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ON JUMP OF A FUNCTION OF HIGHER VARIATION

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ABSTRACT

There is a well known result due to Fejer that for a function of bounded variation with period 2π and for $x \in [0, 2\pi]$, we have

$$\lim_{n\to\infty} \frac{S_n'(x,f)}{n} = \frac{1}{\pi}\left[f(x+0) - f(x-0)\right] = \frac{D(x)}{\pi},$$

where $S_n'(x,f)$ denotes first derivative of n-th partial sum of Fourier series of f at the point x. In the present paper, we extend this result of Fejer into the strictly larger class of functions of higher variation, introduced first by Wiener. In precise, we prove that for a function of higher variation with period 2π and for $x \in [0,2\pi]$, we have again

$$\lim_{n\to\infty} \frac{S_n'(x, f)}{n} = \frac{D(x)}{\pi}$$

where D(x) denotes the jump of f at the point x.

RESUMEN

Existe un resultado bien conocido de Fejer, que muestra que, para una función de variación acotada con período 2π y $x \in [0,2\pi]$, se tiene :

$$\lim_{n\to\infty} \frac{S_n'(x,f)}{n} = \frac{1}{\pi}\left[f(x+0) - f(x-0)\right] = \frac{D(x)}{\pi},$$

donde $S_n'(x,f)$ denota la primera derivada de la n-sima suma parcial de la serie de Fourier de f en el punto x. En el presente trabajo, extendemos este resultado de Fejer a una clase estrictamente más amplia con funciones de variación superior, introducida por primera vez por Wiener. Precisamente, demostramos que para una función de variación superior con período 2π y para $x \in [0,2\pi]$, se tiene

$$\lim_{n\to\infty} \frac{S_n'(x,f)}{n} = \frac{D(x)}{\pi}$$

donde D(x) denota el salto de f en el punto x.

N. Wiener [6] proved that the Fourier series of $f \in V_2$ converges to $\frac{1}{2}[f(x+0) + f(x-0)]$ at every $x \in [0, 2\pi]$. Recently we [5] extended this result by proving that the Fourier series of $f \in V_p$ $(1 \leq p < \infty)$ converges to $\frac{1}{2}[f(x+0)] + f(x-0)]$ at every $x \in [0, 2\pi]$. In this connection there is a well known result due to Fejer [2] (cf. Zygmund [8] Vol. I p. 177) that for a function f of bounded variation with period 2π and for $x \in [0, 2\pi]$, we have

$$\lim_{n\to\infty} \frac{S_n'(x,f)}{n} = \frac{1}{\pi}\left[f(x+0) - f(x-0)\right]$$

where $S_n'(x, f)$ denotes first derivative of n-th partial sum of Fourier series of f at x. In the present paper, we extend this result of Fejer into the strictly larger classes of Wiener.

1.

Let f be a 2π-periodic function defined on $[0, 2\pi]$. We set for $1 \leq p < \infty$

$$V_p(f) = V_p(f;0,2\pi) = \sup \{ \sum_{i=1}^{n} |f(t_i)$$

$$- f(t_{i-1})|^p\}^{1/p}$$

where supremum has been taken with respect to all partitions $P: 0 = t_0 < t_1 < t_2 \ldots < t_n = 2\pi$ of $[0, 2\pi]$. We call $V_p(f; 0, 2\pi)$ the p-th total variation of f on $[0, 2\pi]$. Hence we define Wiener's class simply by

$$V_p = \{f : V_p(f) < \infty\}.$$

It is clear that V_1 is the ordinary class of functions of bounded variation, introduced by Jordan. The class V_p was first introduced by N. Wiener [6]. He [6] showed that the functions of the class V_p could only have simple discontinuities. We [4] note that

$$V_p \subset V_q \quad (1 \leq p < q < \infty)$$

is a strict inclusion. Hence for an arbitrary $1<p<\infty$, Wiener's class V_p is strictly larger than the class V_1.

2.

Let $f \in V_p$ $(1 \leq p < \infty)$ and let

$$\frac{1}{2} a_o + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

be its Fourier series. Recently we [5] proved the following theorem.

THEOREM A. if $f \in V_p$ $(1 \leq p < \infty)$, then the Fourier series of f converges to $1/2\,[f(x+0) + f(x-0)]$ at every $x \in [0, 2\pi]$.

There is also a well known theorem due to Fejer [2] (cf. Zygmund [8] vol I p. 177) which can be stated in the following form.

THEOREM B. If $f \in V_1$, then

$$\lim_{n\to\infty} \frac{S_n'(x, f)}{n} = \frac{1}{\pi} \left[f(x+0) - f(x-0)\right] = \frac{D(x)}{\pi}$$

for every $x \in [0, 2\pi]$ where $S_n'(x,f)$ denotes first derivative of n-th partial sum of Fourier series of f at x.

3.

The main aim of this paper is to extend Theorem B into strictly larger class V_p. Hence we prove that Theorem B is true for all $f \in V_p$ $(1 \leq p < \infty)$. More precisely, we prove the following theorem.

THEOREM 1. If $f \in V_p$ $(1 \leq p < \infty)$, then

$$\lim_{n\to\infty} \frac{S_n'(x,f)}{n} = \frac{D(x)}{\pi}$$

for every $x \in [0, 2\pi]$ where D(x) denotes the jump of f at x.

For the proof of Theorem 1 we need the following lemmas.

LEMMA 1. For any trigonometric polynomial $T_n(x)$ of degree n, one has the inequality

$$V_p(T_n'(x)) \leq n\, V_p(T_n) \quad (1 \leq p < \infty)$$

where $T_n'(x)$ denotes first derivative of $T_n(x)$.

LEMMA 2. Let $f \in V_p$, $g \in V_q$, $p>0$, $q>0$, $\frac{1}{p} + \frac{1}{q} > 1$ and assume that the functions f and g do not have common points of discontinuity. Then for any segment $[a,b]$ one has the inequality

$$\left|\int_a^b f(x) - f(\xi)\, dg\right| \leq M(p,q) Vp(f;a,b) Vq(g;a,b)$$

where ξ is an arbitrary point of $[a, b]$, the integral exists in the sense of Riemann-Stieltjes, and $M(p, q) > 0$ depends only on p and q.

Lemma 1 follows from the result of S.M. Nikol'skii [3] and Lemma 2 is proved by L.C. Young [7]. For the proof of Theorem 1, first we prove the following :

LEMMA 3. For any fixed δ $(0<\delta\leq\pi)$ and $p>1$ one has the estimate

$$V_p(D_n;\delta,\pi) = O(n^{1/p}) \qquad (n\to\infty)$$

where $D_n(x) = -1/2 + \sum_{k=1}^{n} \cos kx \quad (n = 0,1,2,\ldots)$

denotes Dirichlet kernel.

Proof of Lemma 3. We note that

$$D_n(x) = \frac{\sin nx}{x} + g(x)\sin nx + 1/2\cos nx$$

for $|x| < \pi$, where (1)

$$g(x) = \frac{1}{2\,\mathrm{tg}\, x/2} - \frac{1}{x} \quad (0<|x|<\pi,\ g(0)=0)$$

(cf. Bari [1] p. 108). Hence we have

$$V_p(D_n;\delta,\pi) \leq V_p\left(\frac{\sin nx}{x};\delta,\pi\right) + V_p(h_n;\delta,\pi)$$

$$= nV_p(t^{-1}\sin t;n\delta,n\pi) + V_p(h_n;\delta,\pi) \qquad (2)$$

where

$$h_n(x) = g(x)\sin nx + \frac{1}{2}\cos nx \quad (|x|\leq\pi).$$

If we assume that

$$V_p(f;\delta,\infty) = \lim_{b\to\infty} V_p(f;\delta,b),$$

then

$$V_p(t^{-1}\sin t;n\delta,n\pi) \leq V_p(t^{-2}\sin(t-\pi/2);n\delta,\infty). \qquad (3)$$

Taking account of the disposition of the graph of the function $x^{-2}\sin(x-\pi/2)$ between the curves $y=\pm x^{-2}$, it is easy to get the estimate

$$V_p(t^{-2}\sin(t-\pi/2);n\delta,\infty) = O\{\sum_{s=n}^{\infty} s^{-2p}\}^{1/p}$$

$$= O(n^{1/p-1}) \text{ as } n\to\infty. \qquad (4)$$

And also

$$V_p(h_n;\delta,\pi) \leq \max_{\delta\leq x\leq\pi}|\sin nx|\, V_p(g;\delta,\pi)$$

$$+ \max_{\delta\leq x\leq\pi}|g(x)|\, V_p(\sin nx;\delta,\pi) +$$

$$+ \frac{1}{2} V_p(\cos nx;\delta,\pi).$$

But

$$V_p(\sin nx;\delta,\pi) = n^{1/p}\, V_p(\sin x;\delta,\pi),$$

it follows

$$V_p(h_n;\delta,\pi) = 0\,(n^{1/p}) \quad \text{as} \quad n\to\infty. \tag{5}$$

Collecting the terms of (2), (3), (4) and (5) we get the assertion of lemma 3.

Proof of Theorem 1. By change of variables $x - x_o = t$, it is sufficient to prove theorem 1 for $x_o = 0$. We consider the function

$$\phi(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k} \ . \tag{6}$$

Since

$$\phi(x) = \frac{x-\pi}{2} \quad (0<x<2\pi) \ , \quad \phi(x+2\pi) = \phi(x) \ ,$$

and $\phi(+0) - \phi(-0) = \pi$ is the jump of ϕ at zero, hence $\phi \in V_p$.

We shall verify theorem 1 for the function $\phi(x)$ at $x_o = 0$.

We have

$$S_n'(x,\phi) = D_n(x) - \frac{1}{2} \qquad (n = 0,1,2...) \tag{7}$$

where $D_n(x)$ is Dirichlet kernel defined in (1). Hence

$$\lim_{n\to\infty} \frac{S_n'(0,\phi)}{n} = \lim_{n\to\infty} (1-\frac{1}{n}) = 1 = \frac{\phi(+0) - \phi(-0)}{\pi} \ . \tag{8}$$

Hence theorem 1 is proved for the function $\phi(x)$ at $x_o = 0$.

Now suppose given an arbitrary function $f \in V_p$ $(1<p<\infty)$ with jump at the point $x_o = 0$ equal to $d = f(+0) - f(-0)$. We construct

$$F(x) = f(x) - \frac{d}{\pi}\,\phi(x) \ . \tag{9}$$

We can easily verify that $F(x)$ is continuous at $x_o = 0$ and $F \in V_p$.
Since the result of theorem 1 is invariant with respect to subtraction of a constant from the function, hence we can assume $F(0) = 0$.

Now consider

$$S_n'(0,F) = -\frac{1}{\pi} \int_{-\pi}^{\pi} F(t)\, D_n'(t)dt \tag{10}$$

$$= -\int_{-\delta}^{\delta} F(t)\, D_n'(t)dt$$

$$-\frac{1}{\pi}\int_{\delta<|t|<\pi} F(t)D_n'(t)dt = I_n + J_n$$

where integral exists in the Riemann-Stieltjes sense. Now we fix q so that $1/p + 1/q > 1$, $q > 1$. Since $F \in V_p(1<p<\infty)$ and is continuous at zero, for any $\varepsilon>0$, we can find a $\delta>0$ such that

$$V_p(F, -\delta,\delta) < \varepsilon. \tag{11}$$

Since $F(0) = 0$, applying lemma 2 for $\xi = 0$, we obtain

$$|I_n| = \left|\frac{1}{\pi}\int_{-\delta}^{\delta} F(t)\ d\ D_n(t)\right| \le \frac{\varepsilon}{\pi}\, M(p,q) \ \times$$

$$V_q(D_n;\ -\delta,\ \delta) \ . \tag{12}$$

We note that for the polynomials

$$K_n(x) = \int_0^x [D_n(t) - 1/2] \, dt ,$$

L.C. Young [7] proved the inequality

$$V_q(K_n) \leq A_q < \infty \quad (1<q<\infty)$$

where $A_q > 0$ depends only on q. Using lemma 1 on K_n we obtain

$$V_q(D_n) = V_q(K_n') \leq nV_q(K_n) < nA_q \tag{13}$$

From (12) and (13) we obtain

$$|I_n| < \frac{\varepsilon}{\pi} A_q M(p,q)n . \tag{14}$$

Now we estimate J_n. Since

$$J_n = \left[-\frac{1}{\pi} F(t)D_n(t)\right]_{-\pi}^{-\delta} - \left[\frac{1}{\pi} F(t)D_n(t)\right]_{\delta}^{\pi}$$

$$- \left[\frac{1}{\pi} D_n(-\pi)F(t)\right]_{-\pi}^{-\delta} - \left[\frac{1}{\pi} D_n(-\pi)F(t)\right]_{\delta}^{\pi}$$

$$- \frac{1}{\pi} \int_{-\pi}^{-\delta} [D_n(t)-D_n(-\pi)] \, dF(t) - \frac{1}{\pi} \int_{\delta}^{\pi} (D_n(t) -$$

$$- D_n(\pi))dF(t). \tag{15}$$

Taking into account the fact that

$$D_n(\delta) = 0(n) \qquad (n\to\infty) \tag{16}$$

We obtain from (15)

$$J_n = -\frac{1}{\pi} \int_{-\pi}^{-\delta} [D_n(t)-D_n(-\pi)] dF(t) -$$

$$\frac{1}{\pi} \int_{\delta}^{\pi} [D_n(t)-D_n(\pi)] \, dF(t)+0(n) =$$

$$J_{n_1} + J_{n_2} + 0(n).$$

By using lemma 2 and lemma 3 we obtain

$$|J_{n_1}| \leq \frac{1}{\pi} M(p,q) \overset{\pi}{\underset{\delta}{V}}(F) V_q(D_n) = 0(n^{1/q})$$

and analogously we obtain

$$|J_{n_2}| = 0(n^{1/q}) \qquad (n\to\infty)$$

Hence collecting the terms (10), (14) and (15), we obtain

$$\lim_{n\to\infty} \frac{S_n'(0,F)}{n} \leq \frac{\varepsilon}{\pi} A_q M(p,q)$$

which proves

$$\lim_{n\to\infty} \frac{S_n'(0,F)}{n} = 0$$

Hence it completes the proof of theorem 1.

We note that theorem 1 is not true for the class of continuous functions. In fact, if theorem 1 is true for any continuous function, it would follow from (10) that

$$\| D_n'(x) \|_{[-\pi,\pi]} = O(n) \qquad (n\to\infty)$$

which is not true (cf. Zygmund [8] vol. 1 p 115).

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