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A FUNCTIONAL RELATION INVOLVING ψ- FUNCTION

Shyam L. Kalla
División de Postgrado
Facultad de Ingeniería
Universidad del Zulia
Maracaibo, Venezuela

Bader Al-Saqabi
Department of Mathematics,
Kuwait University
Kuwait

ABSTRACT

The object of the present paper is to prove the following functional relation :

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \; {}_2F_1(-\mu,\lambda;n+\lambda;\, -\frac{x}{a})$$

$$= \sum_{k=0}^{\infty} \frac{(-\mu)_k}{k!} (-\frac{x}{a})^k \left[\psi(\lambda+k) - \psi(\lambda-\nu+k)\right]$$

$$Re(\lambda) > Re(\nu) \geq 0, \; |\arg \frac{x}{a}| < \pi$$

by means of the Riemann-Liouville operator of fractional calculus. Some particular cases are mentioned. The results given earlier by Ross and Kalla follow as special cases of our main result. Some results are numerically tabulated for illustration and verification.

RESUMEN

El objeto del presente trabajo es demostrar la siguiente relación funcional

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \; {}_2F_1(-\mu,\lambda;n+\lambda;\, -\frac{x}{a})$$

$$= \sum_{k=0}^{\infty} \frac{(-\mu)_k}{k!} (-\frac{x}{a})^k \left[\psi(\lambda+k) - \psi(\lambda-\nu+k)\right]$$

$$Re(\lambda) > Re(\nu) \geq 0, \; |\arg \frac{x}{a}| < \pi$$

por medio del operador de Riemann - Liouville del cálculo fraccional. Se mencionan algunos casos particulares. Los resultados dados anteriormente por Ross y Kalla-Ross resultan casos particulares del resultado principal. Se dan algunas tablas numéricas para ilustración y verificación.

1. INTRODUCTION

In a recent paper [11] Ross has proved the following result

$$\ln 4 = \sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{n\, 2^n\, n!} \tag{1}$$

by means of fractional calculus. In a follow-up work Kalla to Ross [6] have given the functional relation

$$\psi(\lambda) - \psi(\lambda-\nu) = \frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \tag{2}$$

$$Re(\lambda) > Re(\nu) \geq 0$$

In this paper we prove the following more general functional relation

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \; {}_2F_1(-\mu,\lambda;n+\lambda;- \frac{x}{a})$$

$$= \sum_{k=0}^{\infty} \frac{(-\mu)_k}{k!} (- \frac{x}{a})^k \left[\psi(\lambda+k) - \psi(\lambda-\nu+k)\right] \quad (3)$$

$$Re(\lambda) > Re(\nu) \geq 0, \ |\arg \frac{x}{a}| < \pi$$

by means of Riemann-Liouville operator of fractional calculus. [9,10]

${}_2F_1(\alpha,\beta;r;z)$ is the Gauss'hypergeometric function and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the di-Gamma function [4,7]. We observe that for μ=0,(3) reduces to (2).

The integral

$$\frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)dt, \ Re(\nu) \geq 0 \quad (4)$$

is called the Riemann-Liouville operator of fractional calculus. The operator (4) is denoted by $I_\nu f$ [12] or $D_x^{-\nu} f(x)$ [2]. We shall follow the second notation.

Some particular cases for special values of the parameters are mentioned. For illustration and verification some numerical results are tabulated.

2. THE FUNCTIONAL RELATION

We start with the integral [8]

$$\frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} t^{\lambda-1} (t+a)^\mu \, dt$$

$$= \frac{a^\mu x^{\nu+\lambda-1}\Gamma(\lambda)}{\Gamma(\nu+\lambda)} \ {}_2F_1 (-\mu,\lambda;\nu+\lambda; - \frac{x}{a}) \quad (5)$$

$$Re(\lambda) > Re(\nu) \geq 0 , \ |\arg \frac{x}{a}| < \pi$$

The integral on the left above is differentiated with respect to the parameter according to Leibnitz's rule. The right hand side is also differentiated with respect to λ, getting

$$\frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} \ln t \ t^{\lambda-1} (t+a)^\mu \, dt$$

$$= \frac{(\ln x)\Gamma(\lambda)x^{\lambda+\nu-1}a^\mu}{\Gamma(\lambda+\nu)} \ {}_2F_1(-\mu,\lambda;\nu+\lambda; - \frac{x}{a})$$

$$+ \frac{a^\mu x^{\nu+\lambda-1}\Gamma(\lambda)}{\Gamma(\nu+\lambda)} \sum_{k=0}^{\infty} \frac{(-\mu)_k(\lambda)_k}{(\nu+\lambda)_k \, k!} (- \frac{x}{a})^k \ \times$$

$$\left[\psi(\lambda+k)-\psi(\nu+\lambda+k)\right] \quad (6)$$

For abbreviation, we denote the right side above by $G(x,\lambda,\nu,\mu,a)$ hence the result (6) can be written as

$${}_0D_x^{-\nu}\left[\ln t \ t^{\lambda-1}(t+a)^\mu\right] = G(x,\lambda,\nu,\mu,a) \quad (7)$$

Because of the analyticity and continuity at $\nu = 0$, we can interchange the roles of $-\nu$ and ν. Hence, for differentiation of $\ln t \ t^{\lambda-1}(t+a)^\mu$ to an arbitrary order ν we have

$${}_0D_x^{\nu} \left[\ln t \ t^{\lambda-1}(t+a)^\mu\right] = G(x,\lambda,-\nu,\mu,a) \quad (8)$$

Now we consider the following integral equation of the Volterra type

$$\frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t) \, dt = x^{\lambda-1} \ln x \ (x+a)^\mu$$

$$Re(\lambda) > Re(\nu) \geq 0 , \ |\arg \frac{x}{a}| < \pi \quad (9)$$

The Riemann-Liouville integral (9), can be solved by Laplace transform because of its convolution type. However, we proceed to solve it by the use of the operators of fractional calculus. The use of the operators of fractional integration to obtain new functional relation shows its great power to obtain result in an elegant form by the systematic use of simple notations.

In the notation of the fractional calculus (9) can be written as

$$ {}_0D_x^{-\nu} f(x) = x^{\lambda-1} \ln x\, (x+a)^{\mu} \qquad (10)$$

Operating on both sides with ${}_0D_x^{\nu}$ gives

$$ f(x) = {}_0D_x^{\nu} \left(x^{\lambda-1} \ln x (x+a)^{\mu}\right) \qquad (11)$$

Using (8) we can write the solution of the integral equation (9) as

$$ f(x) = \frac{\ln x\,\Gamma(\lambda)\, x^{\lambda-\nu-1} a^{\mu}}{\Gamma(\lambda-\nu)}\ {}_2F_1(-\mu,\lambda;\lambda-\nu;-\frac{x}{a})$$

$$ + \frac{a^{\mu} x^{\lambda-\nu-1}\Gamma(\lambda)}{\Gamma(\lambda-\nu)} \sum_{k=0}^{\infty} \frac{(-\mu)_k (\lambda)_k (-\frac{x}{a})^k}{(\lambda-\nu)_k\, k!} \times$$

$$ \left[\psi(\lambda+k) - \psi(\lambda-\nu+k)\right] \qquad (12)$$

Now to verify this result we substitute (12) in (9) in terms of the argument t.

We write a series expansion for ln t as follows

$$ t = x + t - x = x(1 + \frac{t-x}{x}) \qquad (13)$$

where x and t are real and x>0. Then

$$ \ln t = \ln x + \ln(1 + \frac{t-x}{x}) \qquad (14)$$

For $\frac{t-x}{x} < 1$, we can expand $\ln(1+\frac{t-x}{x})$ into a Taylor's series expansion. Thus,

$$ \ln t = \ln x - \sum_{n=1}^{\infty} \frac{(x-t)^n}{n x^n} \qquad (15)$$

where the interval of convergence is $0 < t \leq 2x$. Evaluating the corresponding integrals we get the result (3) after little simplification.

3. SPECIAL CASES

(i) For $\mu = 0$ our main result (3) reduces to (2), a result recently given by [6], which itself is a generalization of (1).

(ii) For $\mu = m$, (m is a positive integer), (3) reduces to

$$ \frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}\ {}_2F_1(-m,\lambda;n+\lambda;-\frac{x}{a})$$

$$ = \sum_{k=0}^{m} \frac{(-m)_k}{k!} (-\frac{x}{a})^k \left[\psi(\lambda+k) - \psi(\lambda-\nu+k)\right] \qquad (16)$$

$$ \mathrm{Re}(\lambda) > \mathrm{Re}(\nu) \geq 0, \quad |\arg \frac{x}{a}| < \pi$$

Here ${}_2F_1$ represents a polynomial of degree m. For m=1, (16) becomes

$$ \frac{\Gamma(\lambda)}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)} \left[\frac{\lambda(a+x)+an}{a(\lambda+n)}\right]$$

$$ = \left[\psi(\lambda) - \psi(\lambda-\nu)\right] + (\frac{x}{a}) \left[\psi(\lambda+1) - \psi(\lambda-\nu+1)\right] \qquad (17)$$

Further for $\lambda=1$ and $\nu = \frac{1}{2}$ (17) becomes

$$ \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{2}+n)}{n\Gamma(1+n)} \left[\frac{a+x+an}{a(1+n)}\right] = (1+\frac{x}{a})\ln 4 - \frac{x}{a} \qquad (18)$$

and for $\lambda=1$, $\nu = \frac{1}{3}$, by using [3] we get

$$ \frac{1}{\Gamma(\frac{1}{3})} \sum_{n=1}^{\infty} \frac{\Gamma(n+\frac{1}{3})}{n\Gamma(n+1)} \left[\frac{(a+x)+an}{a(n+1)}\right] \qquad (19)$$

$$= -\frac{\sqrt{3}}{6}\pi(1+\frac{x}{a}) + \frac{3}{2}(\ln 3(1+\frac{x}{a})) - \frac{x}{2a}$$

And for $\lambda=\nu+1$ (17) becomes

$$\sum_{n=1}^{\infty} \frac{\nu}{n(\nu+n)} \left[\frac{(\nu+1)(a+x)+an}{a(\nu+n+1)} \right] \quad (20)$$

$$= (\psi(\nu+1)+\gamma)\ (\frac{a+x}{a}) - \frac{x\nu}{a(\nu+1)}$$

For $x=a$ (20) takes the following form

$$\sum_{n=1}^{\infty} \frac{1}{(\nu+n)(\nu+n+1)} = \frac{1}{\nu+1} \quad (21)$$

(iii) $\mu = -j$ (j is a positive integer)

The result (3) becomes

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum_{n=1}^{\infty} \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}\ {}_2F_1(j,\lambda;n+\lambda;-\frac{x}{a})$$

$$= \sum_{k=0}^{\infty} \frac{(j)_k}{k!} (-\frac{x}{a})^k \cdot [(\psi+k) - \psi(\lambda-\nu+k)] \quad (22)$$

If $j=1$, (22) becomes

$$\frac{\Gamma(\lambda)}{\Gamma(\nu)} \sum \frac{\Gamma(\nu+n)}{n\Gamma(\lambda+n)}\ {}_2F_1(1,\lambda;n+\lambda;-\frac{x}{a})$$

$$= \sum_{k=0}^{\infty} (-\frac{x}{a})^k [\psi(\lambda+k) - \psi(\lambda-\nu+k)] \quad (23)$$

In (23) if we set $\lambda=1$, $\nu = \frac{1}{2}$ we have,

$$\frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{\Gamma(\frac{1}{2}+n)}{n\Gamma(1+n)}\ {}_2F_1(1,1;n+1;-\frac{x}{a})$$

$$= \sum_{k=0}^{\infty} (-\frac{x}{a})^k [\psi(k+1) - \psi(k+\frac{1}{2})]$$

4. NUMERICAL COMPUTATION

In the tables 1 and 2, equations (2) and (17), respectively, are computed by using a VAX/VMS electronic computing machine. The results are given in single precision.

We shall denote, for the sake of simplicity, the left-hand sides of (2) and (17) by R1 and the right-hand sides of the same equations by R2.

In R1, we took n = 500; this is due to slow convergence of the series.

A subroutine called GAMMA [5] was used to compute the gamma function, where its argument x is given by $10^{-6} < x < 34.5$. For $x > 34.5$, we used the first two terms of the Stirling formula from [1] given by

$$\Gamma(x) \sim \sqrt{2\pi}\, e^{-x}\, (x)^{x-\frac{1}{2}}\ (1 + \frac{1}{12x})$$

In R2, we computed the psi-function $\psi(x)$ by usign [1] the following equation

$$\psi(x) = \ln x - f_3(x),$$

where $f_3(n)$, $(n = 1,2,3,\ldots)$ are tabulated in [1]. The linear-interpolation method is used in computing $f_3(x)$.

Table 1

λ	ν	R1	R2
1.0	0.5	1.335862	1.386294
1.0	0.333333	0.732136	0.741018
3.0	1.5	0.886199	0.941109
3.0	2.0	1.496010	1.500000
3.7	1.5	0.622827	0.627459
3.1	2.1	1.536640	1.537735
3.4	1.8	0.943421	0.989308
4.0	3.0	1.830340	1.833333
5.1	2.1	0.605174	0.604904

In Table 1, for the case $\lambda = 1$, $\nu = \frac{1}{2}$ and $\nu = \frac{1}{3}$, we computed R2 by using $\psi(1) - \psi(\frac{1}{2}) = \ln 4$ and

$$\psi(1) - \psi(\tfrac{2}{3}) = -\frac{\sqrt{3}\pi}{6} + \frac{3}{2}\ln 3$$

The first equation from the known relations of the psi-function while the second from [2].

Table 2

We took $x = a = 1$

λ	ν	R1	R2
1.0	0.5	1.727675	1.772589
1.0	0.333333	0.974873	0.982037
2.3	1.1	1.379751	1.450310
3.0	1.5	1.439577	1.548840
4.0	2.5	2.023448	2.132217
6.5	1.1	0.372500	0.373197
10.0	1.5	0.286364	0.326109
2.0	0.333333	0.382075	0.469211
4.5	3.5	3.156024	3.151312
2.2	1.2	1.696832	1.682981

In Table 2, for the case $\lambda = 1$, $\nu = \frac{1}{2}$ and $\nu = \frac{1}{3}$, we use equations (18) and (19), respectively, to compute R2.

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