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ON CERTAIN INFINITE SERIES AND DEFINITE INTEGRALS INVOLVING $\delta_n = \frac{1.3.5. \ldots (2n-1)}{2.4.6. \ldots (2n)}$

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ABSTRACT

The object of this paper is to obtain the sums of certain infinite series and evaluate certain new definite integrals with simple functions. Two combinatorial identities are obtained.

RESUMEN

El objetivo de este trabajo es el de obtener las sumas de ciertas series infinitas y evaluar algunas integrales definidas nuevas con funciones simples. Se obtuvieron dos identidades combinatorias.

1. INTRODUCTION

With the usual notation the numbers δ_n are defined by

$$\delta_0 = 1,\ \delta_n = (-1)^n \binom{-\frac{1}{2}}{n} = 2^{-2n} \binom{2n}{n} = \frac{(2n-1)!!}{(2n)!!}$$

$$= \frac{1.3.5.\ \ldots\ (2n-1)}{2.4.6.\ \ldots\ (2n)} = \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\ \Gamma(n+1)} \quad (n = 1,2,\ldots). \quad (1)$$

It is easily seen that

$$(1-z)^{-1/2} = \sum_0^\infty \delta_n z^n \quad (|z|<1), \quad (2)$$

$$(1+z)^{1/2} = \sum_0^\infty \frac{(-1)^{n+1}\delta_n z^n}{2n-1} \quad (|z| \leq 1). \quad (3)$$

By means of fractional calculus Ross [8] has proved the simple result

$$\sum_1^\infty \frac{\delta_n}{n} = \ln 4 . \quad (4)$$

A probability problem considered by Callan [1] led to the same result.

The extension

$$\sum_{n=1}^\infty \frac{\delta_{n+m}}{n} = \delta_m \left[\ln 4 + \sum_{n=1}^m \frac{1}{n}\right] \quad (5)$$

of (4) was proposed by Shafer [9] and proved by Knuth [5]. In a recent paper Lehmer [6] gave the sums of several interesting infinite series of the types $\sum_0^\infty a_n \delta_n$ and $\sum_0^\infty a_n/\delta_n$, the a_n being simple functions of n.

In this paper we give the sums of certain infinite series involving δ_n and evaluate certain new definite integrals with simple functions in terms of δ_n. Two combinatorial identities are proved.

2. FIRST SET OF SERIES AND INTEGRALS

Let

$$c_0(r,\theta) = \sum_0^\infty r^{2n} \delta_n \cos 2n\theta ,$$

$$S_0(r,\theta) = \sum_0^\infty r^{2n}\delta_n \sin 2n\theta, \quad (6)$$

$$C_1(r,\theta) = \sum_0^\infty r^{2n} \delta_n \cos(2n+1)\theta ,$$

$$S_1(r,\theta) = \sum_{0}^{\infty} r^{2n}\,\delta_n \sin(2n+1)\theta \,, \tag{7}$$

where* $0\le r\le 1$, $0\le\theta\le\pi/2$. Using (2) we have

$$C_o(r,\theta) + i\,S_o(r,\theta) = (1-z^2)^{-1/2} \,, \tag{8}$$

where $z = re^{i\theta}$. This leads to

$$C_o(r,\theta) = \sqrt{\tfrac{1}{2}\,(R^2 + 1 - r^2\cos 2\theta)/R^2} \,, \tag{9}$$

$$S_o(r,\theta) = \sqrt{\tfrac{1}{2}\,(R^2 - 1 + r^2\cos 2\theta)/R^2} \,, \tag{10}$$

$$C_o^2(r,\theta) + S_o^2(r,\theta) = 1/R^2 \,, \tag{11}$$

where

$$R^4 = 1 + r^4 - 2r^2\cos 2\theta \,. \tag{12}$$

From these we get

$$\int_0^{\pi/2} \frac{\cos 2n\theta}{R^2}\sqrt{R^2+1 - r^2\cos 2\theta}\, d\theta =$$

$$= \frac{\sqrt{2}\pi}{4}\, r^{2n}\delta_n\,(n\ge 1), \tag{13}$$

$$\int_0^{\pi/2} \frac{\sin 2n\theta}{R^2}\sqrt{R^2- 1 + r^2\cos 2\theta}\, d\theta =$$

$$= \frac{\sqrt{2}\pi}{4}\, r^{2n}\delta_n\,(n\ge 1)\,, \tag{14}$$

$$\int_0^{\pi/2} \frac{1}{R}\sqrt{R^2+ 1 - r^2\cos 2\theta}\, d\theta = \frac{\sqrt{2}\pi}{2} \quad \text{(independent of r).} \tag{15}$$

Letting r tend to 1 in these results yields the simple formulae

$$\int_0^{\pi/2} \sqrt{\operatorname{cosec}\theta + 1}\,\cos 2n\theta\, d\theta =$$

$$\int_0^{\pi/2} \sqrt{\operatorname{cosec}\theta - 1}\,\sin 2n\theta\, d\theta$$

$$= \frac{\pi}{2}\,\delta_n\,(n\ge 1)\,, \tag{16}$$

$$\int_0^{\pi/2} \sqrt{\operatorname{cosec}\theta + 1}\, d\theta = \pi\,. \tag{17}$$

A similar treatment of (7) gives

$$C_1(r,\theta) = \sqrt{\tfrac{1}{2}\,(R^2 - r^2 + \cos 2\theta)\,/R^2} \,, \tag{18}$$

$$S_1(r,\theta) = \sqrt{\tfrac{1}{2}\,(R^2 + r^2 - \cos 2\theta)\,/R^2} \,, \tag{19}$$

$$C_1^2(r,\theta) + S_1^2(r,\theta) = 1/R^2 \;; \tag{20}$$

$$\int_0^{\pi/2} \frac{\cos(2n+1)\theta}{R^2}\sqrt{R^2 - r^2 + \cos 2\theta}\, d\theta =$$

$$= \frac{\sqrt{2}\pi}{4}\, r^{2n}\,\delta_n\,(n\ge 0)\,, \tag{21}$$

$$\int_0^{\pi/2} \frac{\sin(2n+1)\theta}{R^2}\sqrt{R^2 + r^2 - \cos 2\theta}\, d\theta =$$

$$= \frac{\sqrt{2}\pi}{4}\, r^{2n}\,\delta_n\,(n\ge 0)\,, \tag{22}$$

$$\int_0^{\pi/2} \sqrt{\operatorname{cosec}\theta - 1}\,\cos(2n+1)\theta\, d\theta =$$

$$\int_0^{\pi/2} \sqrt{\operatorname{cosec}\theta + 1}\,\sin(2n+1)\theta\, d\theta = \frac{\pi}{2}\,\delta_n\,(n\ge 0)\,, \tag{23}$$

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* the case r=1 , θ=0 is excluded.

$$\int_0^{\pi/2} \frac{1}{R^2} \sqrt{R^2 - r^2 + \cos 2\theta}\, d\theta = \sqrt{2} \sum_0^{\infty} \frac{(-1)^n r^{2n} \delta_n}{2n+1}$$

$$= \sqrt{2}\, \frac{\sin^{-1} r}{r} \qquad (24)$$

$$\int_0^{\pi/2} \frac{1}{R^2} \sqrt{R^2 + r^2 - \cos 2\theta}\, d\theta = \sqrt{2} \sum_0^{\infty} \frac{r^{2n} \delta_{2n}}{2n+1}$$

$$= \sqrt{2}\, \frac{\sin^{-1} r}{r} \qquad (25)$$

$$\int_0^{\pi/2} \sqrt{\operatorname{cosec} \theta - 1}\, d\theta = 2 \ln (\sqrt{2} + 1) . \qquad (26)$$

Putting r=1 in (25) we obtain (17). The two results (17) and (26) are checked by elementary methods of integration.

3. SECOND SET OF SERIES AN INTEGRALS

Let

$$c_o(r,\theta) = \sum_1^{\infty} \frac{r^{2n} \delta_n}{2n} \cos 2n\theta, \quad s_o(r,\theta) =$$

$$\sum_1^{\infty} \frac{r^{2n} \delta_n}{2n} \sin 2n\theta, \qquad (27)$$

$$c_1(r,\theta) = \sum_0^{\infty} \frac{r^{2n+1} \delta_n}{2n+1} \cos(2n+1)\theta, \quad s_1(r,\theta) =$$

$$\sum_0^{\infty} \frac{r^{2n+1} \delta_n}{2n+1} \sin(2n+1)\theta, \qquad (28)$$

where $0 \leq r \leq 1$, $0 \leq \theta \leq \pi/2$. It is easily shown that

$$c_o + i\, s_o = \ln \frac{2}{1 + \sqrt{1 - z^2}} , \qquad (29)$$

$$c_1 + i\, s_1 = \sin^{-1} z , \qquad (30)$$

where $z = re^{i\theta}$. Thus we have

$$c_o(r,\theta) = \ln \frac{2}{r} - \frac{1}{2} \cosh^{-1} \frac{R^2+1}{r^2} , \qquad (31)$$

$$s_o(r,\theta) = \sin^{-1} \sqrt{\frac{R^2+r^2-1}{2r^2}} - \theta , \qquad (32)$$

$$c_1(r,\theta) = \cos^{-1} \sqrt{\frac{1}{2} (R^2 - r^2 + 1)} , \qquad (33)$$

$$s_1(r,\theta) = \frac{1}{2} \cosh^{-1} (R^2 + r^2) . \qquad (34)$$

Values of the following definite integrals are now obtained :

$$\int_0^{\pi/2} \cos 2n\theta \cosh^{-1} \frac{R^2+1}{r^2}\, d\theta = - \frac{\pi}{4n} r^{2n} \delta_n \ (n \geq 1) , \qquad (35)$$

$$\int_0^{\pi/2} \sin 2n\theta \sin^{-1} \sqrt{\frac{R^2+r^2-1}{2r^2}}\, d\theta =$$

$$\frac{\pi}{4n} \left[(-1)^{n+1} + \frac{1}{2} r^{2n} \delta_n \right] (n \geq 1) , \qquad (36)$$

$$\int_0^{\pi/2} \cosh^{-1} \frac{R^2+1}{r^2}\, d\theta = \pi \ln \frac{2}{r} , \qquad (37)$$

$$\int_0^{\pi/2} \cos(2n+1)\theta \cos^{-1} \sqrt{\frac{1}{2}(R^2 - r^2 + 1)}\, d\theta =$$

$$\frac{\pi}{4(2n+1)} r^{2n+1} \delta_n \ (n \geq 0) , \qquad (38)$$

$$\int_0^{\pi/2} \sin(2n+1)\theta \cosh^{-1}(R^2+r^2)d\theta =$$

$$= \frac{\pi}{2(2n+1)} r^{2n+1}\delta_n \ (n\geq o) \quad , \tag{39}$$

$$\int_0^{\pi/2} \cos^{-1}\sqrt{\frac{1}{2}(R^2-r^2+1)}\ d\theta = \sum_0^{\infty} \frac{(-1)^n r^{2n+1}\delta_n}{(2n+1)^2} =$$

$$\int_0^{r} \frac{\sinh^{-1}x}{x} , \tag{40}$$

$$\int_0^{\pi/2} \cosh^{-1}(R^2+r^2)d\theta = 2\sum_0^{\infty} \frac{r^{2n+1}\delta_n}{(2n+1)^2} =$$

$$= 2\int_0^{r} \frac{\sin^{-1}x}{x} dx \quad . \tag{41}$$

In the special case r =1 these simplify to

$$\int_0^{\pi/2} \cos 2n\theta \sinh^{-1}\sqrt{\sin\theta}d\theta = -\frac{\pi}{8n} \ \delta_n (n\geq 1) \ , \tag{42}$$

$$\int_0^{\pi/2} \sin 2n\theta \sin^{-1}\sqrt{\sin\theta}\ d\theta =$$

$$\frac{\pi}{4n}\left[(-1)^{n+1} + \frac{1}{2}\delta_n\right] (n\geq 1), \tag{43}$$

$$\int_0^{\pi/2} \sinh^{-1}\sqrt{\sin\theta}\, d\theta = \frac{\pi}{2}\ln 2, \tag{44}$$

$$\int_0^{\pi/2} \cos(2n+1)\theta \ \cos^{-1}\sqrt{\sin\theta}\ d\theta = \frac{\pi\delta_n}{4(2n+1)} \ (n\geq o) \tag{45}$$

$$\int_0^{\pi/2} \sin(2n+1)\theta \ \sinh^{-1}\sqrt{\sin\theta}\ d\theta = \frac{\pi\delta_n}{4(2n+1)} \ (n\geq o),$$

$$\int_0^{\pi/2} \cos^{-1}\sqrt{\cos\theta}\ d\theta = \frac{1}{2}\int_0^{\pi/2} \theta\sqrt{\operatorname{cosec}\theta + 1}\ d\theta =$$

$$\int_0^{1} \frac{\sinh^{-1}x}{x}dx = \sum_0^{\infty} \frac{(-1)^n\delta_n}{(2n+1)^2} = 0.955202 \ . \tag{47}$$

We may replace $\sinh^{-1}\sqrt{\sin\theta}$ in (42) and (44) by $\ln(\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$ and the results are to be compared with the following integrals which are given in examples 3 and 4, p. 252 of [2] :

$$\int_0^{\pi/2} \cos 2n\theta \ln \ \sin\theta\ d\theta = -\frac{\pi}{4n} \ (n\geq 1),$$

$$\int_0^{\pi/2} \ln \sin\theta\ d\theta = -\frac{\pi}{2}\ln 2. \tag{48}$$

Addition and subtraction of (42) and (44) yields values for the two integrals

$$\int_0^{\pi/2} \cos^2 n\theta \sinh^{-1}\sqrt{\sin\theta}\ d\theta \ ,$$

$$\int_0^{\pi/2} \sin^2 n\theta \sinh^{-1}\sqrt{\sin\theta}\ d\theta \ .$$

From (44) and (26) we easily obtain

$$\int_0^{\pi/2} \theta\sqrt{\operatorname{cosec}\theta - 1}\ \ d\theta = \pi \ \ln \frac{\sqrt{2}+1}{2} \ , \tag{49}$$

$$\int_0^{\pi/2} \theta\sqrt{\sec\theta - 1}\ d\theta = \pi \ln 2., \tag{50}$$

It can be proved that any of the integrals in (47) equals

$$\int_0^{\pi/2} \sin^{-1} \sin^2\theta\ d\theta = 2\int_0^{\pi/2} \theta\ d\theta / \ \sqrt{\sec^2\theta + 1}$$

$$= -\int_0^1 \frac{\ln x}{\sqrt{1+x^2}}\,dx = \int_0^{\pi/4} \sec\theta \ln\cot\theta\, d\theta. \qquad (51)$$

The integrals appearing in (47) and (51) may be compared with

$$\left.\begin{aligned} &\int_0^1 \frac{\sin^{-1}x}{x}\,dx = \int_0^{\pi/2} \frac{\theta}{\tan\theta}\,d\theta = \sum_0^\infty \frac{\delta_n}{(2n+1)^2} = \\ &= \frac{\pi}{2}\ln 2, \int_0^1 \frac{\tan^{-1}x}{x}\,dx = \frac{1}{2}\int_0^{\pi/2} \frac{\theta}{\sin\theta}\,d\theta = \end{aligned}\right\}(52)$$

$$= \sum_0^\infty \frac{(-1)^n}{(2n+1)^2} = G,$$

where G is Catalan's constant which has the value 0.915966. See Section 4.521, 1, p.606 and 4.531, 1, p. 607 of [3].

Setting r=1 in (32) leads to

$$\sin^{-1}\sqrt{\sin\theta} - \theta = \sum_1^\infty \frac{\delta_n}{2n} \sin 2n\theta \quad (0 \le \theta \le \pi/2). \qquad (53)$$

Integration from $\theta = 0$ to $\theta = \pi/2$ now gives

$$\int_0^{\pi/2} \cos^{-1}\sqrt{\cos\theta}\, d\theta = \frac{\pi^2}{8} - \frac{1}{2}\sum_0^\infty \frac{\delta_{2n+1}}{(2n+1)^2}. \qquad (54)$$

Combining (47) and (54) we obtain the combinatorial identity

$$\sum_0^\infty \frac{(-1)^n\delta_n + \frac{1}{2}\delta_{2n+1} - 1}{(2n+1)^2} = 0. \qquad (55)$$

Using the expansion

$$\sin^{-1} x = \sum_0^\infty \frac{\delta_n x^{2n+1}}{2n+1} \quad (|x| \le 1)$$

and the formula

$$\int_0^{\pi/2} \sin^{2m}\theta\, d\theta = \frac{\pi}{2}\delta_m \quad (m = 0,1,2,\ldots.)$$

we easily deduce that

$$\int_0^{\pi/2} \sin^{-1}\sin^2\theta\, d\theta = \frac{\pi}{2}\sum_0^\infty \frac{\delta_n\delta_{2n+1}}{2n+1}. \qquad (56)$$

(47) and (56) now lead to the combinatorial identity

$$\frac{\pi}{2}\sum_0^\infty \frac{\delta_n\delta_{2n+1}}{2n+1} = \sum_0^\infty \frac{(-1)^n\delta_n}{(2n+1)^2}. \qquad (57)$$

4. THIRD SET OF SERIES AND INTEGRALS

Assuming that $z=re^{i\theta}$ $(0 \le r \le 1)$ in (3) and equating the real an imaginary parts on both sides we get

$$\sum_0^\infty \frac{(-1)^{n-1}\delta_n r^n \cos n\theta}{2n-1} =$$

$$\frac{1}{\sqrt{2}}\left[\sqrt{1+r^2+2r\cos\theta} + 1 + r\cos\theta\right]^{1/2}, \qquad (58)$$

$$\sum_0^\infty \frac{(-1)^{n-1}\delta_n r^n \sin n\theta}{2n-1} =$$

$$\frac{1}{\sqrt{2}}\left[\sqrt{1+r^2+2r\cos\theta} - 1 - r\cos\theta\right]^{1/2}. \qquad (59)$$

From these we obtain

$$\int_0^\pi \left[\sqrt{1+r^2+2r\cos\theta} + 1 + r\cos\theta\right]^{1/2} \cos n\theta\, d\theta =$$

$$= \frac{\pi(-1)^{n-1} r^n \delta_n}{\sqrt{2}\,(2n-1)} \quad (n \ge 1), \qquad (60)$$

$$\int_0^{\pi} \left[\sqrt{1+r^2+2r\cos\theta} - 1 - r\cos\theta\right]^{1/2} \sin n\theta \, d\theta =$$

$$= \frac{\pi(-1)^{n-1} r^n \delta_n}{\sqrt{2}\,(2n-1)} \quad (n\geq 1) \, , \tag{61}$$

$$\int_0^{\pi} \left[\sqrt{1+r^2+2r\cos\theta} + 1 + r\cos\theta\right]^{1/2} d\theta = \sqrt{2}\,\pi$$

(independent of r).

(62)

In the special case $r = 1$ we have

$$\int_0^{\pi/2} \sqrt{\cos\theta + \cos^2\theta}\, \cos 2n\theta \, d\theta = \frac{\pi(-1)^{n-1}\delta_n}{4(2n-1)} \quad (n\geq 1) \, , \tag{63}$$

$$\int_0^{\pi/2} \sqrt{\cos\theta - \cos^2\theta}\, \sin 2n\theta \, d\theta = \frac{\pi(-1)^{n-1}\delta_n}{4(2n-1)} \quad (n\geq 1) \, , \tag{64}$$

$$\int_0^{\pi/2} \sqrt{\cos\theta + \cos^2\theta}\, d\theta = \pi/2 \, . \tag{65}$$

Setting $\theta = \pi/2 - \phi$ in the three foregoing sets of infinite series and definite integrals and making the required changes we obtain corresponding sets of infinite series and definite integrals. Most of these results, though with simple functions, seem to be new. They are not recorded in [3], [4] and [7].

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