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# SOME TRANSFORMATIONS AND INTEGRAL REPRESENTATIONS OF HORN'S FUNCTIONS

## ABSTRACT

Horn gave the general definition of hypergeometric functions of two variables. There are thirty four hypergeometric functions of two variables of this type. Fourteen functions are complete and the remaining twenty are the confluente cases. This includes the four Appell functions and seven Humbert functions. In this work we obtain a number of transformation formulae and integral representations for Horn functions. The transformation formulae for Gauss' hypergeometric function and Kummer function are used to obtain some new rèsults. The results established here give some new formulae for analytic continuation of Horn function in various domains and to study their behaviour in the neighbourhood of singular points.

## RESUMEN

Horn dió la definición general de las funciones hipergeométricas de dos variables. Hay treinta y cuatro funciones hipergeométricas de este tipo. Catorce funciones son completas y las restantes veinte son los casos de confluente. Esto incluye las cuatro funciones de Appell y las siete funciones de Humbért. En este trabajo obtenemos un número de fórmulas de transformación y la representación integral para funciones de Horn. Las fórmulas de transformación para la función hipergeométrica de Gauss y la función de Kummeŕ son usadas para obtener algunos nuevos resultados. Los resultados establecidos aquí, dan nuevas fórmulas para la continuación analítica de las funciones de Horn en varios dominios y para estudiar su comportamiento en la vecindad de puntos singulares.

## 1. INTRODUCCION

Horn gave the general definition of hypergeometric functions of two variables in the following form : A double power series

$$F(x,y) = \sum A_{m,n} x^m y^n \tag{1}$$

where the coefficients satisfy the conditions

$$\frac{A_{m+1,n}}{A_{m,n}} = \frac{P(m,n)}{R(m,n)} \quad \text{and} \quad \frac{A_{m,n+1}}{A_{m,n}} = \frac{Q(m,n)}{S(m,n)} \tag{2}$$

P, Q, R, and S denote polynomials in the indices of summation m and n of degree p, q, r, and s respectively, is of hypergeometric type. There are essentially 34 distinct convergent series, fourteen complete series (non confluent) and twenty distinct limiting forms of them (confluent series). This includes the four Appell functions and seven Humbert functions. Convergence of these series and a compelte list of Horn functions are given in [2].

Many integral representations and transformations of Horn functions have been obtained by several authors [1, 4, 5, 9, 10, 11], using the technique of change of variables in integrals or by changing the forms of the contours. In the present paper, we obtain a number of transformation formulae and integral representations for Horn functions (complete as well as confluent). The transformation formulae for Gauss's hypergeometric functions and Kummer's function are employed to obtain some new formulae for Horn functions. The results established here are important, as they give some new formulae for analytic continuation of Horn function in various domains and to obtain their behaviour in the neighbourhood of singular points.

Moreover, it is interesting to observe that Horn's functions are encountered in many physical applications. For example, the response I(a,b) of an omni-directional radiation detector at a height h directly over a corner of a plane isotropic rectangular (plaque) source of lenght l, width w and uniform strength $\sigma$, may be expressed as the definite integral

$$I(a,b) = \frac{\sigma}{4\pi} \int_0^b \left[\tan^{-1} \frac{a}{\sqrt{x^2+1}}\right] \frac{dx}{\sqrt{x^2+1}}$$

where $a = w/h$ and $b = 1/h$. This integral can be expressed as [SIAM Review, Vol. 26, N° 2,(1984), Problem 83-6]

$$I(a,b) = \frac{\sigma ab}{4\pi} F_2\,(1,\ 1/2,\ 1/2;\ 3/2,\ 3/2;\ -a^2,\ -b^2).$$

## 2. THE TRANSFORMATIONS OF COMPLETE HORN FUNCTIONS

In this section we establish the following transformations for complete Horn Functions.

$$F_1(\alpha,\beta,\alpha-\gamma+1;\gamma;x,y) = (1+y)^{-\alpha} H_3\left[\alpha,\beta,\gamma;\ \frac{y}{(y+1)^2},\ \frac{x}{y+1}\right] \quad (3)$$

$$F_2(\alpha,\beta,\beta',2\beta,2\beta';x,y) = \left(1-\frac{x+y}{2}\right)^{-\alpha}$$

$$F_4\left[\alpha,\alpha+\frac{1}{2};\beta+\frac{1}{2},\beta'+\frac{1}{2};\left(\frac{x}{2-x-y}\right)^2,\left(\frac{y}{2-x-y}\right)^2\right] \quad (4)$$

$$F_3(\alpha,\gamma-\alpha,\beta,\beta';\gamma;x,y) = (1-x)^{-\beta}(1-y)^{-\beta'}$$

$$F_3\left[\gamma-\alpha,\alpha,\beta,\beta';\gamma;\ \frac{x}{x-1},\ \frac{y}{y-1}\right] \quad (5)$$

$$F_3(\alpha,\alpha',\beta,\gamma-\beta;\gamma;x,y) = (1-y)^{\beta-\alpha}$$

$$F_3\left[\beta,\gamma-\beta,\gamma-\alpha-\alpha',\alpha';\gamma;\ \frac{x}{x-1},\ x+y-xy\right] \quad (6)$$

$$= (1-x)^{-\alpha}(1-y)^{\beta-2\alpha}$$

$$F_3\left[\gamma-\beta,\beta,\alpha,\gamma-\alpha-\alpha';\gamma;\ \frac{xy-x-y}{(x-1)(y-1)},\ y\right] \quad (7)$$

$$F_3(\alpha,\gamma-\alpha,\beta,\beta';\gamma;x,y) =$$

$$y^{\beta}\,(1-y)^{\alpha-\beta'}\,(x+y-xy)^{-\beta}$$

$$F_2\left[\gamma-\beta',\beta,\alpha,\gamma-\beta',\gamma;\ \frac{x(y-1)}{xy-x-y},\ y\right] \quad (8)$$

$$F_4(\alpha,\beta,\gamma,\beta;x,y) = (1-x-y)^{-\alpha}$$

$$H_3\left[\alpha,\gamma-\beta,\gamma;\ \frac{x}{x+y-1},\ \frac{xy(y-1)^2}{(x+y-1)^2}\right] \quad (9)$$

$$F_4(\alpha,1-\beta-\beta',1-\beta,1-\beta,;x,y) = (1-x-y)^{-\alpha}$$

$$G_1\left[\alpha,\beta,\beta';\ \frac{x}{1-x-y},\ \frac{y}{1-x-y}\right] \quad (10)$$

$$F_4(\alpha,\alpha+1/2;\gamma,\gamma';x^2,y^2) = (1+x+y)^{-2\alpha}$$

$$F_2\left[2\alpha,\gamma-\frac{1}{2},\gamma'-\frac{1}{2};2\gamma-1,2\gamma'-1;\frac{2x}{x+y+1},\ \frac{2y}{x+y+1}\right] \quad (11)$$

$$G_1(\alpha,\beta,\beta';x,y) = (1+x)^{-\alpha}$$

$$H_1\left[\beta,\alpha,1-\beta-\beta',1-\beta';-\frac{y}{x+1},\ -\frac{x}{x+1}\right] \quad (12)$$

$$G_2(\alpha,1-\beta',\beta,\beta';x,y) =$$

$$= (1+x)^{1-\alpha-\beta-\beta'}\ G_2(1-\beta-\beta',\alpha+\beta,\beta,1-\alpha-\beta;x,y) \quad (13)$$

$$H_1\left[\alpha,\beta,1-\alpha,\ \frac{\alpha+\beta+1}{2};\ x,y\right] =$$

$$= (1-x)^{(1-\alpha-\beta)/2}\ H_1\left[\frac{\alpha-\beta+1}{2},\ \frac{\beta-\alpha+1}{2},\beta,\ \frac{\alpha+\beta+1}{2};x,y\right] \quad (14)$$

$$H_1(\alpha,\beta,\delta-\alpha,\delta;x,y) = (1-x)^{-\beta}$$

$$F_4\left[\delta-\alpha,\beta,\delta,1-\alpha;\ \frac{x}{x-1},\ \frac{y}{x-1}\right] \qquad (15)$$

$$H_1(\alpha,\beta,1-\alpha,\delta;x,y) = (1-x)^{-\beta}$$

$$F_4\left[\delta-\alpha,\beta,\delta,\delta-\alpha;\ \frac{x}{x-1},\ \frac{y}{x-1}\right] \qquad (16)$$

$$H_1\left[\alpha,\beta,\gamma,\ \frac{\alpha+\beta+1}{2};\ x,y\right] = (1-2x)^{-\alpha}$$

$$H_7\left[\alpha,\beta,\gamma,\ \frac{\alpha+\beta+1}{2};\ \frac{x(x-1)}{(2x-1)^2},\ y\,(1-2x)\right] \qquad (17)$$

$$H_3(\alpha,2\gamma-\alpha-1,\gamma;x,y) = (1-4x)^{\gamma-\alpha-1/2}$$

$$H_3(2\gamma-\alpha-1,\alpha,\gamma;x,y) \qquad (18)$$

$$H_4(\alpha,\beta,\gamma,\delta;x,y) = (1-y)^{-\alpha}$$

$$H_4\left[\alpha,\delta-\beta,\gamma,\delta;\ \frac{x}{(y-1)^2},\ \frac{y}{y-1}\right] \qquad (19)$$

$$H_6(\alpha,\beta,1-\alpha;x,y) = (1+4x)^{1/2-\alpha-\beta}$$

$$H_6(1-\alpha-2\beta,\beta,\alpha+2\beta;x,y) \qquad (20)$$

$$H_7(\alpha,\beta,1-\alpha,\delta;x,y) = (1-4x)^{\delta-\alpha-1/2}$$

$$H_4(2\delta-\alpha-1,\beta,\delta,2\delta-\alpha-1;x,(4x-1)y) \qquad (21)$$

$$H_7(\alpha,\beta,2\delta-\alpha-1,\delta;x,y) = (1-4x)^{\delta-\alpha-1/2}$$

$$H_4(2\delta-\alpha-1,\beta,\delta,1-\alpha;x,(4x-1)y) \qquad (22)$$

$$H_7(\alpha,\beta,2\delta-\alpha-1,\delta;x,y) = (1-4x)^{-\alpha/2}$$

$$H_1\left[\alpha,2\delta-\alpha-1,\beta,\delta;\ \frac{1}{2}-\frac{1}{2\sqrt{1-4x}},\ y\sqrt{1-4x}\right] \qquad (23)$$

Similar other transformations can be found in the papers [1,9,10].

Proof of the formula (11) : We have

$$F_4(\alpha,\alpha+\tfrac{1}{2};\gamma,\gamma';x^2,y^2) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha+1/2)_n y^{2n}}{(\gamma')_n\ n!}$$

$$F\left[\alpha+n,\alpha+n+\tfrac{1}{2};\ \gamma;\ x^2\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha+1/2)_n y^{2n}}{(\gamma')_n n!}(1+x)^{-2\alpha-2n}$$

$$F\left[2\alpha+2n,\gamma-\tfrac{1}{2};2\gamma-1;\ \frac{2x}{x+1}\right] =$$

$$= (1+x)^{-2\alpha}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2\alpha)_{m+2n}(\gamma-1/2)_m}{(2\gamma-1)_m(\gamma')_n m!n!}$$

$$\left[\frac{2x}{x+1}\right]^m \left[\frac{y}{2x+2}\right]^{2n} =$$

$$= (1+x)^{-2\alpha}\sum_{m=0}^{\infty}\frac{(2\alpha)_m(\gamma-1/2)_m}{(2\gamma-1)_m m!}\left[\frac{2x}{x+1}\right]^m$$

$$F\left[\alpha+\frac{m}{2},\alpha+\frac{m+1}{2};\gamma';\left(\frac{y}{x+1}\right)^2\right]$$

$$= (1+x)^{-2\alpha} \sum_{m=0}^{\infty} \frac{(2\alpha)_m (\gamma-1/2)_m}{(2\gamma-1)_m m!} \left[\frac{2x}{x+1}\right]^m \left[\frac{x+y+1}{x+1}\right]^{-2\alpha-m}$$

$$. F\left[2\alpha+m, \gamma' - \frac{1}{2}; 2\gamma'-1; \frac{2y}{x+y+1}\right] = (x+y+1)^{-2\alpha}$$

$$F_2\left[2\alpha, \gamma - \frac{1}{2}, \gamma' - \frac{1}{2}; 2\gamma-1, 2\gamma'-1; \frac{2x}{x+y+1}, \frac{2y}{x+y+1}\right]$$

Here we have used twice the quadratic transformation of Gauss's hypergeometric function [2, p.112 (17)].

Proof of the formula (13) : We have

$$G_2(\alpha, \alpha', \beta, \beta'; x, y) =$$

$$\sum_{n=0}^{\infty} \frac{(\alpha')_n (\beta)_n (-y)^n}{(1-\beta')_n n!} F(\alpha, \beta'-n; 1-\beta-n; -x)$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha')_n (\beta)_n (-y)^n}{(1-\beta')_n n!} (1+x)^{1-\alpha-\beta-\beta'}$$

$$F(1-\alpha-\beta-n, 1-\beta-\beta'; 1-\beta-n; -x) = (1+x)^{1-\alpha-\beta-\beta'}$$

$$. \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\beta)_{n-m} (1-\alpha-\beta)_{m-n} (1-\beta-\beta')_m (\alpha')_n (\alpha+\beta)_n x^m y^n}{(1-\beta')_n m!\, n!} \tag{24}$$

on using [2, p.64(23)]

$$F(a,b,c,x) = (1-x)^{c-a-b} F(c-a, c-b; c; x) \tag{25}$$

If we set $\alpha' = 1 - \beta'$ in (24), we obtain (13).

Proof of the formula (17) : We have

$$H_1\left[\alpha, \beta, \gamma, \frac{\alpha+\beta+1}{2}; x, y\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_{-n} (\beta)_n (\gamma)_n y^n}{n!} F\left[\alpha-n, \beta+n; \frac{\alpha+\beta+1}{2}; x\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_{-n} (\beta)_n (\gamma)_n}{n!} y^n (1-2x)^{n-\alpha}$$

$$F\left[\frac{\alpha-n}{2}, \frac{\alpha-n+1}{2}; \frac{\alpha+\beta+1}{2}; \frac{4x(x-1)}{(2x-1)^2}\right] =$$

$$= (1-2x)^{-\alpha} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_{2m-n} (\beta)_n (\gamma)_n}{(\frac{\alpha+\beta+1}{2})_m m!\, n!} \left[\frac{x(x-1)}{(2x-1)^2}\right]^m \left[y(1-2x)\right]^n .$$

on using the transformation [2, p.112(20)] .

The other formulae can be established similarly, by using the linear and/or quadratic transformations of ${}_2F_1$.

## 3. THE TRANSFORMATIONS OF CONFLUENT HORN FUNCTIONS

In this section we establish some new transformation formulae for confluent Horn functions (incomplete Horn functions). Some times, transformation formulae for confluent functions, may be obtained as a limiting cases of the corresponding formulae for their complete functions. Our approach here is to obtain transformations formulae for confluent cases directly by using some known results of the theory of Gauss's hypergeometric function and Kummer's function. There are some formulae for confluent functions wich cannot be obtained by a limiting process. Similar other transformation formulae are given in [1,4,5,11].

$$\emptyset_1(\alpha, \alpha-\gamma+1; x, y) = (1+x)^{-\alpha} H_6\left[\alpha; \gamma; \frac{x}{(x+1)^2}, \frac{y}{x+1}\right] \tag{26}$$

$$\emptyset_1(\alpha, \beta; \gamma; x, y) = (1-x)^{-\beta} e^y \emptyset_1\left[\gamma-\alpha, \beta, \gamma; \frac{x}{x-1}, -y\right] \tag{27}$$

$$\emptyset_1(\alpha, \beta; \gamma; x, y) = (1-x)^{-\beta} \Xi_1\left[\beta, \alpha, \gamma-\alpha, \gamma; \frac{x}{x-1}, y\right] \tag{28}$$

$$\phi_1(\alpha,1-\alpha;\gamma;x,y) = (1-x)^{\gamma-1}(1-2x)^{\alpha-\gamma}\, e^y$$

$$H_6\left[\gamma-\alpha;\gamma;\ \frac{x(x-1)}{(2x-1)^2},\ \frac{y(1-x)}{2x-1}\right] \tag{29}$$

$$\phi_1\left[\alpha,\beta;\gamma;\ \frac{x}{x-1},\ y\right] = (1-x)^{-\beta}\Xi_1(\gamma-\alpha,\gamma,\beta,\gamma;x,y) \tag{30}$$

$$\phi_1(\alpha,\beta;\gamma;x,y) = (1-x)^{-\alpha}e^{y/x}\ \ x$$

$$\psi_1\left[\gamma-\beta,\alpha,\gamma,\gamma-\beta;\ \frac{x}{x-1},\ -\frac{y}{x}\right] \tag{31}$$

$$\phi_1(\alpha,\beta;\gamma;x,y) = \Gamma\begin{bmatrix}\gamma,\gamma-\alpha-\beta\\ \gamma-\alpha,\gamma-\beta\end{bmatrix} \times$$

$$\times\ \psi_1(\alpha,\beta,\alpha+\beta-\gamma+1,\gamma-\beta;1-x,y)+$$

$$+\ (1-x)^{\gamma-\alpha-\beta}\begin{bmatrix}\gamma,\alpha+\beta-\gamma\\ \alpha,\beta\end{bmatrix}\psi_1(\gamma-\beta,\gamma-\alpha,\gamma-\alpha-\beta+1;\gamma-\beta;1-x,y) \tag{32}$$

$$\phi_2(\beta,\beta';\gamma;\ x,y) = \left[1-\frac{x}{y}\right]^{-\beta} e^y$$

$$\psi_1\left[\gamma-\beta',\beta,\gamma-\beta',\gamma;\ \frac{x}{x-y},\ -y\right] \tag{33}$$

$$\phi_2(\beta,\beta';\gamma;x,y) = \left[1-\frac{x}{y}\right]^{\gamma-\beta-\beta'} e^y$$

$$\psi_1\left[\gamma-\beta',\gamma-\beta-\beta',\gamma-\beta',\gamma;\ \frac{x}{y},\ x-y\right] \tag{34}$$

$$\phi_3(\beta;\gamma;x,y) = e^{x+y/x}\ \psi_2\left[\gamma-\beta,\gamma,\gamma-\beta;\ -x,\ -\frac{y}{x}\right] \tag{35}$$

$$\phi_3(\beta;\gamma;\ x,y) = e^{y/x}\ H_4\left[\beta,1-\beta,\gamma;x,\ \frac{y}{x}\right] \tag{36}$$

$$\psi_1(\alpha,\beta,\gamma,\gamma';x,y) = (1-x)^{-\alpha}$$

$$\psi_1\left[\alpha,\gamma-\beta,\gamma,\gamma';\ \frac{x}{x-1},\ \frac{y}{1-x}\right] \tag{37}$$

$$\psi_1(\alpha,\beta,\gamma,\alpha;x,y) = (1-x)^{-\beta}$$

$$H_2\left[\gamma-\alpha,\beta,1+\alpha-\gamma,\gamma;\ \frac{x}{x-1},\ -y\right] \tag{38}$$

$$\psi_1(\alpha,\beta,\gamma,\alpha;x,y) = (1-x)^{\gamma-\alpha-\beta}$$

$$H_2\left[\gamma-\alpha,\gamma-\beta,1-\alpha-\gamma,\gamma;\ x,\ \frac{y}{x-1}\right] \tag{39}$$

$$\psi_1(\alpha,\beta,\gamma,1+\alpha-\gamma;x,y) = (1-x)^{\gamma-\alpha-\beta}$$

$$H_2\left[\gamma-\alpha,\gamma-\beta,\alpha,\gamma;x,\ \frac{y}{x-1}\right] \tag{40}$$

$$\psi_1(\alpha,\beta,\gamma,1+\alpha-\gamma;x,y) = (1-x)^{-\beta}$$

$$H_2\left[\gamma-\alpha,\beta,\alpha,\gamma;\ \frac{x}{x-1},\ -y\right] \tag{41}$$

$$\psi_1(\alpha,\beta,\gamma,1+\alpha-\gamma;x,y) = e^y\ H_{11}(1-\gamma,\alpha,\beta,1+\alpha-\gamma;-y,-x) \tag{42}$$

$$\psi_1(\alpha,\beta,\alpha,\gamma';x,y) = e^y\ H_{11}(\gamma'-\alpha,1+\alpha-\gamma',\beta,\gamma';-y,-x) \tag{43}$$

$$\psi_1(\alpha,\beta,\gamma,\alpha;x,y) = (1-x)^{\gamma-\alpha-\beta}\ e^{y}$$

$$\Xi_1\left[\gamma-\alpha,\beta,\gamma-\beta,\gamma;x,\frac{xy}{1-x}\right] \tag{44}$$

$$\psi_1(\alpha,\beta,\alpha,\gamma';x,y) = (1-x)^{-\beta}\ e^{y}$$

$$\emptyset_2\left[\beta,\gamma'-\alpha,\gamma';\ \frac{xy}{1-x}\ ,\ -y\right] \tag{45}$$

$$\psi_1(\alpha,\beta,2\beta,\gamma;x,y) = (1-\frac{x}{2})^{-\alpha}$$

$$H_7\left[\alpha,\beta+\frac{1}{2},\gamma;\ \left[\frac{x}{4-2x}\right]^2,\ \frac{2y}{y-x}\right] \tag{46}$$

$$\psi_1(\beta,\gamma,1-\alpha,\alpha+\beta;x,y) = (1-x)^{-\gamma}\ e^{y}$$

$$\Gamma_1\left[\gamma,\alpha,1-\alpha-\beta;x,\ \frac{y}{1-y}\right] \tag{47}$$

$$\psi_2(\alpha,\gamma,\alpha;x,y) = e^{x}\ H_4(\gamma-\alpha,1+\alpha-\gamma,\gamma;-x,-y) \tag{48}$$

$$\psi_2(\alpha,\gamma,1+\alpha-\gamma;x,y) = e^{x}\ H_4(\gamma-\alpha,\alpha,\gamma;-x,-y) \tag{49}$$

$$\Xi_1(\alpha,\gamma-\alpha,\beta,\gamma;x,y) = (1-x)^{-\beta}\ e^{y}$$

$$\Xi_1\left[\gamma-\alpha,\alpha,\beta,\gamma;\ \frac{x}{x-1}\ ,\ -y\right] \tag{50}$$

$$= (1-x)^{-\alpha} e^{y-y/x}\ \psi_1\left[\gamma-\beta,\alpha,\gamma,\gamma-\beta;\ \frac{x}{x-1}\ ,\ \frac{y}{x}\right] \tag{51}$$

$$\Xi_1(\alpha,\alpha',\beta,\gamma;x,y) = \Gamma\begin{bmatrix}\beta-\alpha,\gamma\\ \beta,\gamma-\alpha\end{bmatrix}(-x)^{-\alpha}$$

$$H_2\left[1+\alpha-\gamma,\alpha,\alpha',1+\alpha-\beta;\ \frac{1}{x}\ ,\ -y\right] + \Gamma\begin{bmatrix}\alpha-\beta,\gamma\\ \alpha,\gamma-\beta\end{bmatrix}(-x)^{-\beta}$$

$$H_2\left[1+\beta-\gamma,\beta,\alpha',1+\beta-\alpha;\ \frac{1}{x},\ -y\right] \tag{52}$$

$$\Gamma_1(\alpha,\beta,\beta';x,y) = (1+x)^{-\beta}\ e^{-y}$$

$$\cdot\ H_2\left[\beta',1-\alpha-\beta,1-\beta-\beta',1-\beta;\ \frac{x}{x+1}\ ,\ y(x+1)\right] \tag{53}$$

$$\Gamma_1\ (\alpha,\beta,\beta';x,y) = (1+x)^{-\alpha}\ e^{-y}$$

$$\psi_1\left[1-\beta-\beta',\alpha,1-\beta,1-\beta';\ \frac{x}{x+1}\ ,\ -y\right] \tag{54}$$

$$\Gamma_1\ (\alpha,\beta,\beta';x,y) = (1+x)^{1-\alpha-\beta-\beta'}\ e^{-y}$$

$$\cdot\psi_1(1-\beta-\beta',1-\alpha-\beta,1-\beta,1-\beta';-x,-y(x+1)) \tag{55}$$

$$\Gamma_1(\alpha,\beta,\beta';x,y) =$$

$$(1+x)^{-\beta'}\ e^{xy}\ \ \Gamma_1\left[1-\alpha-\beta,\beta,\beta';-\ \frac{x}{x+1}\ ,\ y\ (x+1)\right] \tag{56}$$

$$\Gamma_1(\alpha,\beta,\beta';x,y) = \Gamma\begin{bmatrix}\alpha+\beta,1-\beta'\\ \beta,1+\alpha-\beta'\end{bmatrix}\ x^{-\alpha}$$

$$\emptyset_1\left[\alpha+\beta,\alpha,1+\alpha-\beta';-\ \frac{1}{x}\ ,\ -y\right] \tag{57}$$

$$\Gamma_1(\alpha,\beta,\beta';x,y) = \Gamma\begin{bmatrix}\alpha+\beta,1-\beta'\\ \beta,1+\alpha-\beta'\end{bmatrix}\ x^{-\alpha}e^{-y}$$

$$\cdot\ \left[\emptyset_2\left(\alpha+\beta,1-\beta-\beta',1+\alpha-\beta';\ -\frac{1}{x}\ ,\ -\ y\right)\right] \tag{58}$$

$$\Gamma_1(\alpha,\beta,\beta';x,y) = \Gamma\begin{bmatrix}\alpha+\beta, 1-\alpha-\beta-\beta' \\ \beta, 1-\beta-\beta'\end{bmatrix} e^{y-\pi\alpha i}$$

$$\Gamma_1(\alpha,1-\alpha-\beta-\beta',\beta';-x-1,y) \qquad (59)$$

$$\Gamma_2(\beta,\beta';x,y) = e^{-y} \times$$

$$H_4(\beta',1-\beta-\beta',1-\beta;-x,-y) \qquad (60)$$

$$\Gamma_2(\beta,\beta';x,y) = e^{-x-y}\, \psi_2(1-\beta-\beta',1-\beta,1-\beta';x,y) \qquad (61)$$

$$H_1(\alpha,\beta,\beta';x,y) = (1-x)^{-\alpha}\, \psi_2(\beta,\beta',1-\alpha;xy,y(x-1) \qquad (62)$$

$$H_2(\alpha,\beta,\gamma,\delta;x,y) = (1-x)^{-\alpha}$$

$$H_2\left[\alpha,\delta-\beta,\gamma,\delta;\ \frac{x}{x-1},\ y\,(1-x)\right] \qquad (63)$$

$$H_2(\alpha,\beta,\delta-\alpha,\delta;x,y) = \Gamma\begin{bmatrix}\delta, \delta-\alpha-\beta \\ \delta-\alpha, \delta-\beta\end{bmatrix} \times$$

$$\Gamma_1(\beta,\delta-\alpha-\beta,\alpha;x-1,y) + \Gamma\begin{bmatrix}\delta, \alpha+\beta-\delta \\ \alpha, \beta\end{bmatrix}(1-x)^{\delta-\alpha-\beta} \times$$

$$\emptyset_1(\delta-\alpha,\delta-\beta;\delta-\alpha-\beta+1;1-x,y(x-1)) \qquad (64)$$

$$H_6(\alpha,\gamma;x,y) = \left(\frac{1-\sqrt{1-4x}}{2x}\right)^{\alpha}$$

$$\cdot\ \emptyset_1\left[\alpha,\alpha-\gamma+1;\gamma;\ \frac{1}{2x} - 1 - \sqrt{\frac{1}{4x^2} - \frac{1}{x}},\ y\,\frac{1-\sqrt{1-4x}}{2x}\right] \qquad (65)$$

$$H_6(\alpha,\gamma;x,y) = \Gamma\begin{bmatrix}\gamma, 2\gamma-2\alpha-1 \\ \gamma-\alpha, 2\gamma-\alpha-1\end{bmatrix} \left[\frac{1-\sqrt{1-4x}}{2x}\right]^{\alpha}$$

$$\cdot\psi_1\left[\alpha,\alpha-\gamma+1;2\alpha-2\gamma+2,2\gamma-\alpha-1;\ \frac{4x-1+\sqrt{4x-1}}{2x},\right.$$

$$\left. y\left[\frac{1-\sqrt{1-4x}}{2x}\right]\right] +$$

$$+\ \Gamma\begin{bmatrix}\gamma, 2\alpha-2\gamma+1 \\ \alpha, \alpha-\gamma+1\end{bmatrix} (2x)^{\alpha-2\gamma+1}(1-\sqrt{1-4x})^{\alpha}$$

$$(4x-1+\sqrt{4x-1})^{2\gamma-2\alpha-1}$$

$$\cdot\ \psi_1\left[2\gamma-\alpha-1,\gamma-\alpha,2\gamma-2\alpha,2\gamma-\alpha-1;\ \frac{4x-1+\sqrt{4x-1}}{2x},\right.$$

$$\left. y\left[\frac{1-\sqrt{1-4x}}{2x}\right]\right] \qquad (66)$$

$$H_7(\alpha,\gamma,\alpha;x,y) = (1-4x)^{\gamma-\alpha-1/2}$$

$$H_9\left[2\gamma-\alpha-1,\alpha-2\gamma+2,\gamma;x,\ \frac{y}{4x-1}\right] \qquad (67)$$

$$H_7(\alpha,\gamma,\alpha-2\gamma+2;x,y) = (1-4x)^{\gamma-\alpha-1/2}$$

$$H_9\left[2\gamma-\alpha-1,\alpha,\gamma;x,\ \frac{y}{4x-1}\right] \qquad (68)$$

$$H_8\left[\alpha;\ \frac{1-\alpha}{2};\ x,y\right] = e^{-y}\ H_8\left[\alpha,\ \frac{1-\alpha}{2};\ x,-y\right] \qquad (69)$$

$$H_8(\alpha;\beta;x,y) = \left[\frac{1+\sqrt{1+4x}}{2}\right]^{-\alpha}$$

$$\Gamma_1\left[\beta-\alpha,\beta,\alpha;\ \frac{4x}{(1+\sqrt{1+4x})^2},\ \frac{y(1+\sqrt{1+4x})}{2}\right] \qquad (70)$$

$$H_8(\alpha,\beta;x,y) = 2^{-\alpha-2\beta}\,\Gamma\begin{bmatrix}1-\beta, 1/2-\alpha-\beta \\ 1/2, 1-\alpha-\beta\end{bmatrix}$$

$$H_9\left[\alpha,\beta,\alpha+\beta+\frac{1}{2}; x+\frac{1}{4}, 2y\right] +$$

$$+ (1+4x)^{1/2-\alpha-\beta} 2^{\alpha-1}\,\Gamma\left[\begin{matrix}1-\beta, \alpha+\beta-1/2\\ \alpha, 1/2\end{matrix}\right]$$

$$. H_7(1-\alpha-2\beta, 3/2-\alpha-\beta, 1-\alpha-2\beta; x+1/4, -y) \tag{71}$$

$$H_9(\alpha,\beta,\delta;x^2,y) = (1+2x)^{-\alpha}$$

$$. H_2\left[\alpha, \delta-1/2, \beta, 2\delta-1; \frac{4x^2}{1+2x}, y(1+2x)\right] \tag{72}$$

$$H_{11}(\alpha,\beta,\gamma,\delta;x,y) = \Gamma\left[\begin{matrix}1-\alpha, \gamma-\beta\\ \gamma, 1-\alpha-\beta\end{matrix}\right] y^{-\beta}$$

$$\psi_1\left[\alpha+\beta, \beta, 1+\beta-\gamma, \delta; -\frac{1}{y}, x\right] +$$

$$+ \Gamma\left[\begin{matrix}1-\alpha, \beta-\gamma\\ \beta, 1-\alpha-\gamma\end{matrix}\right] y^{-\gamma}\,\psi_1\left[\alpha+\gamma, \gamma, 1+\gamma-\beta, \delta; -\frac{1}{y}, x\right] \tag{73}$$

PROOF : We have

$$H_8(\alpha,\beta;x,y) = \sum_{n=0}^{\infty} \frac{(\beta)_n (-y)^n}{(1-\alpha)_n n!}$$

$$F\left[\frac{\alpha-n}{2}, \frac{\alpha-n+1}{2}; 1-\beta-n; -4x\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(\beta)_n (-y)^n}{(1-\alpha)_n n!} \left\{\Gamma\left[\begin{matrix}1/2-\alpha-\beta, 1-\beta-n\\ 1-\beta+(n-\alpha)/2, (1-\alpha+n)/2-\beta\end{matrix}\right]\right.$$

$$. F\left[\frac{\alpha-n}{2}, \frac{\alpha-n+1}{2}; \alpha+\beta+1/2; 1+4x\right] +$$

$$+ \Gamma\left[\begin{matrix}\alpha+\beta-1/2, 1-\beta-n\\ (\alpha-n)/2, (\alpha-n+1)/2\end{matrix}\right] . (1+4x)^{1/2-\alpha-\beta}$$

$$\left. F\left[1-\beta+\frac{n-\alpha}{2}, \frac{1-\alpha+n}{2}-\beta; 3/2-\alpha-\beta; 1+4x\right]\right\} =$$

$$= 2^{-\alpha-2\beta}\,\Gamma\left[\begin{matrix}1-\beta, 1/2-\alpha-\beta\\ 1/2, 1-\alpha-\beta\end{matrix}\right] H_9(\alpha,\beta,\alpha+\beta+1/2; x+1/4, 2y) +$$

$$+ 2^{\alpha-1}\,\Gamma\left[\begin{matrix}1-\beta, \alpha+\beta-1/2\\ \alpha, 1/2\end{matrix}\right] (1+4x)^{1/2-\alpha-\beta}$$

$$. H_7(1-\alpha-2\beta, 3/2-\alpha-\beta, 1-\alpha-2\beta; x+1/4, -y)$$

on using the formula for analytic continuation of the Gauss hypergeometric function [2, p.108(1)]. This completes the proof of (71). The formula (32) may be established similarly. On the other hand the result (66) may be obtained by an appeal to the formulae (32) and (65).

Further, we have

$$\phi_3(\beta;\gamma;x,y) = \sum_{n=0}^{\infty} \frac{y^n}{(\gamma)_n n!}\,\phi(\beta;\gamma+n;x) =$$

$$e^x \sum_{n=0}^{\infty} \frac{y^n}{(\gamma)_n n!}$$

$$. \phi(\gamma-\beta+n;\gamma+n;-x) = e^x \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma-\beta)_{m+n}(-x)^m y^n}{(\gamma)_{m+n}(\gamma-\beta)_n m!\,n!} =$$

$$= e^x \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma-\beta)_{m+n}(-m-n)_n(-x)^{m+n}(y/x)^n}{(\gamma)_{m+n}(\gamma-\beta)_n (m+n)!\,n!}$$

$$= e^x \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\gamma-\beta)_k(-k)_n(-x)^k(y/x)^n}{(\gamma)_k(\gamma-\beta)_n k!\,n!}$$

$$= e^x \sum_{k=0}^{\infty} \frac{(\gamma-\beta)_k(-x)^k}{(\gamma)_k\,k!}$$

$$\cdot\ \emptyset\left[-k;\gamma-\beta;\ \frac{y}{x}\right] =$$

$$= e^{x+y/x}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\gamma-\beta)_k(-x)^k}{(\gamma)_k\ k!}\ \emptyset\left[\gamma-\beta+k;\gamma-\beta;-\frac{y}{x}\right]$$

$$= e^{x+y/x}\ \psi_2\left[\gamma-\beta,\gamma,\gamma-\beta;\ -x,\ -\frac{y}{x}\right]$$

on using the Kummer's transformation [2,p.253].This completes the proof of (35).

The result (36) can be derived from (35) and (48). The other results can be derived similarly. It is interesting to observe that the formulae (66), (70) and some others can't be derived from their corresponding complete cases.

## 4. THE INTEGRAL REPRESENTATION OF THE HORN FUNCTIONS

Several integral representations of the Horn functions, based on the Euler integrals of the first and second kind, the Mellin-Barnes integrals, the Laplace integral and several other contour integrals are known. Recently, some authors [16] have obtained integral representations in context with the homological groups. In this section we establish some new integral representations.

$$G_1\left[\alpha,\beta,\beta';\ \frac{x(1-y)}{1-x-y+2xy},\ \frac{y(1-x)}{1-x-y+2xy}\right]$$

$$= \Gamma\left[\begin{matrix}1-\beta,1-\beta'\\ \alpha,\beta,1-\alpha-\beta,1-\beta-\beta'\end{matrix}\right]$$

$$\cdot(1-x-y+2xy)^{\alpha}\int_0^1\int_0^1 u^{\alpha-1}v^{-\beta-\beta'}(1-u)^{\alpha-\beta}(1-v)^{\beta-1}$$

$$\cdot(1-ux)^{\alpha+\beta+\beta'-1}\ (1-ux-vy)^{-\alpha}\ dudv,$$

$$\mathrm{Re}\alpha,\ \mathrm{Re}\beta,\ \mathrm{Re}(1-\alpha-\beta),\ \mathrm{Re}(1-\beta-\beta')>0 \qquad (74)$$

$$G_2(\alpha,\alpha',\beta,\beta';x,y) =$$

$$= \Gamma\left[\begin{matrix}1-\beta,1-\beta'\\ \alpha,\alpha',1-\alpha-\beta,1-\alpha'-\beta'\end{matrix}\right](1+x)^{-\alpha}(1+y)^{-\alpha'}$$

$$\cdot\int_0^1\int_0^1 u^{\alpha-1}v^{\alpha'-1}(1-u)^{-\alpha-\beta}(1-v)^{-\alpha'-\beta'}\left[1-\frac{ux}{x+1}-\frac{vy}{y+1}\right]^{\beta+\beta'-1}dudv$$

$$\mathrm{Re}\,\alpha,\ \mathrm{Re}\ \alpha',\ \mathrm{Re}(1-\alpha-\beta),\ \mathrm{Re}(1-\alpha'-\beta')>0 \qquad (75)$$

$$G_3\left[\alpha,\alpha';\ \frac{x(1-y)(1-y+xy)}{(1-x+xy)^2},\ \frac{y(1-x)(1-x+xy)}{(1-y+xy)^2}\right] =$$

$$\Gamma\left[\begin{matrix}1-\alpha,1-\alpha'\\ \alpha+\alpha',\alpha+\alpha',1-\alpha-2\alpha',1-\alpha'-2\alpha\end{matrix}\right](1-x+xy)^{\alpha'}\ (1-y+xy)^{\alpha}$$

$$\cdot\int_0^1\int_0^1(uv)^{\alpha+\alpha'-1}(1-u)^{-2\alpha-\alpha'}(1-v)^{-2\alpha'-\alpha}(1-ux-vy)^{\alpha+\alpha'-1}dudv$$

$$\mathrm{Re}(\alpha+\alpha'),\ \mathrm{Re}(1-\alpha-2\alpha'),\ \mathrm{Re}(1-\alpha'-2\alpha)>0. \qquad (76)$$

$$H_2(\alpha,\beta,\gamma,\delta,\varepsilon;x,y) =$$

$$\Gamma\left[\begin{matrix}\varepsilon,1-\alpha\\ \beta,\varepsilon-\beta,\gamma,1-\alpha-\gamma\end{matrix}\right]\int_0^1\int_0^1 u^{\beta-1}v^{\gamma-1}(1-u)^{\varepsilon-\beta-1}$$

$$\cdot\ (1-v)^{-\alpha-\gamma}(1-ux)^{-\alpha}(1+(1-ux)vy))^{-\delta}\ dudv$$

$$\mathrm{Re}\ \beta,\ \mathrm{Re}\ \gamma,\ \mathrm{Re}(\varepsilon-\beta),\ \mathrm{Re}(1-\alpha-\gamma)>0 \qquad (77)$$

$$H_3(\alpha,\beta,\gamma;x,y) =$$

$$= \Gamma\left[\begin{matrix}\gamma\\ \alpha/2,\beta,\gamma-\alpha/2-\beta\end{matrix}\right]\int_0^1\int_0^1 u^{\beta-1}\ v^{\alpha/2-1}(1-u)^{\gamma-\beta-1}$$

$$\cdot\,(1-v)^{\gamma-\alpha/2-\beta-1}(1-uy)^{-\alpha}\left[1-\frac{4vx(1-u)}{(1-uy)^2}\right]^{-(\alpha+1)/2}dudv$$

$\mathrm{Re}\,\alpha,\ \mathrm{Re}\,\beta,\ \mathrm{Re}(\gamma-\alpha/2-\beta) > 0$ (78)

$$H_4(\alpha,\beta,\gamma,\delta;x,y)=\Gamma\begin{bmatrix}\delta,\gamma\\ -\alpha/2,\beta,\delta-\beta,\gamma-\alpha/2\end{bmatrix}\int_0^1\int_0^1 u^{\beta-1}\,v^{\alpha/2-1}$$

$$.\,(1-u)^{\delta-\beta-1}(1-v)^{\gamma-\alpha/2-1}(1-uy)^{-\alpha}\left[1-\frac{4vx}{(1-uy)^2}\right]^{-(\alpha+1)/2}dudv$$

$\mathrm{Re}\,\alpha,\ \mathrm{Re}\,\beta,\ \mathrm{Re}(\gamma-\alpha/2),\ \mathrm{Re}(\delta-\beta) > 0$ (79)

$$H_6\left[\alpha,\beta,\gamma;\ \frac{y(y+1)}{(2y+1)^2},\ \frac{x(2y+1)}{y+1}\right]=$$

$$=\Gamma\begin{bmatrix}1-\alpha,1-\beta\\ \alpha,\gamma,1-\alpha-\beta,1-\alpha-\gamma\end{bmatrix}(1+y)^{\beta}\,(1+2y)^{\alpha}$$

$$.\int_0^1\int_0^1 u^{\gamma-1}\,v^{-\alpha-\beta}(1-u)^{-\alpha-\gamma}(1-v)^{\alpha-1}(1-ux)^{-\beta}(1+(1-ux)vy)^{-\alpha-\beta}dudv$$

$\mathrm{Re}\,\alpha,\ \mathrm{Re}\,\gamma,\ \mathrm{Re}(1-\alpha-\beta),\ \mathrm{Re}(1-\alpha-\gamma) > 0$ (80)

$$H_7\left[\alpha,\beta,\gamma,\delta;\ \frac{x^2}{(4-2x)^2},\ y\,(1-\frac{x}{2})\right]=$$

$$\Gamma\begin{bmatrix}1-\alpha,2\delta-1\\ \beta,\delta-1/2,\delta-1/2,\ 1-\alpha-\beta\end{bmatrix}$$

$$\cdot\left[1-\frac{x}{2}\right]^{\alpha}\int_0^1\int_0^1 u^{\delta-3/2}\,v^{\beta-1}(1-u)^{\delta-3/2}(1-v)^{-\alpha-\beta}(1-ux)^{-\alpha}$$

$.(1+(1-ux)vy)^{-\gamma}dudv$, $\mathrm{Re}\,\beta,\ \mathrm{Re}(\delta-1/2),\ \mathrm{Re}(1-\alpha-\beta) > 0$

(81)

$$\phi_3(\beta,\gamma;x,y)=$$

$$=\Gamma\begin{bmatrix}\gamma\\ 1/2,\beta,\gamma-\beta-1/2\end{bmatrix}\int_0^1\int_0^1 u^{\beta-1}\,v^{-1/2}\,(1-u)^{\gamma-\beta-1}$$

$$.\,(1-v)^{\gamma-\beta-3/2}\,e^{ux}\,\mathrm{ch}(2\sqrt{(1-u)vy})\,dudv$$

$\mathrm{Re}\,\beta,\ \mathrm{Re}(\gamma-\beta-1/2) > 0$ (82)

$$\psi_1(\alpha,\beta,\gamma,\gamma';x,y)=$$

$$=\Gamma\begin{bmatrix}\gamma,\gamma'\\ \alpha,\beta,\gamma'-\alpha,\gamma-\beta\end{bmatrix}\int_0^1\int_0^1 u^{\beta-1}\,v^{\alpha-1}$$

$$.\,(1-u)^{\gamma-\beta-1}(1-v)^{\gamma'-\alpha-1}(1-ux)^{-\alpha}\,e^{vy/(1-ux)}dudv$$

$\mathrm{Re}\,\alpha,\ \mathrm{Re}\,\beta,\ \mathrm{Re}(\gamma'-\alpha),\ \mathrm{Re}(\gamma-\beta) > 0$ (83)

$$\psi_2(\alpha,\gamma,\gamma',;x,y)=\Gamma\begin{bmatrix}\gamma,\gamma'\\ 1/2,\alpha-1/2,\alpha,\gamma-\alpha,\gamma'-\alpha\end{bmatrix}$$

$$.\int_0^1\int_0^1\int_0^1 u^{\alpha-1}v^{\alpha-1}\tau^{-1/2}(1-u)^{\gamma-\alpha-1}(1-v)^{\gamma'-\alpha-1}(1-\tau)^{\alpha-3/2}$$

$$.\ e^{ux+vy}\,\mathrm{ch}(2\sqrt{uv\tau xy})\,dudvd\tau$$

$\mathrm{Re}(\alpha-1/2),\ \mathrm{Re}(\gamma-\alpha),\ \mathrm{Re}(\gamma'-\alpha) > 0$ (84)

$$\Xi_1(\alpha,\alpha',\beta,\gamma;x,y)=$$

$$=\Gamma\begin{bmatrix}\gamma\\ \alpha,\alpha',\gamma-\alpha-\alpha'\end{bmatrix}\int_0^1\int_0^1 u^{\alpha-1}\,v^{\alpha'-1}\,(1-u)^{\gamma-\alpha-1}$$

$$.\,(1-v)^{\gamma-\alpha-\alpha'-1}(1-ux)^{-\beta}\,e^{(1-u)vy}\,dudv$$

$$\mathrm{Re}\,\alpha,\ \mathrm{Re}\,\alpha',\ \mathrm{Re}(\gamma-\alpha-\alpha') > 0 \qquad (85)$$

$$\Xi_2(\alpha,\beta,\gamma;x,y) =$$

$$= \Gamma\left[\begin{matrix}\gamma\\ 1/2,\alpha,\gamma-\alpha-1/2\end{matrix}\right]\int_0^1\int_0^1 u^{\alpha-1}v^{-1/2}(1-u)^{\gamma-\alpha-1}$$

$$\cdot\ (1-v)^{\gamma-\alpha-3/2}(1-ux)^{-\beta}\mathrm{ch}(2\sqrt{(1-u)vy})\,dudv$$

$$\mathrm{Re}\,\alpha,\ \mathrm{Re}(\gamma-\alpha-1/2) > 0 \qquad (86)$$

$$\Gamma_1(\alpha,\beta,\beta';x,y) =$$

$$= \Gamma\left[\begin{matrix}1-\beta,1-\beta'\\ \alpha,\beta,1-\alpha-\beta,1-\beta-\beta'\end{matrix}\right](1+x)^{1-\alpha-\beta-\beta'}e^{-y}$$

$$\cdot\int_0^1\int_0^1 u^{-\alpha-\beta}v^{-\beta-\beta'}(1-u)^{\alpha-1}(1-v)^{\beta-1}(1+ux)^{\beta+\beta'-1}$$

$$\cdot\ e^{-vy(x+1)/(1+ux)}\,dudv$$

$$\mathrm{Re}\,\alpha,\ \mathrm{Re}\,\beta,\ \mathrm{Re}(1-\alpha-\beta),\ \mathrm{Re}(1-\beta-\beta') > 0 \qquad (87)$$

$$\Gamma_2\ (\beta,\beta';x,y) =$$

$$= \Gamma\left[\begin{matrix}1-\beta,1-\beta'\\ 1/2,\beta,\beta',1/2-\beta-\beta',1-\beta-\beta'\end{matrix}\right]$$

$$\cdot\int_0^1\int_0^1\int_0^1 (\tau(1-\tau))^{-1/2}(uv(1-\tau))^{-\beta-\beta'}(1-u)^{\beta'-1}(1-v)^{\beta-1}$$

$$\cdot\ e^{(u-1)x+(v-1)y}\ \mathrm{ch}(2\sqrt{uv\tau xy})\,dudvd\tau$$

$$\mathrm{Re}\,\beta,\ \mathrm{Re}\,\beta',\ \mathrm{Re}(1-\beta-\beta') > 0 \qquad (88)$$

$$H_1(\alpha,\beta,\beta;x,y) =$$

$$= \Gamma\left[\begin{matrix}1-\alpha\\ 1/2,\beta-1/2,1-\alpha-\beta\end{matrix}\right](1-x)^{-\alpha}e^{xy}\int_0^1\int_0^1 u^{\beta-1}v^{-1/2}$$

$$\cdot\ (1-u)^{-\alpha-\beta}(1-v)^{\beta-3/2}e^{uy(x-1)}\mathrm{ch}(2\sqrt{x(x-1)uv}\,y)\,dudv$$

$$\mathrm{Re}(\beta-1/2),\ \mathrm{Re}(1-\alpha-\beta) > 0 \qquad (89)$$

$$H_2(\alpha,\beta,\gamma,\delta;x,y) =$$

$$= \Gamma\left[\begin{matrix}\delta,1-\alpha\\ \beta,\gamma,\delta-\beta,1-\alpha-\gamma\end{matrix}\right]\int_0^1\int_0^1 u^{\beta-1}v^{\gamma-1}$$

$$\cdot\ (1-u)^{\delta-\beta-1}(1-v)^{-\alpha-\gamma}(1-ux)^{-\alpha}e^{(ux-1)vy}\,dudv$$

$$\mathrm{Re}\,\beta,\ \mathrm{Re}\,\gamma,\ \mathrm{Re}(\delta-\beta),\ \mathrm{Re}(1-\alpha-\gamma) > 0 \qquad (90)$$

$$H_3(\alpha,\beta,\delta;x,y) =$$

$$= \Gamma\left[\begin{matrix}1-\alpha,\delta\\ 1/2,1/2-\alpha,\beta,\delta-\beta\end{matrix}\right]\int_0^1\int_0^1 u^{\beta-1}v^{-1/2}(1-u)^{\delta-\beta-1}$$

$$\cdot\ (1-v)^{-\alpha-1/2}(1-ux)^{-\alpha}\cos(2\sqrt{(1-ux)vy})\,dudv$$

$$\mathrm{Re}(1/2-\alpha),\ \mathrm{Re}\,\beta,\ \mathrm{Re}(\delta-\beta) > 0 \qquad (91)$$

$$H_4(\alpha,1-\alpha,\delta;x,y) =$$

$$= \Gamma\left[\begin{matrix}\delta\\ 1/2,\alpha,\delta-\alpha-1/2\end{matrix}\right]e^{x-y}\int_0^1\int_0^1 u^{\delta-\alpha-1}v^{-1/2}$$

$$\cdot\ (1-u)^{\alpha-1}(1-v)^{\delta-\alpha-3/2}e^{-ux}\ \mathrm{ch}(2\sqrt{uvxy})\,dudv$$

$$\mathrm{Re}\ \alpha,\ \mathrm{Re}(\delta-\alpha-1/2) > 0 \qquad (92)$$

$$H_6\left[\alpha,\gamma;\ \frac{x}{(x+1)^2},\ \frac{y}{x+1}\right]$$

$$= \Gamma\left[\begin{matrix}\gamma\\ \alpha,\gamma-\alpha\end{matrix}\right]\int_0^1 u^{\alpha-1}(1-u)^{\gamma-\alpha-1}$$

$$\cdot\ (1-ux)^{\gamma-\alpha-1}\, e^{uy}du \quad \mathrm{Re}\ \alpha,\ \mathrm{Re}(\gamma-\alpha) > 0 \qquad (93)$$

$$H_6\left[\alpha,\gamma;\ \frac{x}{(x+1)^2},\ \frac{y}{x+1}\right] =$$

$$= \Gamma\left[\begin{matrix}\gamma\\ \alpha,\alpha-\gamma+1,2\gamma-\alpha-\beta-1\end{matrix}\right]\int_0^1\int_0^1 u^{\alpha-\gamma}\, v^{\beta-1}$$

$$\cdot\ (1-u)^{2\gamma-\alpha-2}(1-v)^{2\gamma-\alpha-\beta-2}(1-ux-(1-u)vy)^{-\alpha}dudv$$

$$\mathrm{Re}\ \beta,\ \mathrm{Re}(1+\alpha-\gamma),\ \mathrm{Re}(2\gamma-\alpha-\beta-1) > 0 \qquad (94)$$

$$H_7\left[\alpha,\gamma,\delta;\ \frac{x^2}{(4-2x)^2},\ \frac{2y}{2-x}\right] =$$

$$= \Gamma\left[\begin{matrix}\delta,2\gamma-1\\ \alpha,\delta-\alpha,\gamma-1/2,\gamma-1/2\end{matrix}\right](1-\frac{x}{2})^{\alpha}$$

$$\int_0^1\int_0^1 u^{\gamma-3/2}v^{\alpha-1}(1-u)^{\gamma-3/2}(1-v)^{\delta-\alpha-1}(1-ux)^{-\alpha}e^{vy/(1-ux)}dudv$$

$$\mathrm{Re}\ \alpha,\ \mathrm{Re}(\delta-\alpha),\ \mathrm{Re}(\gamma-1/2) > 0 \qquad (95)$$

$$H_8(\alpha,\beta;x,y) =$$

$$= \Gamma\left[\begin{matrix}1-\alpha,1-\beta\\ \beta,\alpha+\beta,1-\alpha-\beta,1-\alpha-2\beta\end{matrix}\right](\frac{1+\sqrt{1+4x}}{2})^{-\alpha}$$

$$\cdot(1+\frac{4x}{(1+\sqrt{1+4x})^2})^{1-2\alpha-2\beta}e^{-(1+\sqrt{1+4x})y/2}\int_0^1\int_0^1 u^{-\alpha-2\beta}v^{-\alpha-\beta}$$

$$\cdot\ (1-u)^{\alpha+\beta-1}(1-v)^{\beta-1}(1+\frac{4ux}{(1+\sqrt{1+4x})^2})^{\alpha+\beta-1}$$

$$\cdot\ \exp\left[-\frac{vy(1+\sqrt{1+4x})(1+4x/(1+\sqrt{1+4x})^2}{2+8ux/(1+\sqrt{1+4x})^2}\right]dudv$$

$$\mathrm{Re}\ \beta,\ \mathrm{Re}(\alpha+\beta),\ \mathrm{Re}(1-\alpha-2\beta) > 0 \qquad (96)$$

$$H_9(\alpha,\beta,\delta;x^2,y) =$$

$$= \frac{4^{\delta-1}}{\sqrt{\pi}}\Gamma\left[\begin{matrix}1-\alpha,\delta\\ \beta,\delta-1/2,1-\alpha-\beta\end{matrix}\right]\int_0^1\int_0^1 u^{\delta-3/2}v^{\beta-1}$$

$$\cdot\ (1-u)^{\delta-3/2}(1-v)^{-\alpha-\beta}(1+2x-4ux)^{-\alpha}e^{(4ux-2x-1)vy}dudv$$

$$\mathrm{Re}\ \beta,\ \mathrm{Re}(\delta-1/2),\ \mathrm{Re}(1-\alpha-\beta) > 0 \qquad (97)$$

$$H_{10}(\alpha,\delta;x^2,y^2) =$$

$$= \frac{4^{\delta-1}}{\pi}\Gamma\left[\begin{matrix}1-\alpha,\delta\\ 1/2-\alpha,\delta-1/2\end{matrix}\right]\int_0^1\int_0^1 u^{\delta-3/2}v^{-1/2}$$

$$\cdot\ (1-u)^{\delta-3/2}(1-v)^{-\alpha-1/2}(1+2x-4ux)^{-\alpha}\cos(2y\sqrt{v(1+2x-4ux)})dudv$$

$$\mathrm{Re}(\delta-1/2),\ \mathrm{Re}(1/2-\alpha) > 0 \qquad (98)$$

$$H_{11}(\alpha,\beta,\gamma,\delta;x,y) =$$

$$= \Gamma\left[\begin{matrix}1-\alpha,\delta\\ \alpha,\beta,\gamma,1-\alpha-\gamma,\delta-\alpha-\beta\end{matrix}\right]e^x\int_0^1\int_0^1\int_0^1 u^{\beta-1}v^{\gamma-1}$$

$$\cdot\ \tau^{\delta-\alpha-1}(1-u)^{\delta-\alpha-\beta-1}(1-v)^{-\alpha-\gamma}(1-\tau)^{\alpha-1}(1+uvy)^{\alpha-\delta}$$

$$\cdot\ e^{-\tau x/(1+uvy)}\,dudvd\tau$$

$Re\ \alpha$, $Re\ \beta$, $Re\ \gamma$, $Re(1-\alpha-\gamma)$, $Re(\delta-\alpha-\beta) > 0$ (99)

PROOF : The result (74) can be obtained by using the tie formula of the functions $G_1$ and $F_2$ [10] and the integral representation of the function $F_2$ [1]. The results (75), (80), (81), (87), (93), (94), (95), (96), (97), and (99) can be derived similarly.

We have

$$\phi_3\ (\beta,\gamma;x,y) =$$

$$\sum_{m,n=0}^{\infty} \frac{(\beta)_m x^m y^n}{(\gamma)_{m+n} m!n!} = \sum_{n=0}^{\infty} \frac{y^n}{(\gamma)_n\ n!}\ {}_1F_1(\beta;\gamma+n;x) =$$

$$= \sum_{n=0}^{\infty} \frac{y^n}{(\gamma)_n n!}\Gamma\begin{bmatrix}\gamma+n\\ \beta,\gamma-\beta+n\end{bmatrix}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta+n-1}e^{ux}du$$

$$= \Gamma\begin{bmatrix}\gamma\\ \beta,\gamma-\beta\end{bmatrix}\int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}\ e^{ux}{}_0F_1(\gamma-\beta;(1-u)y)du$$

From this and the integral representation of the Bessel function we can establish (82). The results (84), (86), (88), and (91) follow in the same way. By employing together the above methods, we can easily establish (89), (92) and (98). The results (77), (78), (79), (83); (85) and (90) can be derived by each term integration of the corresponding series. The other representations can be derived similarly.

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