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NOTA TECNICA

ON THE DIFFERENTIATION OF VECTOR VALUED COMPOSITE LIPSCHITZ CONTINUOUS FUNCTIONS

K.P. Gilliam
Department of Economics/Finance
Kennesaw College
Marietta
Georgia, U.S.A.

C.D. Vinante
Facultad de Ingeniería
División de Postgrado
Universidad del Zulia
Maracaibo, Venezuela

ABSTRACT

It is well known that there exists in economic what is referred to as "kinked" demand curves. That is, curves that are Lipschitz continuous. This paper presents a technical theorem on the existence of partial derivatives (and the applicability of the chain rule to compute them) of certain class of composite Lipschitz continuous vector valued functions that appear in Mathematical Economics.

RESUMEN

Es bien conocido en economía la existencia de curvas de demanda "kinked". Esto es, curvas que son Lipschitz contínuas. En este trabajo se presenta un teorema sobre la existencia de derivadas parciales (y la aplicabilidad de la regla de la cadena para calcularla) de una clase de funciones vectoriales compuestos y Lipschitz contínuos.

The purpose of this article is to present a theorem on the existance of partial derivatives of certain class of composite Lipschitz continuous vector valued functions. Lipschitz functions often appear in economics when the demand curve is "kinked". That is, functions that may be only piecewise differentiable. A phenomena not uncommon in the real world.

In order to simplify the proof of the theorem, we will present first two preliminary lemmas and the definition of a Lipschitz continuous function for the purpose of completeness.

Let the vector $p \in R^n$ represents prices, the function $m = M(p)$ represents income and $f(p,m)$ the vector valued demand function.

Definition 1.1. A function M that satisfies the following inequality,

$$|M(p_1)-M(p_2)| \leq K\,|p_1 - p_2| \quad , \qquad (1.1)$$

for all (p_1,p_2) in a region D, and K a fixed positive number, is said to be a Lipschitz continuous function in D.

Lemma 1.1. Let M be a Lipschitz continuous function on U, an open subset of R^1. Let E be a Borel set in R^1 with measure zero, then $m(E)=0$ implies that $M' = 0$ a.e., on $M^{-1}(E)$.

Proof.: Let g be any function which has derivative (finite of infinite) on a set E with $m(g(E))=0$, then $g' = 0$ a.e., on E. (Ref. 1).

Since M is Lipschitz continuous on U it is absolutely continuous on U and M' exists a.e. on U. Let F be the set of measure zero on which M' does not exists. Then $M' = 0$ a.e., on $M^{-1}(E) - F$. Hence a.e., on $M^{-1}(E)$.

We will now generalize lemma 1.1 as follow:

Lemma 1.2. Let M be Lipschitz continuous on the open set $\Omega \subset R^n$. Let E be a Borel set in R^1 with measure zero. Then on $M^{-1}(E)$ we have $dM(p)=0$ a.e., where dM is the Frechet derivative of M.

Proof: Fix a parallel to the p_i axis (i.e., fix all but the ith coordinate). Then M as a function of a single variable defined in this line satisfies the same Lipschitz condition as M on Ω, so by the previous lemma

$\frac{\partial M}{\partial p} = 0$ a.e. on $M^{-1}(E)$ intersected with any such a line. Since $\frac{\partial M}{\partial p_i} < K$ a.e., where K is the Lipschitz constant, $\frac{\partial M}{\partial p_i}$ (defined a.e., in Ω) is integrable over E.

By Fubini's Theorem

$$\int \left|\frac{M}{p_i}\right| dx = \int dx_1 \ldots dx_{i-1} \ldots dx_n \int \left|\frac{\partial M}{\partial p_i}\right| dx_i = 0. \qquad (1.2)$$

Hence, $\frac{\partial M}{\partial p_i} = 0$ a.e., on E. Since M is differentiable a.e., see (Ref. 2), the lemma follows.

We will now present the theorem that proves the existance of partial derivatives of composite Lipschitz vector valued functions.

Theorem 1.1. Let f(p,m) be Lipschitz continuous on $\Omega x R^1$, where Ω is an open subset of R^n; let B_p and B_m be Borel sets of measure zero in Ω and R^1 respectively. (Ref. 3). Assume that:

(i) df exists on $(\Omega x R^1) - (A_p \cup A_m)$,

(ii) $\frac{df}{dp_i}$ exists on $(\Omega x R^1) - A_p$, i= 1,2,...n,

where $A_p = (B_p x R^1)$,

$A_m = (\Omega x B_m)$.

If M= M(p) is a Lipschitz continuous function on Ω, then for the composite function we have:

$$\frac{\partial foM}{\partial p_i}(p) = \frac{\partial f}{\partial p_i} oM(p) + \frac{\partial f}{\partial m} oM(p) \frac{\partial M}{\partial p_i}(p) \text{ a.e.,} \quad (1.3)$$

where $\frac{\partial f}{\partial m}$ o M(p) $\frac{\partial M}{\partial p_i}(p)$ is understood to be zero if $\frac{\partial M}{\partial p_i} = 0$.

It should be noted that foM means the composite function foM(p)= f(p,M(p)).

Proof: If $p \in (\Omega - B_p) \cap (\Omega - M^{-1}(B_m))$ then df exists at (p,M(p)) and dM exists a.e., on this set. (Ref. 2). Then by the ordinary chain rule the desired formula for $\frac{\partial foM}{\partial p_i}$ (p) holds for almost all p in this set. Since $B_p \cap M^{-1}(B_m)$ is a set of measure zero it remains only to prove the theorem for the following case, $p \in (\Omega - B_p) \cap (M^{-1}(B_m))$.

By lemma 2.1, $M'(p) = 0$ a.e., on this set so by our notational convention we must prove that if $M'(p) = 0$ then $\frac{\partial (foM)}{\partial p_i}(p) = \frac{\partial f}{\partial p_i} oM(p)$.

Let e_i be the vector with the i component equal to 1 and all other components 0.

Then

$$f(p+te_i, M(p+te_i)) = f(p+te_i, M(p) + O(t)), \quad (1.4)$$

and by Lipschitz continuity

$$f(p+te_i, M(p+te_i)) = f(p+te_i, M(p)) + O(t). \quad (1.5)$$

Therefore

$$\frac{\partial (foM)}{\partial p_i}(p) = \lim_{t \to 0} (f(p+te_i, M(p+te_i)) - f(p,M(p)))/t$$

$$= \lim_{t \to 0} (f(p+te_i, M(p)) - f(p,M(p)))/t \quad (1.6)$$

$$= \frac{\partial f}{\partial p_i}(p,m) \Big|_{m=M(p)}$$

and the theorem is proved.

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