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# A MODIFIED OPERATOR OF THE HANKEL TRANSFORMATION ON CERTAIN SPACES OF FUNCTIONS

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## ABSTRACT

In this papaer we study the image of a certain space of smooth functions by a modified operator of the Hankel transform defined by

$$S_{\eta,\alpha}\{f(x)\}(y) = y^{\alpha} \int_0^{\infty} x^{1-\alpha} J_{2\eta+\alpha}(xy)\, f(x)\, dx$$

For every $\eta, \alpha \in R$, this space of functions is equipped with a locally convex topology. These spaces with be denoted $D_{\eta,\alpha}$. We prove that $S_{\eta,\alpha}$ is an isomorphism from $D_{\eta,\alpha}$ onto a space of entire functions denoted by $G_{\eta,\alpha}$, if the three inequalities $\eta > -1$ $2\eta+\alpha \geq 0$, $2\eta+2\alpha \geq 0$ are true. The $S_{\eta,\alpha}$-transform is extended to the dual spaces $D'_{\eta,\alpha}$ and $G'_{\eta,\alpha}$. Finally we solve a functional equation involving the Bessel type operator

$$\Delta_{\eta,\alpha} = x^{-2\eta-1} D x^{2(2\eta+\alpha)+1} D\, x^{-2\eta-2\alpha}\ .$$

## RESUMEN

En este trabajo se estudia la imagen de algún espacio de funciones lisas mediante un operador modificado de Hankel definido como

$$S_{\eta,\alpha}\{f(x)\}(y) = y^{\alpha} \int_{\infty}^{\infty} x^{1-\alpha} J_{2\eta+\alpha}(xy)\, f(x)dx$$

Para cada $\eta, \alpha \in R$, este espacio de funciones es equipado con una topología convexa localmente. Estos espacios se denota por $D_{\eta,\alpha}$. Se demuestra que $S_{\eta,\alpha}$ es un isomorfismo de $D_{\eta,\alpha}$ "onto" un espacio de funciones enteras, denotado por $G_{\eta,\alpha}$, si las desiguales $\eta>-1$, $2\eta+\alpha \geq 0$, $2\eta+2\alpha \geq 0$ son ciertas. La transformada $-S_{\eta,\alpha}$ es extendida a espacios duales $D'_{\eta,\alpha}$ y $G'_{\eta,\alpha}$. Finalmente se resuelve una ecuación funcional que involucra el operador de tipo Bessel

$$\Delta_{\eta,\alpha} = x^{-2\eta-1} Dx^{2(2\eta+\alpha)+1} Dx^{-2\eta-2\alpha}$$

## 1. INTRODUCTION

The Hankel transformation defined by

$$F(y) = h_{\mu}\{f(x)\}(y) = \int_0^{\infty} xJ_{\mu}(xy)f(x)\, dx \qquad (1)$$

where $J_{\mu}$ is the Bessel function of the first kind and order $\mu$, and several of its variants have been extensively studied in the last years.

An inversion formula for this integral transformation was derived by G.N. Watson [10] in the following

Theorem 1 : If $\sqrt{x}f(x) \in L_1(0,\infty)$, $f(x)$ is of bounded variation in a neigborhood of the point $x=x_0>0, \mu \geq -\frac{1}{2}$ and $F(y)$ is defined by (1), then

$$\frac{1}{2}(f(x_0+0)+ f(x_0-0))=h^{-1}\{F(y)\}(x)=$$

$$= \int_0^{\infty} F(y)yJ_{\mu}(x_0y)\, dy$$

Another useful result is the following Parseval's formula :

Theorem 2 : If the functions $f(x)$ and $g(x)$ satisfy the conditions in Theorem 1 and if $F(y)$ and $G(y)$ denote their Hankel transforms of order $\mu \geq -\frac{1}{2}$, then

$$\int_0^\infty xf(x)g(x)dx = \int_0^\infty yF(y)G(y)dy$$

J.M. Méndez [7] extended the $h_\mu$-transformation to a space of generalized functions of slow growth.

He considered (together with the $h_\mu$-transform) the integral transformation defined by

$$H_\mu\{f(x)\}(y) = y \int_0^\infty J_\mu(xy)f(x)dx$$

closely connected with (1). $h_\mu$ and $H_\mu$ satisfy the mixed Parseval equation that can be deduced from Theorem 2 :

Theorem 3 : Let $\mu \geq -\frac{1}{2}$. If $\sqrt{x}f(x) \in L_1(0,\infty)$ and $\sqrt{y}G_2(y) \in L_1(0,\infty)$ then

$$\int_0^\infty f(x)g(x)dx = \int_0^\infty F_1(y)G_2(y)dy \quad , \qquad (2)$$

where $F_1(y) = h_\mu\{f(x)\}(y)$ and $G_2(y) = H_\mu\{g(x)\}(y)$.

Méndez introduced two new spaces $H_{\mu,1}$ and $H_{\mu,2}$ of testing functions with a structure similar to the space $H_\mu$ defined by A.H. Zemanian [11]. $h_\mu$ and $H_\mu$ are automorphism onto $H_{\mu,1}$ and $H_{\mu,2}$, respectively.

As a generalization of the Parseval's equation (2), Méndez defined the generalized $h'_\mu$-transformation $h'_\mu f$ for $f \in H'_{\mu,2}$ as the adjoint of the classical $H_\mu$-transform, so that,

$$\langle h'_\mu f, \psi\rangle = \langle f, H_\mu \psi\rangle, \text{ for every } \psi \quad H_{\mu,2} \quad .$$

Also, for every f $H'_{\mu,1}$, the $H'_\mu$ transforma- of f was defined by the relation

$$\langle H'_\mu f, \psi\rangle = \langle f, h_\mu \psi\rangle \text{ , for every } \psi \quad H_{\mu,1} \text{ .}$$

In a recent paper [3] we define a new generalized Hankel transformation following the ideas presented by L.S. Dube and J.N. Pandey [4]. We introduced a space $L_\mu^{\delta,\beta}$ of testing functions such that the function $xJ_\mu(xy)$ is in $L_\mu^{\delta,\beta}$ for every $y>0$. Then, we defined the $h'_\mu$-transform $h'_\mu f$ of f, for every $f \in (L_\mu^{\delta,\beta})'$, as

$$(h'_\mu f)(y) = \langle f(x), xJ_\mu(xy)\rangle \quad \text{, for every } y>0$$

We derived a distributional inversion formula for this transformation. Moreover, the generalized Abelian theorems due to Zemanian [12] were extended to the space $(L_\mu^{\delta,\beta})'$ of generalized functions.

In this paper, we consider the modified operator of the Hankel transform defined by

$$F(y) = S_{\eta,\alpha}\{f(x)\}(y) = y^\alpha \int_0^\infty x^{1-\alpha} J_{2\eta+\alpha}(xy)f(x)dx \qquad (3)$$

if $2\eta+\alpha \geq -\frac{1}{2}$. ($S_{\eta,\alpha}$ is a variant of the operator introduced by $I^{\eta,\alpha}$N. Sneddon [9]). In certain theoretical investigations is more convenient to use $S_{\eta,\alpha}$ instead of the operator $h_\mu$.

We study the image by $S_{\eta,\alpha}$ of a certain space of functions defined on $(0,\infty)$ having superiorly bounded support. Said space is endowed with a locally convex lopology for every $\eta$ and $\alpha$. We denoted these spaces by $D_{\eta,\alpha}$. We prove that the $S_{\eta,\alpha}$-transform is and isomorphism form $D_{\eta,\alpha}$ onto a certain space $G_{\eta,\alpha}$ of entire functions, for $2\eta+2\alpha \geq 0$, $2\eta+\alpha \geq 0$ and $\eta>-1$. The $S_{\eta,\alpha}$-transformation is extended to the dual spaces of $D_{\eta,\alpha}$ and $G_{\eta,\alpha}$. Moreover, a functional differential equation involving the Bessel type operator

$$\Delta_{\eta,\alpha} = x^{-2\eta-1} D_x x^{2(2\eta+\alpha)+1} D_x x^{-2\eta-2\alpha} \quad \text{is solved.}$$

Finally, we state an open problem related to this paper.

## 2. THE SPACES OF FUNCTIONS $D_{\eta,\alpha}$ AND $G_{\eta,\alpha}$ AND THE $S_{\eta,\alpha}$ TRANSFORMATION.

J.L. Griffith [6] proved a theorem analogous to Paley-Wiener Theorem concerning the ordinary Hankel transformation.
Through some changes of variables his theorem takes on the following form

Theorem 4 : Let

$$G(y) = \text{l.i.m. } y^\alpha \int_0^\infty x^{1-\alpha} J_{2\eta+\alpha}(xy)f(x)dx, \quad 2\eta+\alpha \geq -\frac{1}{2}$$

where $J_\mu$ is the Bessel function of the first kind and order $\mu$.

Then :

δ) $f(t)$ is zero almost all $t>A$, and

β) $t^{-\alpha+1/2} f(t)$ is in $L_2(0,A)$

when, and only when

i) $s^{\alpha}G(s)$ is analytic in s for $0\leq \arg s\leq\pi$, $|s|>\varepsilon>0$,

ii) $s^{-\alpha+1/2}G(s) = o(e^{A\,\mathrm{Im}\,s})$, as $|s|\to\infty$, $\mathrm{Im}\,s\geq 0$

iii) $G(u) = G(ue^{i\pi})$, $u>0$,

iv) $u^{-\alpha+1/2}G(u)$ is in $L_2(0,\infty)$, and

v) $|s^{-2\eta-2\alpha}G(s)| = o(1)$ as $s\to 0$

This statement is the background of this paper.

## 2.1. THE SPACES OF FUNCTIONS $D_{\eta,\alpha}$ (a) AND $D_{\eta,\alpha}$

Let a denote a positive real number and $\eta$ and $\alpha$ any real numbers. Then we define $D_{\eta,\alpha}(a)$ as the space of functions $\psi(x)$ which are defined and smooth on $0<x<\infty$, such that $\psi(x) = 0$ for $a<x<\infty$, and

$$\lim_{x\to 0} D^k(x^{-2\eta-2\alpha}\Delta^m_{\eta,\alpha}\psi(x))=0, \text{ for every } k,m\in N.$$

We define $\gamma^k_{\eta,\alpha}(\psi)=\sup_{x\in I}|\Delta^k_{\eta,\alpha}\psi(x)|$ for $k\in N$

where $\Delta_{\eta,\alpha} = x^{-2\eta-1}Dx^{2(2\eta+\alpha)+1}Dx^{-2\eta-2\alpha}$ and $I=(0,\infty)$.

We assign to $D_{\eta,\alpha}(a)$ the topology generated by the countable multinorm $\{\gamma^k_{\eta,\alpha}\}_{k\in N}$. $D_{\eta,\alpha}(a)$ is a Hausdorff space that satisfies the first axiom of countability. The dual space $D'_{\eta,\alpha}(a)$ consists of all continuos lineal functionals on $D_{\eta,\alpha}(a)$. The dual is a linear space to which we assign the weak topology generated by the seminorms $\{\xi_\psi\}_\psi$, when $\xi_\psi(f)=|\langle f,\psi\rangle|$ and $\psi$ varies through $D_{\eta,\alpha}(a)$.

If $0<a<b$, then $D_{\eta,\alpha}(a)\subset D_{\eta,\alpha}(b)$, and the topology of $D_{\eta,\alpha}(a)$ is identical to the topology induced in it by $D_{\eta,\alpha}(b)$. Hence, we can construct strict countably union space $D_{\eta,\alpha}= \bigcup_{a>0} D_{\eta,\alpha}(a)$.

## 2.2. THE SPACES OF FUNTIONS $G_{\eta,\alpha}$ (a) AND $D_{\eta,\alpha}$.

Let now $\eta$, $\alpha$, a be real numbers, with $a>0$ and $2\eta+\alpha\geq 0$. We define a topological linear space $G_{\eta,\alpha}(a)$ as follows. $\phi$ is a member of $G_{\eta,\alpha}(a)$ if and only if

a) $s^{-2\eta-2\alpha}\phi(s)$ is an even entire function

b) for every $k\in N$, $|s^{k-\alpha}\phi(s)|<C_k e^{a|\mathrm{Im}\,s|}$ for $|s|$ enough large, $C_k$ being a positive constante depending on $\phi$.

c) $\int_0^\infty s^{2\eta+1+k}\phi(s)ds = 0$, for $k = 0,2,4,\ldots$ and

d) $|s^{-2\eta-2\alpha}\phi(s)| = O(1)$ as $s\to 0$.

$G_{\eta,\alpha}(a)$ is endowed with the topology generated by the multinorm $\{w^k_{\eta,\alpha}\}_{k\in N}$ where

$$w^k_{\eta,\alpha}(\phi) = \sup_{s\in C} |s^{k-\alpha}\phi(s)|\, e^{-a|\mathrm{Im}\,s|}, \text{ for every } k\in N$$

(Note that $w^k_{\eta,\alpha}(\phi)<\infty$, for every $\phi\in G_{\eta,\alpha}(a)$ and $k\in N$).

$G_{\eta,\alpha}(a)$ is a locally convex, Haudorff topological vector space. The dual space of $G_{\eta,\alpha}(a)$ is denoted by $G'_{\eta,\alpha}(a)$ and is equipped with the weak topology.

Moreover, if $0<a<b$, then $G_{\eta,\alpha}(a)\subset G_{\eta,\alpha}(b)$ and the topology of $G_{\eta,\alpha}(a)$ is stronger than the one induced in it by $G_{\eta,\alpha}(b)$. This allows to define the countably union space.

## 2.3. THE MODIFIED OPERATOR $S_{\eta,\alpha}$.

The modified operator $S_{\eta,\alpha}$ of Hankel transform is given by (3).

The main result of this paper is the next one.

THEOREM 5 : Let $\eta$, $\alpha$, a be real numbers, with $a>0$, $2\eta+\alpha\geq 0$, $2\eta+2\alpha\geq 0$ and $\eta>-1$. $S_{\eta,\alpha}$ is an algebraical and topological isomorohism of $D_{\eta,\alpha}(a)$ onto $G_{\eta,\alpha}(a)$.

PROOF : Let $\psi$ be in $D_{\eta,\alpha}(a)$. We denote

$$\phi(s) = S_{\eta,\alpha}\{\psi(x)\}(s) = s^{\alpha}\int_0^\infty t^{1-\alpha}\psi(t)J_{2\eta+\alpha}(st)dt =$$

$$= s^{2\eta+2\alpha}\int_0^\infty \psi(t)t^{2\eta+1}b_{2\eta+\alpha}(ts)dt$$

where $b_\nu(z)=z^{-\nu}J_\nu(z)$. Since $b_\nu$ is an even entire function, then $s^{-2\eta-2\alpha}\phi(s)$ is also an even entire function.

The following operational ruler holds :

$$S_{\eta,\alpha}\{\Delta^k_{\eta,\alpha}\psi(t)\}\ (s) = (-s^2)^k S_{\eta,\alpha}\{\psi(x)\}\ (s) \qquad (4)$$

for every $k\in N$ and $\psi\in D_{\eta,\alpha}(a)$.

The function $\psi(t)$ satisfies the conditions $\delta$) and $\beta$) given in Theorem 4 and in virtue of (4) we get

$$|s^{-\alpha+5/2}S_{\eta,\alpha}\{\psi(t)\}(s)| =$$

$$|s^{-\alpha+1/2}S_{\eta,\alpha}\{\Delta_{\eta,\alpha}\psi(t)\}\ (s)|\leq C\exp(a|\mathrm{Im}\ s|)$$

for $|s|$ enough large. Hence, the condition (b) is satisfied for k=0,1,2.

Also

$$|s^{-\alpha+9/2}S_{\eta,\alpha}\{\psi(t)\}(s)|=$$

$$|s^{-\alpha+1/2}S_{\eta,\alpha}\{\Delta^2_{\eta,\alpha}\psi(t)\}\ (s)|\leq C'\exp(a|\mathrm{Im}\ s|)$$

for $|s|$ enough large. Hence the condition b) is fulfilled for k=3,4.

By induction of k we can see that $\phi(s)$ satisfies b).

By invoking the inversion formula we can obtain

$$\psi(t) = t^{2\eta+2\alpha}\int_0^\infty b_{2\eta+\alpha}\ (ts)\ s^{2\eta+1}\ \phi(s)ds$$

Differentiation under the integral sign leads to

$$D_t^k(t^{-2\eta-2\alpha}\Delta^m_{\eta,\alpha}\psi(x)) =$$

$$\int_0^\infty s^{2\eta+1+k+2m}b^{(k)}_{2\eta+\alpha}\ (st)\ \phi(s)ds$$

An application of Lebesgue's dominated convergence Theorem and by letting $t\to0$ allows to write

$$0=\lim_{t\to0} D^k(t^{-2\eta-2\alpha}\Delta^m_{\eta,\alpha}\psi(x))=$$

$$b^{(k)}_{2\eta+\alpha}(0)\int_0^\infty s^{2\eta+1+2m+k}\ \phi(s)ds$$

(note that the differentiation under the integral sign is justified).

Since $b^{(k)}_{2\eta+\alpha}(0)\neq0$, for $k=0,2,4,\ldots$, then

$\int_0^\infty s^{2\eta+1+k}\phi(s)ds = 0$, for every nonnegative integer k. Therefore, $\phi$ satisfies the condition c).

The property d) is equal to v) in Theorem 4.

On the other hand, let $\phi$ be in $G_{\eta,\alpha}(a)$. We denote

$$\psi(t) = t^{2\eta+2\alpha}\int_0^\infty s^{2\eta+1}B_{2\eta+\alpha}(st)\psi(s)ds$$

Again, by differentiation under the integral sign and in virtue of the condition c),

$$\lim_{t\to0} D^k(t^{-2\eta-2\alpha}\Delta^m_{\eta,\alpha}\psi(t))=0,\ \text{for}\ k,\ m\in N.$$

Moreover, the contitions a)-d) imply the properties i)-v) in Theorem 4, hence $\psi(t)=0$, for $t>a$.

Therefore $\psi(t)\in D_{\eta,\alpha}(a)$.

To complete the proof of this theorem we now prove the continuity of the mappings

$$S_{\eta,\alpha}\ \text{and}\ S^{-1}_{\eta,\alpha}\ .$$

Assume $k\in N$, $\psi\in D_{\eta,\alpha}(a)$ and denote

$$\phi(s) = S_{\eta,\alpha}\{\psi(t)\}\ (s).$$

If $2\eta+\alpha+k = 2p$ for a certain nonnegative integer p, then, by applying (4) one has

$$s^{k-\alpha}\phi(s) = s^{k+\alpha+2\eta}$$

$$\int_0^\infty t^{2\eta+1} b_{2\alpha+\eta}(ts)\psi(t)dt =$$

$$= \int_0^a t^{2\eta+1} b_{2\eta+\alpha}(st)\Delta^p_{\eta,\alpha}\,\psi(t)dt$$

Hence, since $|\exp(-a|\mathrm{Im}\ s|)b_\nu(xs)| \leq C_\nu$, for every $s\in C$ and $x\in I$, where $C_\nu$ is a suitable positive constant, we get

$$\sup_{s\in C} |s^{k-\alpha}\phi(s)\ |\exp(-a|\mathrm{Im}\ s|) \leq M \sup_{t\in I}|\Delta^p_{\eta,\alpha}\ \psi(t)| \quad (5)$$

where M is a positive number.

Moreover, if $2\eta+\alpha+k$ is not even it is enough to note that

$$|s|^{2\eta+\alpha+k} \leq |s|^p + |s|^j$$

where p and j are even nonnegative integers such that $p \leq k+2\eta+\alpha \leq j$, proving in this case an inequality similar to (5).

Therefore $S_{\eta,\alpha}: D_{\eta,\alpha}(a) \to G_{\eta,\alpha}(a)$ is a continuous linear mapping.

Assume now $\phi\in G_{\eta,\alpha}(a)$ and let

$\psi(t) = S^{-1}_{\eta,\alpha}\{\phi(s)\}(t)$. We can write

$$\Delta^k_{\eta\ \alpha}\psi(t) = t^{2\eta+2\alpha} \int_0^\infty s^{2\eta+1} b_{2\eta+\alpha}(st)(-s^2)^k\phi(s)ds$$

for every $k\in N$. By taking into account that $b_\nu(z)$ is bounded on $(0,\infty)$, we get

$$|\Delta^k_{\eta,\alpha}\psi(t)| \leq K \int_0^\infty \frac{s^{2\eta+1+\alpha}}{1+s^p}\{s^{2k-\alpha+2p}+s^{2k-\alpha}\}|\phi(s)|ds \leq$$

$$\leq K'\{\sup_{t\in I}|s^{2k-\alpha+p}\phi(s)| + \sup_{t\in I}|s^{2k-\alpha}\phi(s)|\} \leq$$

$$\leq K'\{\sup_{s\in C}|s^{2k-\alpha+p}\phi(s)|\exp(-a|\mathrm{Im}\ s|) +$$

$$+ \sup_{s\in C}|\ s^{2k-\alpha}\phi(s)\ \exp(-a|\mathrm{Im}\ s|)\}$$

where p is nonnegative integer such that $p-(2\alpha+\eta+1)>1$, and K and K' are suitable positive constants.

Hence, the mapping $S^{-1}_{\eta,\alpha} = S_{\eta,\alpha}$ :

$G_{\eta,\alpha}(a) \to D_{\eta,\alpha}(a)$ is continuous.

This theorem can be extended to the respective union spaces.

Theorem 6 : This transformation $S_{\eta,\alpha}$ is an isomorphism of $D_{\eta,\alpha}$ on to $G_{\eta,\alpha}$, its inverse is $S^{-1}_{\eta,\alpha} = S_{\eta,\alpha}$, provided that $\eta>-1$, $2\eta+\alpha \geq 0$ and $2\eta+2\alpha \geq 0$.

## 3. A GENERALIZED TRANSFORMATION $S'_{\eta,\alpha}$

Let $u\in D'_{\eta,\alpha}$. We define the generalized transformation $S'_{\eta,\alpha}u$ of u as the adjoint of the classical transform, so that

$$\langle S'_{\eta,\alpha}u,\phi\rangle = \langle u,S_{\eta,\alpha}\phi\rangle$$

for every $\phi\in G_{\eta,\alpha}$.

Also, for every $v\in G'_{\eta,\alpha}$ the generalized transformation $S'_{\eta,\alpha}v$ of v is defined by

$$\langle S'_{\eta,\alpha}v,\psi\rangle = \langle v,S_{\eta,\alpha}\psi\rangle$$

for each $\psi\in D_{\eta,\alpha}$.

The following statement can be easily proved by invoking Theorem 6.

Theorem 7 : The operator $S'_{\eta,\alpha}$ is an isomorphism of $D'_{\eta,\alpha}$ on to $G'_{\eta,\alpha}$, provided that $\eta>-1$, $2\eta+\alpha \geq 0$ and $2\eta+2\alpha \geq 0$.

## 4. OPERATIONAL CALCULUS

We now prove a generalized operational ruler analogous to the classical one (4). This operational ruler is useful in solving of certain generalized differential equations.

Theorem 8 : Let u be in $D'_{\eta,\alpha}$, then

$$S'_{\eta,\alpha}(\Delta^*_{\eta,\alpha}u) = -s^2 S'_{\eta,\alpha}u$$

where $\Delta^*_{\eta,\alpha}$ denotes the adjoint operator of $\Delta_{\eta,\alpha}$.

PROOF: For every $\psi$ in $D_{\eta,\alpha}$, one has

$$<\Delta^*_{\eta,\alpha}u,\psi> = <u,\Delta_{\eta,\alpha}\psi>$$

By using the operational ruler (4), we get

$$<S'_{\eta,\alpha}(\Delta^*_{\eta,\alpha}u),\psi> = <u,\Delta_{\eta,\alpha}S_{\eta,\alpha}\psi = <u,S_{\eta,\alpha}(-s^2)\psi> =$$

$$<-s^2 S'_{\eta,\alpha}u,\psi> \quad , \text{ for every } \psi \in G_{\eta,\alpha} .$$

We consider the differential equation of the form

$$P(\Delta^*_{\eta,\alpha})u = g \tag{6}$$

where P is a polynomial and $2\eta+2\alpha \geq 0$, $2\eta+\alpha \geq 0$ and $\eta > -1$. g is in $D'_{\eta,\alpha}$. A formal application of the $S'_{\eta,\alpha}$-transform leads to

$$S'_{\eta,\alpha}u = \frac{1}{P(-s^2)} S'_{\eta,\alpha}g$$

and invoking the inversion formula we get

$$u = S'_{\eta,\alpha}\{ \frac{1}{P(-s^2)} S'_{\eta,\alpha}g\} \tag{7}$$

We now introduce the following space of functions

$$A_{\eta,\alpha,P} = \{\psi \in C^\infty(I) \ / \ \frac{1}{P(-s^2)} S_{\eta,\alpha}\psi \in G_{\eta,\alpha} \}$$

$A_{\eta,\alpha,P}$ is equipped with the topology induced in it by $G_{\eta,\alpha}$ (namely, a sequence $\{\psi_\nu\}_{\nu \in N}$ converges to $\psi$, as $\nu\to\infty$ in $A_{\eta,\alpha,P}$ if, and only if, $\frac{1}{P(-s^2)} S_{\eta,\alpha}\psi_\nu$

converges to $\frac{1}{P(-s^2)} S_{\eta,\alpha}\psi$ as $\nu \to \infty$,

Note that $A_{\eta,\alpha,P}$ is algebraically contained in $D_{\eta,\alpha}$.

The functional u given by (7) is in $A_{\eta,\alpha,P}$. In effect, let $\{\psi_\nu\}_{\nu \in N}$ be a sequence in $A_{\eta,\alpha,P}$ which converges to $\psi \in A_{\eta,\alpha,P}$, as $\nu\to\infty$. Then

$$<u \ \psi_\nu> = <g,S_{\eta,\alpha}\{\frac{1}{P(-s^2)} S_{\eta,\alpha}\psi_\nu\}\}>$$

$$>_{\nu \to \infty} < g,S_{\eta,\alpha}\{ \frac{1}{P(-s^2)} S_{\eta,\alpha}\psi\}> = <u,\psi> .$$

since $\frac{1}{P(-s^2)} S_{\eta,\alpha}\psi_\nu \to \frac{1}{P(-s^2)} S_{\eta,\alpha}\psi$ as

$\nu\to\infty$ in $G_{\eta,\alpha}$, $S_{\eta,\alpha}$ is

a continuous mapping from $G_{\eta,\alpha}$ onto $D_{\eta,\alpha}$ and

$g \in D'_{\eta,\alpha}$.

Moreover, one can show that :

$$<P(\Delta^*_{\eta,\alpha})u,\psi> = <u,P(\Delta_{\eta,\alpha})\psi> =$$

$$= <g,S_{\eta,\alpha}\{\frac{1}{P(-s^2)} S_{\eta,\alpha}\{P(\Delta_{\eta,\alpha})\psi\}\}> =$$

$$= <g,\psi>, \text{ for every } \psi \in A_{\eta,\alpha,P}.$$

Hence, u is a solution for (6) in $A'_{\eta,\alpha,P}$

REMARK : If $\eta=\mu$, $\eta=-\mu$, $S_{\eta,\alpha}$ coincides with the Hankel-Schwartz transform (see [1]). In that case, the results obtained in this paper reduce to that presented in our previous paper [2].

An open problem.

Thes spaces $D_{\eta,\alpha}$ are different to the space D(I) introduced by L. Schwartz [8]. To describe the behaviour of the operator $S_{\eta,\alpha}$ on D(I) is an open problem.

## REFERENCES


[1] ALTENGURG, G.: "*Bessel transformationen in Raumen von Grund-functionen uber dem intervall Ω=[0,∞] und derem dual raumen*". Math. Nach. 108(1982), 197-218.

[2] BENTANCOR, J.J. : "*The Hankel-Schwartz transformation of functions of compact support*". Preprint N°5, Department of Math. Anal. University of La Laguna, Tenerife, Spain, 1987.

[3] BENTANCOR, J.J. : "*Sobre una transformación generalizada de Hankel*". Preprint N°8, Department of Math. Anal., University of La Laguna, Tenerife, Spain, 1987.

[4] DUBE, L.S. and PANDEY, J.N. : "*On the Hankel transform of distributions*". Tohoku Math. J. 27 (1975), 337-354.

[5] FENYO, I. : "*On the Hankel transformation of the Schwartz distributions*", To appear in Stud. Hung., Vol. 22.

[6] GRIFFITH, J.L. : "*Hankel transform of functions zero autside a finite interval*". J. Proc. Roy. New South Wales, 89(1955), 109-115.

[7] MENDEZ, J.M. : "*A mixed Parseval equation and the generalized Hankel transformation*". To appear in Proc. Amer. Math. Soc.

[8] SCHWARTZ, L. : "*Theorie des distributions*". Hermann, Paris, 1966.

[9] SNEDDON, I.N. : "*The use of integral transforms*", McGraw Hill Publishing Co.Ltd. New Delhi, 1974.

[10] WATSON, G.N. : "*A Treatise on the Theory of Bessel functions*". Cambridge Univ. Press. Cambridge, England, 1966.

[11] ZEMANIAN, A.H. : "*A distributional Hankel transformation*". SIAM Appl. Math. 14(1966), 561-576.

[12] ZEMANIAN, A.H. : "*Some Abelian theorems for the distributional Hankel and K-transformations*", SIAM J. Appl. Math., Vol. 14, N°6, (1966) 1255-1265.

[13] ZEMANIAN, A.H. : "*Generalized Integral Transformation*", Interscience Publishers, New York, 1968.


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