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A CLASS OF INDEX INTEGRAL TRANSFORMS

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ABSTRACT

The present paper deals with a class of integral transforms by index or parameters of the special function kernels. By invoking the Parseval relation for Mellin transform, a new integral transform is defined and some of its properties and special cases are obtained. The inversion theorem is established in a special defined space of functions $m^{-1}_{c,\gamma}(L)$. A Plancherel type theorem for a partially symmetric case of W-transform with Meijer's G-function as kernel is given.

RESUMEN

Este trabajo se trata de una clase de transformadas integrales por el parámetro ó índice de la función especial del núcleo. Se define una nueva transformada integral mediante el uso de la relación de Parseval para la transformada de Mellin. Se dan algunas propiedades y casos especiales de la nueva transformada. Se establece el teorema de inversión en un espacio de funciones $m^{-1}_{c,\gamma}(L)$. Además, se demuestra un teorema de tipo Plancherel para un caso parcialmente simétrico de la transformada w, con la función G- de Meijer como núcleo.

1. INTRODUCTION

Let

$$g(x) = \int_0^\infty G^{m,n+2}_{p+2,q}\left(t\,\middle|\,{1-\nu+ix,1-\nu-ix,(\alpha_p) \atop (\beta_q)}\right)f(t)dt, \qquad 0<x<\infty, \qquad (1)$$

is the integral transform with Meijer's G- function as the kernel, where $(\alpha_p) = \alpha_1,\alpha_2,\ldots,\alpha_p$; $(\beta_q) = \beta_1, \beta_2, \ldots, \beta_q$ and ν are some complex parameters. This integral transform was introduced and it's formal inversion formula was obtained in 1964 by Wimp [12]. Some special cases of (1), such as Kontorovich - Lebedev, Mehler-Fock [4 - 6] and other integral transforms were given. The inversion formulas for such transforms were obtained even earlier, see [7].

We continue the investigation of such integral transforms. We introduce the W-transform which is more general than (1). Parseval equality for the Mellin integral transform is used to construct the W-transform. It is proved corresponding inversion formulas for (1) in special defined space $m^{-1}_{c,\gamma}(L)$. Also the transform (1) is studied in L_2-space, where the theorem of Plancherel type is proved for some special symmetric case of (1). In conclusion it is given few examples of some special cases of (1).

2. W-transform in the space $m^{-1}_{c,\gamma}(L)$.

A more compact view of the inversion formula of (1) was indicated in [13]. It is different from such formulas in [12]. The proof of this inversion formula was given later [10]. Using the Parseval equality for Mellin transform, the W-transform, is defined as [10] :

Definition 1. Let $f^*(s)$ is Mellin transform of $f(x)$ [1,6], which is defined by the formulae $f^*(s)= \int_0^\infty f(x)x^{s-1}\,dx$.

Let ν be some complex parameter with $\mathrm{Re}\,\nu >1/2$ and $(\alpha_n) = \alpha_1,\alpha_2,\ldots,\alpha_n$; $(\beta_m) = \beta_1, \beta_2, \ldots, \beta_m$;

$(\alpha_p^{n+1}) = \alpha_{n+1}, \alpha_{n+2},\ldots, \alpha_p$; $(\beta_q^{m+1}) = \beta_{m+1}, \beta_{m+2}, \ldots, \beta_q$ are the vectors with complex components, moreover,

$$\mathrm{Re}\,\alpha_j \neq 1/2+k,\ j=1,2,\ldots,n;\quad \mathrm{Re}\,\alpha_j\neq -1/2-k,\quad j=n+1,n+2,\ldots,p;$$

$$\mathrm{Re}\,\beta_j \neq -1/2-k, j=1,2,\dots,m; \quad \mathrm{Re}\,\beta_j \neq 1/2+k,$$
$$j=m+1,m+2,\dots,q;\ k=0,1,2,\dots \qquad (2)$$

then the integral

$$(Wf)(x) \equiv \left(W^{mn}_{pq} \left| \begin{matrix} \nu,(\alpha_p) \\ (\beta_q) \end{matrix} \right| f(t) \right)(x) = \qquad (3)$$

$$= \frac{1}{2\pi i} \int_\sigma \Gamma\left[\nu-ix-s, \nu+ix-s\right]$$

$$\Gamma\left[\begin{matrix} (\beta_m)+s, 1-(\alpha_n)-s \\ (\alpha_p^{n+1})+s, 1-(\beta_q^{m+1})-s \end{matrix}\right] f^*(1-s)\, ds,$$

where $\sigma = \{s, \mathrm{Re}\, s = 1/2\}$, defines the W-transform of the function $f(x)$.

The following class of the functions is appropiate for W-transform.

Definition 2. Let $c,\gamma \in R$ and $2\ \mathrm{sgn}\ c + \mathrm{sgn}\gamma \geq 0$. The space of functions $f(x)$ which can be represented in the form of

$$f(x) = \frac{1}{2\pi i} \int_\sigma \phi(s)\, x^{-s}\, ds, \qquad x \in (0,\infty) , \qquad (4)$$

where

$$\phi(s) = s^{-\gamma} e^{-c\pi|\mathrm{Im}s|} F(s) , \quad F(s) \in L(\sigma),$$

is denoted as $m^{-1}_{c,\gamma}(L)$, and we write $f(x) \in m^{-1}_{c,\gamma}(L)$.

The space $m^{-1}_{c,\gamma}(L)$ with the norm

$$\|f\|_{m^{-1}_{c,\gamma}(L)} = \int_\sigma |F(s)\, ds|$$

converts into a Banach space. Now , we prove the follwing result.

Theorem I. If and only if,

$$2\mathrm{sgn}(c+c^*+1) + \mathrm{sgn}(\gamma+\gamma^*+2\mathrm{Re}\nu-1) \geq 0, \qquad (5)$$

where

$$c^* = \frac{m+n}{2} -p-q, \quad \gamma^* = \mathrm{Re}\left(\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j \right). \qquad (6)$$

then the W-transform exist in the space $m^{-1}_{c,\gamma}(L)$.

Moreover, if

$$\mathrm{Re}\alpha_j < 1/2,\ j=1,2,\dots,n;\ \mathrm{Re}\alpha_j > -1/2,\ j=n+1,\dots,p;$$
$$\mathrm{Re}\beta_j > -1/2,\ j=1,2,\dots,m;\ \mathrm{Re}\beta_j < 1/2,\ j=m+1,\dots,q, \qquad (7)$$

and

$$4\ \mathrm{sgn}(c^*+1)+2\mathrm{sgn}(\gamma^*-2\mathrm{Re}\nu+2)+\mathrm{sgn}|2+p-q| > 0, \qquad (8)$$

then the W-transform (3) may be represented in the form (1).

Proof. By the asymptotic expansion of the gamma-function [2] we have ,

$$\Gamma\left[\nu-ix-s, \nu+ix-s\right] \Gamma\left[\begin{matrix} (\beta_m)+s,\ 1-(\alpha_n)-s \\ (\alpha_p^{n+1})+s, 1-(\beta_q^{m+1})-s \end{matrix}\right]$$

$$\sim C \exp\left[-(c^*+1)\pi|\mathrm{Im}s|\right] |s|^{-\gamma^*-2\mathrm{Re}\nu+1} \qquad (9)$$
$$s \in \sigma,\ |\mathrm{Im}s| \to \infty .$$

Consequently, the integral (3) is convergent at infinity if and only if the condition (5) is fulfilled. If the conditions (7) and (8) are fulfilled then for all $s \in \sigma$ and $0<\varepsilon<E<\infty$ we have the inequality

$$\left| \int_\varepsilon^E G^{m,n+2}_{p+2,q} \left(t \left| \begin{matrix} 1-\nu+ix, 1-\nu-ix, (\alpha_p) \\ (\beta_q) \end{matrix} \right. \right) t^{s-1}\, dt \right| < c_1 \qquad (10)$$

This follows from the asymptotic expansions of G-function [6, p.117] and (8). So if $f(t) \in m^{-1}_{c,\gamma}(L)$ then $f^*(s) \in L(\sigma)$ and we have :

$$\int_0^\infty G^{m,n+2}_{p+2,q}\left(t\left|\begin{matrix}1-\nu+ix,1-\nu-ix,(\alpha_p)\\(\beta_q)\end{matrix}\right.\right) f(t)\,dt =$$

$$= \lim_{\substack{\varepsilon\to 0\\ E\to\infty}} \frac{1}{2\pi i} \times$$

$$\int_\varepsilon^E G^{m,n+2}_{p+2,q}\left(t\left|\begin{matrix}1-\nu+ix,1-\nu-ix,(\alpha_p)\\(\beta_q)\end{matrix}\right.\right)\int_\sigma f^*(1-s)t^{s-1}ds\,dt=$$

$$= \frac{1}{2\pi i}\int_\sigma f^*(1-s)\int_0^\infty G^{m,n+2}_{p+2,q}\left(t\left|\begin{matrix}1-\nu+ix,1-\nu-ix,(\alpha_p)\\(\beta_q)\end{matrix}\right.\right)$$

$$t^{s-1}\,dt\,ds = \frac{1}{2\pi i}\int_\sigma \Gamma\left[\nu+ix-s,\nu-ix-s\right] \times$$

$$\Gamma\begin{bmatrix}(\beta_m)+s,1-(\alpha_n)-s\\(\alpha_p^{n+1})+s,1-(\beta_q^{m+1})-s\end{bmatrix} f^*(1-s)\,ds =$$

$$= \left(W^{m\,n}_{p\,q}\left|\begin{matrix}\nu,(\alpha_p)\\(\beta_q)\end{matrix}\right| f(t)\right)(x).$$

3. Inversion of W-transform in the space $m^{-1}_{c,\gamma}(L)$

The theorem 7 of [10] porvides the conditions for the inversion of W-transform. But it contains a lack - the condition on the shape of W-transform. Here we present an explicit inversion formula.

Theorem 2. Let $5/8<\mathrm{Re}\nu<3/4$, $p\le q$,

$f(x)\in m^{-1}_{c,\gamma}(L)$ and

$$2\,\mathrm{sgn}\,(c+c^*) + \mathrm{sgn}\,(\gamma+\gamma^*) \ge 0 \qquad (11)$$

$$2\,\mathrm{sgn}c^* + \mathrm{sgn}(1+2\gamma^*+3(q-p)/2) + \mathrm{sgn}(q-p-2) \le 0$$

$$(q>p),$$

$$2\,\mathrm{sgn}c^* + \mathrm{sgn}(\gamma^*+1) < 0 \qquad (q=p) \qquad (12)$$

Further, the parameters of the vectors (α_p) and (β_q) satisfy the conditions

$$\mathrm{Re}\,\alpha_j<1/2,\ j=1,2,\dots,n;\quad \min_{n+1\le j\le p}\mathrm{Re}\alpha_j>1/4, \qquad (13)$$

$$\mathrm{Re}\,\beta_j>-1/2,\ j=1,2,\dots,m;\quad \max_{m+1\le j\le q}\mathrm{Re}\beta_j<-3/4$$

Then the inversion formula for the w-transform (3) is given by

$$f(x) = \frac{1}{\pi^2}\int_0^\infty \tau\,\mathrm{sh}\,2\pi\tau \times$$

$$G^{q-m,p-n+2}_{p+2,q}\left(x\left|\begin{matrix}\nu+i\tau,\nu-i\tau,-(\alpha_p^{n+1}),\ -(\alpha_n)\\-(\beta_q^{m+1}),\ -(\beta_m)\end{matrix}\right.\right) \times \qquad (14)$$

$$\times\ (Wf)\ (\tau)\ d\tau\ ,\quad x>0,$$

Proof. First we shall prove the following lemma.

Lemma 1. Let the conditions of the theorem 2 are fulfilled. Then the following integral representation

$$\frac{1}{4\pi}\,\mathrm{sh}\pi\tau\,\Gamma\left[\nu-s+\frac{i\tau}{2},\nu-s-\frac{i\tau}{2}\right] \times$$

$$G^{q-m,p-n+2}_{p+2,q}\left(x\left|\begin{matrix}\nu-\frac{i\tau}{2},\nu+\frac{i\tau}{2},\ -(\alpha_p^{n+1}),-(\alpha_n)\\-(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right.\right) = \qquad (15)$$

$$= \int_0^\infty z^{-s} G^{q-m,p-n}_{p,q}\left(xz \left| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right.\right) dz \times$$

$$\int_{-\infty}^{\infty} e^{2(\nu-s)\theta}\, d\theta \quad \times$$

$$\times \int_{|\theta|}^{\infty} J_o(2\sqrt{z}\, e^{\theta/2} \sqrt{2chu-2ch\theta})\, \sin \tau u\, du, \; x>0 ,$$

takes place.

Proof of this lemma is based on the following integral representation of the Bessel function [4]

$$\frac{2}{\pi} K_{i\tau}(2\sqrt{z})K_{i\tau}(2\sqrt{y})\, sh\pi\tau =$$

$$\int_{\left|\frac{1}{2}\ell n \frac{y}{z}\right|}^{\infty} J_o(2\sqrt{2\sqrt{zy}\, chu - z - y})\, \sin \tau u\, du . \tag{16}$$

For the proof of the lemma we shall perform the following operations. We multiply both sides of (16) by $y^{\nu-s-1}$ with Re $s=1/2$, and integrate with respect to y from 0 to ∞, and then multiply the result by

$$z^{-\nu} G^{q-m,p-n}_{p,q}\left(xz \left| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right.\right)$$

(with $5/8 < \mathrm{Re}\nu < 3/4$, $x<0$). Integration of this expression with respect to z, from 0 to ∞, and the transformation
$\frac{1}{2}\, \ell n \, \frac{y}{z} = \theta$ leads to the desired result (15). Here

we have also used the formulae [6;9.3(1)] and [2,p. 210]. The existance of the repeated integral of (15), follows from the estimation

$$\int_0^\infty \left| z^{-s} G^{q-m,p-n}_{p,q}\left(xz \left| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right.\right)\right| dz \quad \times$$

$$\int_{-\infty}^{\infty} \left| e^{2(\nu-s)\theta} \right| d\theta \times \int_{|\theta|}^{\infty} \left| J_o(2\sqrt{z}\, e^{\theta/2}\sqrt{2chu-2ch\theta})\, \sin\tau u \right| du <$$

$$< \frac{1}{2} \int_0^\infty z^{-3/4} \left| G^{q-m,p-n}_{p,q}\left(xz \left| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right.\right)\right| dz \quad \times$$

$$\times \int_{-\infty}^{\infty} \frac{e^{(2\mathrm{Re}\nu-5/4)\theta}}{(ch\theta)^{1/4}} d\theta \int_1^\infty (v-1)^{-1/4} (v^2-1)^{-1/2} dv < Ax^{-1/4}$$

This estimation followed from the boundedness of the function

$\sqrt{x}\; J_o(x)$ for $x>0$, the conditions (12), (13) of the theorem 2 which were derived from the asymptotic formulas for the G-function [6]. Lemma is proved.

Let us consider the following integral, which is similar to (14) :

$$I(\lambda,x) = \frac{1}{4\pi^2} \int_0^\lambda \tau sh\pi\tau \quad \times$$

$$\times \quad G^{q-m,p-n+2}_{p+2,q}\left(x \left| \begin{matrix} \nu-i\tau/2, \nu+i\tau/2, -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right.\right) \cdot$$

$$\times\; (Wf)(\tau/2)\; d\tau . \tag{18}$$

This result exist for finite λ. So, we replace the value of $(Wf)(\tau/2)$ from the formulae (3). Changing the order of integrals by Fubini theorem and using the formula (15), we obtain

$I(\lambda,x) =$

$$-\frac{1}{2\pi i}\int_{\sigma} \Gamma\left[\begin{matrix}(\beta_m)+s, 1-(\alpha_n)-s\\ (\alpha_p^{n+1})+s, 1-(\beta_q^{m+1})-s\end{matrix}\right] f^*(1-s)ds \times \quad (19)$$

$$\times \frac{1}{\pi}\int_0^{\infty} z^{-s} G_{p,q}^{q-m,p-n}\left(xz \left| \begin{matrix}-(\alpha_p^{n+1}), -(\alpha_n)\\ -(\beta_q^{m+1}), -(\beta_m)\end{matrix}\right.\right) dz \times$$

$$\times \int_{-\infty}^{\infty} e^{2(\nu-s)\theta} d\theta \int_{|\theta|}^{\infty} J_o(2\sqrt{2z}\, e^{\theta/2}\sqrt{chu-ch\theta}) \frac{\partial}{\partial u}\left(\frac{\sin\lambda u}{u}\right) du \,.$$

Now we shall interchange the order of integration. This leads to the formula :

$$I(\lambda,x) = -\frac{1}{\pi}\int_0^{\infty} \frac{\partial}{\partial u}\left(\frac{\sin\lambda u}{u}\right) du \int_{-u}^{u} e^{2\nu\theta}\, d\theta \int_0^{\infty} (Gf)(ze^{2\theta}) \times \quad (20)$$

$$G_{p,q}^{q-m,p-n}\left(xz \left| \begin{matrix}-(\alpha_p^{n+1}), -(\alpha_n)\\ -(\beta_q^{m+1}), -(\beta_m)\end{matrix}\right.\right) J_o(2\sqrt{2z}\, e^{\theta/2}\sqrt{chu-ch\theta})dz,$$

where $(Gf)(w)$ is the G-transform of the function $f(x)$ [10]

$$(Gf)(w) = \frac{1}{2\pi i}\int_{\sigma} \Gamma\left[\begin{matrix}(\beta_m)+s, 1-(\alpha_n)-s\\ (\alpha_p^{n+1})+s, 1-(\beta_q^{m+1})-s\end{matrix}\right] f^*(1-s) w^{-s} ds,$$

which is calculated at the point $w = ze^{2\theta}$, (21)

After integration by parts, the equality (20) takes the form

$$I(\lambda,x) = \frac{1}{\pi}\int_0^{\infty} \phi(u)\, \frac{\sin\lambda u}{u}\, du \,, \quad (22)$$

where

$$\phi(u) = e^{2\nu u}\int_0^{\infty} (Gf)(ze^{2u}) \times \quad (23)$$

$$G_{p,q}^{q-m,p-n}\left(xz \left| \begin{matrix}-(\alpha_p^{n+1}), -(\alpha_n)\\ -(\beta_q^{m+1}), -(\beta_m)\end{matrix}\right.\right) dz + e^{-2\nu u}\int_0^{\infty} (Gf)(ze^{-2u})$$

$$\times G_{p,q}^{q-m,p-n}\left(xz \left| \begin{matrix}-(\alpha_p^{n+1}), -(\alpha_n)\\ -(\beta_q^{m+1}), -(\beta_m)\end{matrix}\right.\right) dz \times$$

$$+ \int_{-u}^{u} e^{2\nu\theta} d\theta \frac{d}{du}\int_0^{\infty} (Gf)(ze^{2\theta}) \times$$

$$\times G_{p,q}^{q-m,p-n}\left(xz \left| \begin{matrix}-(\alpha_p^{n+1}), -(\alpha_n)\\ -(\beta_q^{m+1}), -(\beta_m)\end{matrix}\right.\right) \times$$

$$\times J_o(2\sqrt{2z}\, e^{\theta/2}\sqrt{chu-ch\theta})\, dz \,.$$

It can be shown that the outerintegral terms are equal to zero on the ends; for $u\to 0$ and $u\to\infty$. It is obvious for $u\to 0$, because the interval $(-u, u)$ is vanished. The case, $u\to\infty$ is more difficult. We have the following inequalities

$$\left|\int_{-u}^{u} e^{2\nu\theta} d\theta \int_0^{\infty} (Gf)(ze^{2\theta}) \times\right.$$

$$\times G_{p,q}^{q-m,p-n}\left(xz \left| \begin{matrix}-(\alpha_p^{n+1}), -(\alpha_n)\\ -(\beta_q^{m+1}), -(\beta_m)\end{matrix}\right.\right) \times$$

$$\left.\times J_o(2\sqrt{2z}\, e^{\theta/2}\sqrt{chu-ch\theta})\, dz\right| <$$

$$A_1 x^{-1/4} \int_{-u}^{u} \frac{e^{(2\mathrm{Re}\nu - 5/4)\theta}}{(\mathrm{ch}u - \mathrm{ch}\theta)^{1/4}} d\theta \times$$

$$\times \int_0^\infty z^{-3/4} \left| G_{p,q}^{q-m,p-n}\left(z \left| \begin{matrix} -(\alpha_p^{n+1}), -(\alpha_n) \\ -(\beta_q^{m+1}), -(\beta_m) \end{matrix} \right.\right) \right| dz <$$

$$A_2\ x^{-1/4} \int_{-u}^{u} \frac{e^{(2\mathrm{Re}\nu - 5/4)\theta}}{(\mathrm{ch}u - \mathrm{ch}\theta)^{1/4}} d\theta =$$

$$= 2A_2\ x^{-1/4} \int_0^u \frac{\mathrm{ch}(2\mathrm{Re}\nu - 5/4)\theta}{(\mathrm{ch}u - \mathrm{ch}\theta)^{1/4}} d\theta\ , \quad A_1,\ A_2 - \text{const.}$$

They follow from the boundedness of the function $\sqrt{x}\ J_o(x)$ with $x > 0$, conditions of the theorem 2 and the estimation

$$|(Gf)(w)| < A_3 w^{-1/2}\ , \quad A_3\text{-const.}, \tag{24}$$

which is received from the condition (11).

The last integral is divided in two, by the intervals (0,1) and (1,u). It is obvious that the first integral is bounded for $u > 1$. For the estimation of the second integral, we shall use the properties $\mathrm{ch}\ \alpha\theta = O(\mathrm{ch}^\alpha \theta)$, $\mathrm{sh}\theta = O(\mathrm{ch}\theta)$ for $\alpha > 0$, $\theta > 1$ and then calculate the beta-integral

$$\int_1^u \frac{\mathrm{ch}(2\mathrm{Re}\nu - 5/4)}{(\mathrm{ch}u - \mathrm{ch}\theta)^{1/4}} d\theta =$$

$$O\left(\int_1^u \frac{(\mathrm{ch}\theta)^{2\mathrm{Re}\nu - 5/4 - 1} d\mathrm{ch}\theta}{(\mathrm{ch}u - \mathrm{ch}\theta)^{1/4}} \right) = O\left((\mathrm{ch}u)^{2\mathrm{Re}\nu - 3/2}\right).$$

The last expression tends to zero for $u \to \infty$, as $5/8 < \mathrm{Re}\nu < 3/4$. So we have proved the vanishing of outerintegral term at infinity.

Let us study the function $\phi(u)$ of (23). If we apply the theorem 6 of [10] about the inversion of G-transform, we arrive at the following simple form

$$\phi(u) = e^{(2\nu - 2)u} f(xe^{-2u}) + e^{(2-2\nu)u} f(xe^{2u}) + F(x,u), \tag{25}$$

where

$$F(x,u) = \int_{-u}^{u} e^{(2\nu-2)\theta} d\theta \int_0^\infty (Gf)(z) \times \tag{26}$$

$$G_{p\ ,\ q}^{q-m,p-n}\left(xze^{-2\theta} \left| \begin{matrix} -(\alpha_p^{n+1}), -(\alpha_n) \\ -(\beta_q^{m+1}), -(\beta_m) \end{matrix} \right.\right) \times$$

$$\times \frac{\partial}{\partial u} J_o(2\sqrt{2z}\ e^{-\theta/2}\sqrt{\mathrm{ch}u - \mathrm{ch}\theta})\ dz\ ,$$

Moreover the differentiation under the sign of the integral is valid, because of the conditions of the theorem, inequality (24) and

$|x^{-1/2}\ J_1(x)| < M$, M - const.,are fulfilled.

Further we have [4]

$$\frac{\partial}{\partial u} J_o(2\sqrt{2z}\ e^{-\theta/2} \sqrt{\mathrm{ch}u - \mathrm{ch}\theta}) = \frac{\sqrt{2}(e^{\theta - u} - 1)\sqrt{z}\ e^{-3\theta/2}}{\sqrt{\mathrm{ch}u - \mathrm{ch}\theta}} \times$$

$$\times J_1(2\sqrt{2z}\ \ e^{-\theta/2}\sqrt{\mathrm{ch}u - \mathrm{ch}\theta}) - \tag{27}$$

$$\frac{\partial}{\partial \theta} J_o(2\sqrt{2z}\ e^{-\theta/2}\sqrt{\mathrm{ch}u - \mathrm{ch}\theta}).$$

We use this result in (26), interchange the order of integration and then perform the integration by parts, to get

$$F(x,u) = \sqrt{2}e^{-u}\int_{-u}^{u}\frac{e^{(2\nu-5/2)\theta}}{\sqrt{chu-ch\theta}}\,d\theta\int_{0}^{\infty}(Gf)(z)\sqrt{z}\;\times$$

$$G^{q-m,p-n}_{p,q}\left(xze^{-2\theta}\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right)\times \qquad (28)$$

$$\times\, J_1(2\sqrt{2z}\;e^{-\theta/2}\sqrt{chu-ch\theta})dz - \sqrt{2}\int_{-u}^{u}\frac{e^{(2\nu-7/2)\theta}}{\sqrt{chu-ch\theta}}\,d\theta\;\times$$

$$\int_{0}^{\infty}\sqrt{z}\;(Gf)(z)\times G^{q-m,p-n}_{p,q}\left(xze^{-2\theta}\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),\ (\beta_m)\end{matrix}\right)\times$$

$$J_1(2\sqrt{2z}\;e^{-\theta/2}\sqrt{chu-ch\theta})\;dz - e^{(2\nu-2)u}\times\int_{0}^{\infty}(Gf)(z)$$

$$G^{q-m,p-n}_{p,q}\left(xze^{-2u}\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right)dz\;+$$

$$+\;e^{-(2\nu-2)u}\int_{0}^{\infty}(Gf)(z)\;\times$$

$$\times\;G^{q-m,\,p-n}_{p,q}\left(xze^{2u}\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right)dz\;+$$

$$+\int_{0}^{\infty}(Gf)(z)\;dz\int_{-u}^{u}J_o(2\sqrt{2z}\;e^{-\theta/2}\sqrt{chu-ch\theta})\;\times$$

$$\Big[2(\nu-1)e^{2(\nu-1)\theta}\;\times$$

$$\times\;G^{q-m,p-n}_{p,q}\left(xze^{-2\theta}\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right)+\frac{e^{2\nu\theta}}{xz}\;\times$$

$$\times\;G^{q-m,p-n+1}_{p+1,\;q+1}\left(xze^{-2\theta}\,\middle|\,\begin{matrix}0,-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),-(\beta_m),1\end{matrix}\right)\Big]\,d\theta$$

By invoking the theorem 6 of [10] about the inversion of G-transform, we get

$$F(x,u) = F_1(x,u) - F_2(x,u) + F_3(x,u) - \qquad (29)$$

$$-\;e^{(2\nu-2)u}\,f(xe^{-2u}) + e^{(2-2\nu)u}\,f(xe^{2u})\;,$$

where

$$F_1(x,u) = \sqrt{2}\;e^{-u}\int_{-u}^{u}\frac{e^{(2\nu-5/2)\theta}}{\sqrt{chu-ch\theta}}\,d\theta\qquad\times$$

$$\times\int_{0}^{\infty}\sqrt{z}\quad G^{q-m,p-n}_{p,q}\left(xze^{-2\theta}\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right)\times$$

$$\times\;(Gf)(z)\;J_1(2\sqrt{2z}\;e^{-\theta/2}\sqrt{chu-ch\theta})\quad dz\;,$$

$$F_2(x,u) = \sqrt{2}\int_{-u}^{u}\frac{e^{(2\nu-7/2)\theta}}{\sqrt{chu-ch\;\theta}}\,d\theta$$

$$\int_{0}^{\infty}\sqrt{z}\,G^{q-m,p-n}_{p,q}\left(xze^{-2\theta}\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),-(\alpha_n)\\ -(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right)\times \qquad (30)$$

$$\times \ (Gf)(z)\, J_1(2\sqrt{2z}\, e^{-\theta/2} \sqrt{chu-ch\theta})\, dz \ ,$$

$$u) = \int_0^{\infty} (Gf)(z)\, dz \ ,$$

$$\int_{-u}^{u} J_o(2\sqrt{2z}\, e^{-\theta/2} \sqrt{chu-ch\theta}) \Big[2(\nu-1) e^{2(\nu-1)\theta} \ \times$$

$$\times \ G^{q-m,\,p-n}_{p\,,\,q} \left(xze^{-2\theta} \middle| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right) +$$

$$\frac{\nu\theta}{xz} \ G^{q-m,\,p-n+1}_{p+1,\,q+1} \left(xze^{-2\theta} \middle| \begin{matrix} 0,-(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m),1 \end{matrix} \right) \Big]\, d\theta \ .$$

The results (25) and (29) leads to

$$\phi(u) = 2e^{(2-2\nu)u} f(xe^{2u}) + F_1(x,u) - F_2(x,u) + F_3(\alpha,u) \tag{31}$$

Let us show that for each $x>0$, the functions $\frac{1}{u} F_i(x,u)$, $i = 1,2,3$, are absolutely integrable on the interval $0<u<\infty$. This follows from the boundedness of the functions $x^{1/2} J_o(x)$ and $x^{-1/2} J_1(x)$, the conditions of the theorem 2 and the following estimations

$$\text{(i)} \quad |F_1(x,u)| < M_1 e^{-u} \int_{-u}^{u} \frac{e^{(2Re\nu - 5/2)\theta}}{(chu-ch\theta)^{1/4}}\, d\theta \ \times \tag{32}$$

$$\int_0^{\infty} z^{1/4} \left| G^{q-m,p-n}_{p\,,\,q} \left(xze^{-2\theta} \middle| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right) \right| e^{-\theta/4}\, dz =$$

$$= M_2\, x^{-5/4}\, e^{-u} \int_{-u}^{u} \frac{e^{(2Re\nu-1/4)\theta}}{(chu-ch\theta)^{1/4}}\, d\theta \ , M_1,\ M_2 - \text{const},$$

$$\int_0^{\infty} \frac{|F_1(x,u)|}{u}\, du < M_2\, x^{-5/4} \int_0^{\infty} \frac{e^{-u}}{u}\, du \ \times$$

$$\times \int_{-u}^{u} \frac{e^{(2Re\nu - 1/4)\theta}}{(chu-ch\theta)^{1/4}}\, d\theta <$$

$$M_2\, x^{-5/4} \int_{-\infty}^{\infty} \frac{e^{(2Re\nu - 1/4)\theta - |\theta|}}{(ch\theta)^{1/4}}\, d\theta \int_1^{\infty} \frac{dy}{\ell ny \sqrt{y^2-1}\, \sqrt[4]{y-1}} < \infty \ ,$$

$$\text{(ii)} \ |F_2(x,u)| < M_3 x^{-5/4} \int_{-u} \frac{e^{(2Re\nu - 5/4)\theta}}{(chu-ch\theta)^{1/4}}\, d\theta \ , \tag{33}$$

$$\int_0^{\infty} \frac{|F_2(x,u)|}{u}\, du < M_4\, x^{-5/4} \int_{-\infty}^{\infty} \frac{e^{(2Re\nu - 5/4)\theta}}{(ch\theta)^{1/4}}\, d\theta \ \times$$

$$\times \int_1^{\infty} \frac{dy}{\ell ny \sqrt[4]{y-1}\, \sqrt{y^2-1}} < \infty, M_3,\ M_4 - \text{const},$$

$$\text{(iii)} \ |F_3(x,u)| < M_5 \int_{-u}^{u} \frac{e^{2(Re\nu-1)\theta}}{(chu-ch\theta)^{1/4}}\, d\theta \ \times$$

$$\int_0^{\infty} \left| G^{q-m,p-n}_{p\,,\,q} \left(xze^{-2\theta} \middle| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix} \right) \right| \times \tag{34}$$

$$\times z^{-3/4} e^{\theta/4} dz + \frac{M_6}{x} \int_{-u}^{u} \frac{e^{2\theta Re\nu}}{(chu-ch\theta)^{1/4}} d\theta \quad \times$$

$$\times \int_0^\infty z^{-7/4} e^{\theta/4} \times$$

$$\times \left| G^{q-m,p-n+1}_{p+1,q+1}\left(xze^{-2\theta} \left| \begin{matrix} 0,-(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m),1 \end{matrix} \right.\right)\right| dz \quad <$$

$$< M_7 x^{-1/4} \int_{-u}^{u} \frac{e^{(2Re\nu-5/4)\theta}}{(chu-ch\theta)^{1/4}} d\theta + M_8\, x^{3/4} \times$$

$$\int_{-u}^{u} \frac{e^{(2Re\nu-5/4)\theta}}{(chu-ch\theta)^{1/4}} d\theta ,$$

$$\int_0^\infty \frac{|F_3(x,u)|}{u} du < \infty \quad , M_j\text{-const} \ , \ j = 5,6,7,8.$$

From the results (32)-(34) it follows that $\frac{1}{u} F_i(x,u) \in L(0,\infty)$, $i=1,2,3$ and from the Riemann lemma we have the equality

$$\lim_{\lambda\to\infty} \int_0^\infty F_i(x,u) \frac{\sin\lambda u}{u} du = 0, \qquad i = 1,2,3 \qquad (35)$$

To obtain the final result we must to study the integral

$$I_1(\lambda,x) = \frac{2}{\pi} \int_0^\infty e^{(2-2\nu)u} f(xe^{2u}) \frac{\sin\lambda u}{u} du . \quad (36)$$

In connection with the formulas (4) and $f(x)\in m^{-1}_{c,\gamma}(L)$ we have the equality

$$f(xe^{2u}) = \frac{1}{2\pi i} \int_\sigma \phi(s) (xe^{2u})^{-s} ds \qquad (37)$$

So, if we substitute (37) into (36) and interchange the order of integration which is valid due to the absolute convergence of the repeated integral, we get

$$I_1(\lambda,x) = \frac{2}{\pi}\frac{1}{2\pi i} \int_\sigma \phi(s)x^{-s} ds \times \qquad (38)$$

$$\times \int_0^\infty e^{(2-2\nu-2s)u} \frac{\sin\lambda u}{u} du =$$

$$= \frac{1}{\pi^2 i} \int_\sigma \phi(s) \operatorname{arctg} \frac{\lambda}{2\nu-2+2s} x^{-s} ds.$$

Let, $s=1/2 + i\tau$, then we have

$$\left|\operatorname{arctg} \frac{\lambda}{2\nu-2+2s}\right| \le \frac{1}{2} \left|\operatorname{arctg} \frac{\lambda+2(\tau+ Im\nu)}{2Re\nu-1}\right| +$$

$$\frac{1}{2} \left|\operatorname{arctg} \frac{\lambda-2(\tau+ Im\nu)}{2 Re\nu-1}\right| + \qquad (39)$$

$$+ \frac{1}{4}\left|\ell n \left(\frac{(2Re\nu-1)^2+ (\lambda+2(Im\nu+\tau))^2}{(2Re\nu-1)^2 + (\lambda-2(Im\nu+\tau))^2}\right)\right|$$

and for last item the estimation

$$\left|\ell n \left(\frac{(2Re\nu-1)^2+ (\lambda+2(Im\nu+\tau))^2}{(2Re\nu-1)^2+ (\lambda-2(Im\nu+\tau))^2}\right)\right|$$

$$\le \ell n \left(\frac{(2Re\nu-1)^2+ (\sqrt{\lambda^2+(2Re\nu-1)^2}+ \lambda)^2}{(2Re\nu-1)^2+ (\sqrt{\lambda^2+(2Re\nu-1)^2}-\lambda)^2}\right) ,$$

$$-\infty < \tau < \infty \quad . \qquad (40)$$

Now taking into account (38) and (39), we shall calculate the limit for $\lambda\to\infty$ in the result (36) by the Lebesgue theorem :

$$\lim_{\lambda\to\infty} I_1(\lambda,x) = \frac{1}{2\pi i}\int_\sigma \phi(s)\, x^{-s}\, ds = f(x) \tag{41}$$

Hence from the formulae (18), (22), (35) and (41) we obtain

$$\lim_{\lambda\to\infty} I(\lambda,x) = f(x) = \frac{1}{4\pi^2}\int_0^\infty \tau \operatorname{sh}\pi\tau\, (Wf)(\tau/2) \times$$

$$\times\, G^{q-m,p-n+2}_{p+2,q}\left(x \,\middle|\, \begin{matrix} \nu+i\tau/2,\ \nu-i\tau/2, -(\alpha_p^{n+1}), -(\alpha_n) \\ -(\beta_q^{m+1}), -(\beta_m) \end{matrix}\right) d\tau. \tag{42}$$

Now, $t=\tau/2$ leads us to the formula (14). The theorem 2 is proved.

Remark 1. In addition to the conditions of the Theorem 2, if we add the restriction (8), then the result (14) can be written in the following Fourier type integral form :

$$f(x) = \frac{1}{\pi^2}\int_0^\infty \tau \operatorname{sh} 2\pi\tau \times \tag{43}$$

$$\times\, G^{q-m,p-n+2}_{p+2,q}\left(x \,\middle|\, \begin{matrix} \nu+i\tau, \nu-i\tau, -(\alpha_p^{n+1}) -(\alpha_n) \\ -(\beta_q^{m+1}), -(\beta_m) \end{matrix}\right) d\tau \times$$

$$\times \int_0^\infty G^{m,n+2}_{p+2,q}\left(y \,\middle|\, \begin{matrix} 1-\nu+i\tau, 1-\nu-i\tau,\ (\alpha_p) \\ (\beta_q) \end{matrix}\right) f(y)\, dy.$$

Remark 2. In the case $q<p$ the conditions corresponding to (11), (12) may be written by analogy with the help of the formulas (8.37) of [6].

4. L_2-theory

Now we shall study the integral transform (1) in the space $L_2(0,\infty)$. The following theorem takes place.

Theorem 3. Let the conditions (7) are fulfilled and

$$c^* = \gamma^* = 0, \quad p \neq q. \tag{44}$$

Then the integral transform

$$g(x) = \underset{M,N\to\infty}{\ell.i.m} \int_{1/M}^{N} K(x,y)\, f(y)\, dy\ , \tag{45}$$

where

$$K(x,y) = \frac{\sqrt{x \operatorname{sh} 2\pi x}}{\pi}\, G^{m,n+2}_{p+2,q}\left(y \,\middle|\, \begin{matrix} 1/2-ix, 1/2+ix, (\alpha_p) \\ (\beta_q) \end{matrix}\right) \tag{46}$$

is automorfism in the space $L_2(0,\infty)$ and its inversion formulae has the form

$$f(y) = \underset{M,N\to\infty}{\ell.i.m.} \frac{\sqrt{x \operatorname{sh} 2\pi x}}{\pi} \times$$

$$\times \int_{1/M}^{N} G^{q-m,p-n+2}_{p+2,q}\left(y \,\middle|\, \begin{matrix} 1/2-ix, 1/2+ix, -(\alpha_p^{n+1}), -(\alpha_n) \\ -(\beta_q^{m+1}), -(\beta_m) \end{matrix}\right)$$

$$\times\ g(x)dx. \tag{47}$$

If, moreover,

$$\alpha_j \in R,\ j = 1,2,\ldots,p;\ \beta_j \in R,\ j = 1,2,\ldots q; \quad p = 2n,\ q = 2m, \tag{48}$$

$$\alpha_{n+j} = -\alpha_j\ ,\quad j = 1,2,\ldots,n\ ;$$

$$\beta_{m+j} = -\beta_j,\ j = 1,2,\ldots,m,$$

then the transform (45) is unitary.

Proof. Proof of the theorem is based on the following three lemmas.

Lemma 2. The transform

$$g(x) = \underset{M,N\to\infty}{\ell.i.m} \int_{1/M}^{N} p(x,y)f(y)dy, \tag{49}$$

where

$$p(x,y) = \frac{2}{\pi}\sqrt{x sh 2\pi x}\;\frac{K_{2ix}(2\sqrt{y})}{\sqrt{y}}, \tag{50}$$

is unitary in the space $L_2(0,\infty)$ and its inversion formula has the form

$$f(y) = \underset{M,N\to\infty}{\ell.i.m.} \int_{1/M}^{N} p(x,y)\,g(x)\,dx. \tag{51}$$

For the proof of this lemma, see [5]

Lemma 3. Let the condition (44) is fulfilled Then the transform

$$g(x) = \underset{M,N\to\infty}{\ell.i.m} \int_{1/M}^{N} h(xy)\,f(y)\,dy\,, \tag{52}$$

where

$$h(x) = G^{m,n}_{p,q}\left(x\,\middle|\,\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right), \tag{53}$$

is automorfism in the space $L_2(0,\infty)$ and its inversion formula has the form

$$f(y) = \underset{M,N\to\infty}{\ell.i.m} \int_{1/M}^{N} G^{q-m,p-n}_{p\,,\,q}\left(xy\,\middle|\,\begin{matrix}-(\alpha_p^{n+1}),\ -(\alpha_n)\\-(\beta_q^{m+1}),\ -(\beta_m)\end{matrix}\right)$$

$$\times\ g(x)\,dx. \tag{54}$$

If, moreover, the condition (48) is fulfilled, then the transform (52) is unitary in the space $L_2(0,\infty)$.

Proof of this lemma is based on the results of [3,11].

Lemma 4. For the functions (46), (50), (53) the following equality.

$$K(x,y) = \int_0^\infty p(x,t)\,h(ty)\,dt \tag{55}$$

holds, moreover, the integral (55) is uniformly convergent in x, $y \in [1/M,\ N]$ for all $M,N > 0$.

For the proof, of this Lemma, see [2].

The uniform convergence of the integral (55) follows from the asymptotic properties of the functions $h(x)$ of (53) and $K_{ix}(t)$.

Proof of the theorem 3. The function $K(x,y)$ is bounded for $y \in [1/M,N]$ and $f(y) \in L_2(0,\infty)$ So, the integral

$$g_{M,N}(x) = \int_{1/M}^{N} K(x,y)\,f(y)\,dy \tag{56}$$

is absolute convergent. From the uniform convergence of the integral (55) for $y \in [1/M,N]$, we have the equalities

$$g_{M,N}(x) = \int_{1/M}^{N} f(y)\int_0^\infty p(x,t)h(ty)\,dtdy =$$

$$= \int_0^\infty p(x,t)\int_{1/M}^{N} h(ty)\,f(y)\,dy. \tag{57}$$

Let

$$f_{MN}(y) = \begin{cases} f(y), & y \in (1/M,N),\\ 0\,, & y \notin (1/M,N).\end{cases} \tag{58}$$

Then it is obvious that, $f_{MN}(y) \in L_2(0,\infty)$ and hence the integral

$$\phi_{MN}(t) = \int_{1/M}^{N} h(ty)\,f(y)\,dy \tag{59}$$

is the transform (52) of the function $f_{MN}(y) \in L_2(0,\infty)$ and so it belonges to $L_2(0,\infty)$ too. But, then the integral

$$\int_0^\infty p(x,t)\ \phi_{MN}(t)\ dt \tag{60}$$

is convergent not only in the conventional sense but also in the sense of $L_2(0,\infty)$. So,

$$\int_0^\infty p(x,t)\ \phi_{MN}(t)\ dt = \underset{P,Q\to\infty}{\ell.i.m} \int_{1/P}^{Q} p(x,t)\ \phi_{MN}(t)\ dt \tag{61}$$

and

$$g_{MN}(x) = \underset{P,Q\to\infty}{\ell.i.m} \int_{1/P}^{Q} p(x,t)\ \phi_{MN}(t)\ dt. \tag{62}$$

The following inequalities

$$\int_0^\infty \left|g_{MN}(x)\ g_{M_1N_1}(x)\right|^2 dx \le$$

$$\le .C_1 \int_0^\infty \left|\phi_{MN}(t) - \phi_{M_1N_1}(t)\right|^2 dt \le \tag{63}$$

$$\le C \int_0^\infty \left|f_{MN}(y) - f_{M_1N_1}(y)\right|^2 dy$$

follow from the lemma 2, 3. Hence

$$\underset{M,N\to\infty}{\ell.i.m}\ g_{MN}(x) = g(x) \tag{64}$$

exist and

$$\int_0^\infty |g(x)|^2\ dx \le c \int_0^\infty |f(y)|^2\ dy, \tag{65}$$

$$g(x) = \underset{M,N\to\infty}{\ell.i.m}\ \underset{P,Q\to\infty}{\ell.i.m} \int_{1/P}^{Q} p(x,t) \int_{1/M}^{N} h(ty)\ f(y)\ dy\ dt \tag{66}$$

$$= \underset{P,Q\to\infty}{\ell.i.m} \int_{1/p}^{Q} p(x,t)\ \underset{M,N\to\infty}{\ell.i.m} \int_{1/M}^{N} h(ty)\ f(y)\ dy\ dt$$

In a similar way, the following result can be obtained from (45)

$$f(y) =$$

$$\underset{M,N\to\infty}{\ell.i.m}\ \underset{P,Q\to\infty}{\ell.i.m} \int_{1/M}^{N} G^{q-m,p-n}_{p,q}\left(ty \middle| \begin{matrix} -(\alpha_p^{n+1}),-(\alpha_n) \\ -(\beta_q^{m+1}),-(\beta_m) \end{matrix}\right)$$

$$\int_{1/P}^{Q} p(x,t)g(x)dx\ dt. \tag{67}$$

But the formulas (66), (67) are inverse to each other. This follows from the lemmas 2, 3. Under the conditions (48), the components (49), (52) of the transform (45) are unitary and so the transform (45) is also unitary. The theorem 3 is proved.

5. Examples

Finally we shall study some particular cases of the transform (1).

A. Let $m = n = p = q = 0$. Then after some simplification in the formula (1), we get the Kontorovich - Lebedev transform. The conditions of the theorem 2 are not fulfilled in this case. But, in [9] it is shown that for functions

$f(x) \in m^{-1}_{0,\frac{1}{4}}(L)$ the following representation

$$f(x) = \frac{2}{\pi^2 x} \int_0^\infty \tau sh\pi\tau\ K_{i\tau}(x)\ d\tau \int_0^\infty K_{i\tau}(y)\ f(y)\ dy \tag{68}$$

takes place.

B. Let $m=n=p=0$, $q=1$, $\nu=1/2+\alpha$, $\beta_1=x-\alpha$.

Then $c^* = -1/2$, $\gamma^* = -Re\beta_1$. Let the conditions of the theorem 2 are fulfilled : $1/8 < Re\alpha < 1/4$, $Re x < Re\alpha - 3/4$, $f(x) \in m^{-1}_{c,\gamma}(L)$, $2\ sgn(c-1/2) + sgn(\gamma - Re(x-\alpha)) \ge 0$. Then the following representation

$$f(x) = \frac{x^\alpha}{\pi^2}\ e^{\frac{1}{2x}} \int_0^\infty \tau sh\ 2\pi\tau$$

$$\Gamma\left[1/2-x-i\tau,1/2-x+i\tau\right]W_{x,i\tau}\left(\frac{1}{x}\right)d\tau \times \quad (69)$$

$$\times \int_0^\infty y^{-\alpha} e^{-\frac{1}{2y}} W_{x,i\tau}\left(\frac{1}{y}\right) f(y)\, dy\ , \quad x > 0\ ,$$

takes place. This can be derived, from the formulas (8.53), (8.54) of [6].

C. Let $m=1$, $q=2$, $p=n=0$, $\nu=\mu-\alpha$, $\beta_1=\alpha$, $\beta_2=1-c+\alpha$ and $c^*=0$, $\gamma^* = \mathrm{Re}c-2\mathrm{Re}\alpha-1$; $5/8<\mathrm{Re}(\mu-\alpha)<$

$<3/4$; $\mathrm{Re}\alpha>11/4$; $7/4 + \mathrm{Re}\alpha<\mathrm{Re}c<2\mathrm{Re}\alpha-1$,

$f(x) \in m^{-1}_{c,\gamma}(L)$, $2\mathrm{sgn}c + \mathrm{sgn}\,(\gamma+ \mathrm{Re}c - 2\mathrm{Re}\alpha -1) \geq 0$.

Then the following representation

$$f(x) = \frac{x^{c-\alpha-1}}{\pi^2\Gamma^2(c)} \int_0^\infty \tau \mathrm{sh}\, 2\pi\tau\, \Gamma\left[c-\mu+i\tau,c-\mu-i\tau\right]$$

$${}_2F_1(c-\mu+i\tau,c-\mu-i\tau;c;-x)\, d\tau \quad \times \quad (70)$$

$$\times\int_0^\infty \Gamma\left[\mu-i\tau,\mu+i\tau\right] {}_2F_1(\mu-i\tau,\mu+i\tau;\ c;-y)y^{\alpha}f(y)\, dy,\ x>0,$$

takes place. It is connected with the formulas (8.55), (8.56) of [6] which is related to Olevskii transform.

D. Let $b > -1/2$. Then the integral transform

$$g(x) = \underset{M,N\to\infty}{\mathrm{l.i.m}} \int_{1/M}^{N} \frac{\sqrt{x\mathrm{sh}2\pi x}}{\pi\Gamma(1+2b)}\, \Gamma\left[1/2+ix+b,\ 1/2-ix+b\right] \times \quad (71)$$

$$\times\ {}_2F_1(1/2+ix+b,\ 1/2-ix+b;1+2b;-y)y^{b}f(y)\, dy.$$

is unitary in the space $L_2(o,\infty)$ and its inversion formula has the form

$$f(y) = \underset{M,N\to\infty}{\mathrm{l.i.m}} \int_{1/M}^{N} \frac{\sqrt{x\mathrm{sh}2\pi x}}{\pi\Gamma(1+2b)}\ \Gamma\left[1/2+ix+b,\frac{1}{2} - ix+b\right] \quad (72)$$

$$\times\ {}_2F_1(1/2+ix+b,1/2-ix+b;1+2b;-y)\ y^{b}\ g(x)\, dx\ .$$

In conclusion we may point out that a large number of other particular cases of the transform (1) and its inversion formulae can be derived from the formulas of the tables of the monographs [6,8].

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