1}(x) \qquad (2.12')$$

Of special importance are two linearization formulas for the product of two members of the CON $\{\rho_n\}$:

$$2\sqrt{\pi}\ \ \rho_n\rho_k = \rho_{n+k} - \rho_{n+k+1}\ , \qquad (2.13)$$

and

$$2\sqrt{\pi}\ \rho_n\rho_k^* = \rho_{n-k} - \rho_{n-k-1}\ . \qquad (2.13')$$

In addition to the results cited in [3] one can easily calculate the relations

$$\rho_n(x)\ \rho_n(y) = \frac{1}{\sqrt{\pi}\,i(x+y)}\ \rho_n\left(\frac{xy-1}{x+y}\right)$$

and

$$\rho_n(x)\rho_n^*(y) = \frac{-1}{\sqrt{\pi}\,i(y-x)}\ \rho_n\left(\frac{xy+1}{y-x}\right)\ . \qquad (2.14')$$

Directly from the definition we can get a generating function for the sequence $\{\rho_n\}$:

$$\sum_{n=0}^{\infty} \rho_n(x)z^{-n} = \frac{-iz}{\sqrt{\pi}(z-1)}\ \frac{1}{i\left(\frac{1+z}{1-z}\right)+x}\ , \quad |z| > 1 \qquad (2.15)$$

and

$$\sum_{n=0}^{\infty} \rho_{-n-1}(x)\,z^{-n} = \frac{-iz}{\sqrt{\pi}(z-1)}\ \frac{1}{-i\left(\frac{1+z}{1-z}\right)+x}\ , \quad |z| > 1 \qquad (2.15')$$

Now let S be the differential expression defined by

$$Su(x) = -\left[(x^2+1)u(x)\right]' + xu(x), \qquad (2.16)$$

then we have the linear first order differential equation

$$S\ \rho_n = (2n+1)i\ \ \rho_n, \quad n \in Z\ . \qquad (2.17)$$

With

$$S^2u(x) = S(Su(x)) = (x^2+1)\ u''(x) + 4x(x^2+1)u'(x) + (2x^2+1)u(x) \qquad (2.16')$$

from (2.16) it follows at once, that the functions ρ_n are solutions of the linear differential equation of second order

$$S^2 \rho_n = -(2n+1)^2 \rho_n, \quad n \in Z . \qquad (2.17')$$

Similarly, as in the case of the functions $e_n(x) = e^{inx}$, $n \in Z$, we can consider (2.16') from another point of view. Let

$$S^2 u + (2n+1)^2 u = 0, \quad n \in N_0 \qquad (2.18)$$

then from the considerations above we see, that the set $\{\rho_n , \rho_{-n-1}\}$ forms a fundamental system for (2.18). We return to this result in the section 5.

Let R denote a right inverse of S, i.e.

$$SR\, f = f$$

and let $y = Rf$, so y is solution of the differential equation

$$Sy = f.$$

The solution of this inhomogeneous linear differential equation of first order is

$$y(x) = (x^2+1)^{-1/2}\left[\int_x^\infty f(t)(t^2+1)^{-1/2} dt + C\right], \quad C \in C_0.$$

The special solution with $C = 0$ is denoted by R, so that

$$Rf(x) = (x^2+1)^{-1/2} \int_x^\infty f(t)(t^2+1)^{-1/2} dt \qquad (2.19)$$

3. THE TRANSFORM

As the Wiener-Laguerre-transform (WLT) of a function f we define the sequence of Fourier coefficients of f with respect to the CON $\{\rho_n\}_{n=-\infty}^{\infty}$:

$$\tau[f](n) = F(n) = \int_{-\infty}^{\infty} f(t)\, \rho_n(t) dt, \quad n \in Z, \qquad (3.1)$$

if the integrals exist.

Let $C^k(R) =: C^k$, $k \in N_o$ with $C^o = C$ be the spaces of functions with continuous derivatives up to the order k. Then we consider the following original spaces of functions. Let $\nu > 0$, then

$$E_\nu^k := \{f : f \in C^k : f^{(j)}(t) = O(|t|)^{-\nu-j}, \quad |t| \to \infty, \ j = 0,1,\dots,k\} \qquad (3.2)$$

and

$$E_\nu := \{f : f \in L_{1,loc}(R) ; f(t) = O(|t|^{-\nu}), \ |t| \to \infty\}, \qquad (3.2')$$

where $L_{1,loc}(R)$ is the space of locally integrable functions on the real line. Obviously we have

$$E_\nu \supset E_\nu^0 \supset E_\nu^1 \supset \dots \supset E_\nu^k \supset E_\nu^{k+1} \supset \dots . \qquad (3.3)$$

Remark : In the following all results are formulated in the largest spaces in which they are valid.

From (3.1) we get the

THEOREM 3.1. Let $f \in E_\nu$. Then $\tau[f]$ exists, τ is a linear transformation . If $f \in E_\nu^k$ then

$$F(n) = O(|n|^{-k}), \quad n \to \pm\infty \qquad (3.4)$$

For the proof we refer to (3.1) and (3.2) and from the first of these formulas we get (3.4) by means of integration by parts, using (see [11], 1.2.6,4.)

$$\int \frac{(x+a)^p}{(x+b)^q} dx = -\frac{(x+a)^p}{(q-1)(x+b)^{q-1}} + \frac{p}{(q-1)} \int \frac{(x+a)^{p-1}}{(x+b)^{q-1}} dx$$

In preparation for an inversion formula we want to remark, that instead of (3.1) the Fourier coefficients (in the usual notation) are the numbers

$$(f, \rho_n) = \int_{-\infty}^{\infty} f(t)\, \rho_n^*(t)\, dt.$$

Because of (2.6') and $n \in Z$ we can choose the definition (3.1), but one has to be careful in formulating the inversion formula in the Hilbert space $L_2(R)$, that is the Fourier expansion of f with respect to the CON $\{\rho_n\}_{n=-\infty}^{\infty}$. From (2,6')we get immediately for all

$f \in L_2(R)$:

$$f(x) = - \sum_{n=-\infty}^{\infty} F(-n-1)\, \rho_n(x). \qquad (3.5)$$

The validity of the inversion formula (3.5) in the sense of ordinary convergence follows easily from (2.8) and (3.4) :

THEOREM 3.2. Let $f \in F_\nu^2$ and $F = \zeta[f]$ then the inversion formula (3.5) holds and the convergence is absolute and uniform.

Remark : With the help of (2.6') instead of (3.5) we can write

$$f(x) = \sum_{n=-\infty}^{\infty} F(n)\, \rho_n^*(x) \qquad (3.5')$$

and Theorem 3.2 is also valid.

A connexion with well tabulated integral transforms can be derived in the following manner: Let z be the z -transformation of a sequence $a = \{a_n\}_{n=0}^{\infty}$, that is

$$z[a](z) = \sum_{n=0}^{\infty} a_n z^{-n}, \qquad (3.6)$$

and γ the Stieltjes transformation, defined by

$$\gamma[f](x) = \int_0^{\infty} \frac{f(t)}{x+t}\, dt, \quad |\arg(x)| < \pi \qquad (3.7)$$

Furthermore let F^+ resp. F^- be the restrictions of $F = \zeta[f]$ to N_0 resp. N^-, then we have(first of all purely formal) by means of the generating function (2.15)

$$z[F^+](z) = \int_{-\infty}^{\infty} f(t) \left(\sum_{n=0}^{\infty} z^{-n} \rho_n(t) \right) dt =$$

$$\frac{-iz}{\sqrt{\pi}(z-1)} \int_{-\infty}^{\infty} \frac{f(t)}{i\left(\frac{1+z}{1-z}\right)+t}\, dt$$

$$= \frac{iz}{\sqrt{\pi}(1-z)} \{\gamma[f](a) - \gamma[f(-t)](-a)\}, \quad a = i\,\frac{1+z}{1-z}$$

Analogous by means of (2.15") we have

$$z[F^-](z) = \int_{-\infty}^{\infty} f(t) \left(\sum_{n=0}^{\infty} \rho_{-n-1}(t) z^{-n} \right) dt =$$

$$= \frac{-iz}{\sqrt{\pi}\,(z-1)} \int_{-\infty}^{\infty} \frac{f(t)}{t-a}\, dt$$

$$= \frac{iz}{\sqrt{\pi}(1-z)} \{\gamma[f](-a) - \gamma[f(-t)](a)\}.$$

Let $f \in E_\nu$ then from (3.1) we get immediately that $F(n) = O(1)$ if n tends to $\pm\infty$. Therefore the z - transforms $z[F^+]$ and $z[F^-]$ exist for all z with $|z| > 1$. With $z = re^{i\rho}$, $r > 1$, and $N = 1 + r^2 - 2r\cos\phi > 0$ we have

$$a = i\,\frac{1+re^{i\phi}}{1-re^{\phi}} = N^{-1}(-2r\sin\phi + i(1-r^2))$$

and therefore $\mathrm{Im}(\pm a) \underset{(>)}{<} 0$ if $r>1$, that is $|\arg(\pm a)| < \pi$ is satisfied and hence the Stieltjes transforms $\gamma[f(\pm t)](\pm a)$ exist. Summarizing the consideration above we have the

THEOREM 3.3. Let $f \in E_\nu$ and F^+ resp. F^- the restrictions of F to N_0 resp. N , $a = i\,\frac{1+z}{1-z}$, then

$$z[F(\pm)](z) = \frac{iz}{\sqrt{\pi}(1-z)} \{\gamma[f](\pm a) - \gamma[f(-t)](\mp a)\}, \quad |z| > 1. \qquad (3.8)$$

Because of $\gamma[f] = L[L[f]]$ one can use the tables of the Stieltjes transforms (see for example [2], vol. II) as well as the tables of the Laplace transforms ([2], vol. I). Tables of z - transforms are contained in [10].

4. OPERATIONAL CALCULUS

For applications of the WLT the operational rules are of great importance. From (2.7) and (2.6') we have

$$\zeta[f(-x)](n) = -F(-n-1) \tag{4.1}$$

Integrating by parts by means of (2.11) resp. (2.12) we can derive

$$\zeta[f'](n) = \frac{i}{2}[(2n+1)F(n)-nF(n-1)-(n+1)F(n+1)], \tag{4.2}$$

$$f \in E_\nu^1,$$

and

$$\zeta[xf'(x)] = \frac{1}{2}[(n+1)F(n+1)-nF(n-1)-F(n)], \tag{4.3}$$

$$f \in E_\nu^1.$$

From (2.16) and (2.17) we have by means of integration by parts the differentiation rule

Proposition 4.1. Let $f \in E_{\nu+1}$ and S the differential expression of (2.16). Then $Sf \in E_\nu^0$ and

$$\zeta[Sf](n) = -(2n+1)\, iF(n). \tag{4.4}$$

By complete induction we have the

Conclusion 4.1. Let $f \in E_{\nu+k}^k$, $k \in N_0$, then $S^k f \in E_\nu^0$

$$\zeta[S^k f](n) = [-(2n+1)i]^k F(n). \tag{4.4'}$$

Conversely with the notation of (2.19) we have for the right inverse operator R of S the

Lemma 4.1. Let $f \in E_\nu^0$ Then $Rf \in E_1^1$ and $SRf = f$.

Proof. From (2.19) and $f \in C$ we have $Rf \in C^1$ and $Rf(x) = O|x|^{-1}$ as $x \to \pm\infty$. Now we have

$$[Rf(x)]' = \frac{-1}{x^2+1}\left(\frac{x}{x^2+1}\left[\int_x^\infty f(t)(t^2+1)^{-1/2}dt+C\right]+f(x)\right)$$

and therefore $[Rf(x)]' = O(x^{-2})$ as $x \to \pm\infty$, hence $Rf \in E_1^1$.

From Proposition 4.1 and Lemma 4.1 we also have the

Conclusion 4.2. Let $f \in E_\nu^0$ and $k \in N_0$, then we have the integration rule

$$\zeta[R^k f](n) = \left(\frac{i}{2n+1}\right)^k F(n). \tag{4.5}$$

The construction of a convolution in the original domain will be prepared by a generalized translation operator τ_y, $y \in R$, fixed with domain E_1 and

$$(\tau_y f)(x) = \frac{1}{\sqrt{\pi}\, i(y-x)} f\left(\frac{xy+1}{y-x}\right). \tag{4.6}$$

Proposition 4.2. Let $f \in E_1$. Then for each fixed $y \in R$ the operator τ_y is a linear operator of E_1 into E_1 and the formulas

$$\tau_y \rho_n^*(x) = \rho_n^*(x)\,\rho_n(y) \tag{4.7}$$

$$\zeta[\tau_y f](n) = F(n)\,\rho_n(y) \tag{4.8}$$

are satisfied.

Proof. From (4.6) we have immediately, that τ_y is a linear operator of E_1 into E_1. Also (4.7) follows from (2.14') by taking the complex conjugate. Substituting $t := (xy+1)/(y-x)$ and using (2.14) we get

$$\zeta[\tau_y f](n) = \frac{1}{\sqrt{\pi}\, i}\int_{-\infty}^{\infty} f(t)\,\frac{\rho_n\left(\frac{ty-1}{y+1}\right)}{(y+t)}\,dt = \rho_n(y)\,F(n)$$

In terms of the operator τ_y we are able to define a convolution in E_1^0 by

$$(f*g)(x) = \int_{-\infty}^{\infty} f(t)\,\tau_t\, g(x)\,dt. \tag{4.9}$$

Theorem 4.1 (Convolution Theorem). Let $f, g \in \check{E}_1$, Then $f*g \in E_1^0$ and the convolution is commutative, associative and distributive.
Furthermore we have

$$\tau[f*g] = FG. \tag{4.10}$$

Proof. From (4.6) we conclude, that also $(\tau_t f)(x)$ is $O(|t|^{-1})$ as $|t| \to \infty$ for each fixed $x \in R$ and therefore the integrand in (4.9) is $O(t^{-2})$ as $t \to \infty$, that is, the convolution $f*g$ exists. Obviously it belongs to E_1^0 too. From the theorem and (4.8) we have

$$\tau[f*g](n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\tau_t g(x)dt\rho_n(x)\, dx =$$

$$\int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty} (\tau_t g)(x)\rho_n(x)\, dxdt$$

$$= G(n)\int_{-\infty}^{\infty} f(t)\rho_n(t)\, dt = F(n)G(n)$$

As usual, working in the image domain, one can derive the properties of the convolution stated in the theorem.

Similarly the convolution in the image domain depends on a generalized translation operator T_k, $k \in Z$, fixed, defined (see (2.13')) by

$$(T_k F)(n) = \frac{1}{2\sqrt{\pi}}[F(n-k)-F(n-k-1)]. \tag{4.11}$$

Here we have

Theorem 4.2. Let $f \in E_\nu$ and $F = \tau[f]$, then the equalities

$$T_k\rho_n = \rho_n\rho_k^*, \tag{4.12}$$

and

$$\tau^{-1}[T_k F] = f\rho_k^* \tag{4.13}$$

are satisfied

Proof. The first of the results follows directly from (2.13'). From (4.11) and (2.13') we have

$$\tau[f\rho_k^*](n) = \int_{-\infty}^{\infty} f(t)\rho_k^*(t)\rho_n(t)dt =$$

$$\frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} f(t)[\rho_{n-k}(t) - \rho_{n-k-1}(t)]\, dt$$

$$= \frac{1}{2\sqrt{\pi}}[F(n-k)-F(n-k-1)] = (T_k F)(n).$$

Hence (taking τ^{-1}) the result (4.13) follows.

By means of the operator T_k we define a convolution in the image domain by

$$(F\otimes G)(n) = \sum_{k=-\infty}^{\infty} F(k)T_k G(n). \tag{4.14}$$

Theorem 4.2. Let $f, g \in E_1^1$ and $F = \tau[f]$, $G = \tau[g]$. Then $F\otimes G$ exists and the convolution is commutative, associative and distributive. Furthermore we have for $f \in E_\nu^2$ (or $g \in E_\nu^2$)

$$\tau[fg] = F\otimes G, \tag{4.15}$$

that is

$$\tau^{-1}[F\otimes G] = fg. \tag{4.15'}$$

Proof. From (4.11) and (3.4) we conclude, that $F(k)$ and $T_k G$ is $O(|k|)^{-1}$ as $|k| \to \infty$ and therefore the sum at the right hand side of (4.14) exists. With the help of (2.13') and (3.5') and because of $f \in E_\nu^2$ we have :

$$\tau[fg](n) = \int_{-\infty}^{\infty} f(t)\, g(t)\rho_n(t)\, dt =$$

$$\int_{-\infty}^{\infty} g(t)\sum_{k=-\infty}^{\infty} F(k)\rho_k^*(t)\rho_n(t)\, dt$$

$$= \sum_{k=-\infty}^{\infty} F(k)\int_{-\infty}^{\infty} g(t)\rho^*(t)\rho_n(t)\, dt =$$

$$\frac{1}{2\sqrt{\pi}} \sum_{k=-\infty}^{\infty} F(k) \int_{-\infty}^{\infty} g(t).(\rho_{n-k}(t)-\rho_{n-k-1}(t))dt =$$

$$\sum_{k=-\infty}^{\infty} F(k) \frac{1}{2\sqrt{\pi}} [G(n-k)-G(n-k-1)]$$

$$= \sum_{k=-\infty}^{\infty} F_k(T_k G)(n) = F*G$$

5. REAL VERSION

Analogous to the theory of the Fourier transformation where, besides of the complex version exist real transforms (Fourier-sine and Fourier-cosin-transforms), we would be able to give a real version of the Wiener-Laguerre transformation. We include no formulas for the transforms, but, following Christov [3], only some definitions and results on the kernels.

Instead of the complex valued functions ρ_n Christov considered the real valued functions S_n, C_n, which are connected with ρ_n by

$$S_n = (\rho_n + \rho_{-n-1})/i\sqrt{2} \quad , \quad n \in Z \qquad (5.1)$$

$$C_n = (\rho_n - \rho_{-n-1})/\sqrt{2} \quad , \quad n \in Z \text{ or } (5.1')$$

$$\rho_n = (C_n + iS_n)/\sqrt{2} \quad , \quad n \in Z \qquad (5.2)$$

Because of

$$S_{-n} = S_{n-1}, \; C_{-n} = -C_{n-1} \; , \quad n \in Z \qquad (5.3)$$

we see, that the set $\{S_n, C_n\}_{n=0}^{\infty}$ forms a CON in $L_2(R)$. The set $\{S_n, C_n\}$ forms a real-valued fundamental system for the linear second order differential equation (2.18). From (2.6) we find that the behaviour at $\pm\infty$ is given by

$$S_n(x) \sim -\sqrt{\frac{2}{\pi}} \; x^{-1} \; , \quad x \to \pm\infty \qquad (5.4)$$

$$C_n(x) \sim \sqrt{\frac{2}{\pi}} \; (2n+1)x^{-2} \; , \quad x \to \pm\infty \qquad (5.4')$$

Analogous to the sine- and cosine- functions, the S_n are odd and the C_n are even functions. The recurrences (2.11, 11', 12) and the linearization formulas (2.13, 13', 14, 14') can be transferred to the real valued functions S_n and C_n and therefore the results of the sections 3 and 4 can be translated to transforms of the type

$$\mathcal{F}_S [f] (n) = \int_0^{\infty} S_n(x)f(x)dx \qquad (5.5)$$

and

$$\mathcal{F}_C [f] (n) = \int_0^{\infty} C_n(x)f(x)dx \qquad (5.5')$$

We will not explain this in detail.

6. CONCLUDING REMARKS

The WLT with the complex valued functions ρ_n in the kernel investigated in this paper can of course be generalized to suitable spaces of generalized functions. For applications, specially for original functions defined in R^+ only, the WLT with the real-valued kernels S_n resp. C_n (when the originals are to be continued odd resp. even to R^-) is of interest. These problems and some applications are considered in two following papers.

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