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APPLICATION OF FRACTIONAL CALCULUS TO ORDINARY DIFFERENTIAL EQUATIONS OF FUCHS TYPE

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ABSTRACT

In this paper, an application of the fractional calculus to a differential equation

$$\phi_2 \cdot (z^2 - \nu z) + \phi_1 \cdot (2\nu z - \nu^2 + \nu) + \phi \cdot \nu(\nu - 1) = f$$

$z \neq o$, ν is discussed, where $f = f(z)$, $\phi = \phi(z)$, $\phi_2 = \phi''(z)$, $\phi_1 = \phi'(z)$, $z \in C$ and ν is arbitrary.

A particular solution of the above equation is given as

$$\phi = \left((f_{1-\nu} \cdot \frac{z-\nu}{z})_{-1} \cdot \frac{1}{(z-\nu)^2} \right)_{\nu-2}$$

if f_ν exists and $f_\nu \neq o$, where $f_\nu = f_\nu(z)$ means the differintegration of arbitrary order ν of the function f(z)

RESUMEN

En este trabajo se aplica el método del calculo fraccional para resolver la ecuación diferencial

$$\phi_2 (z^2 - \nu z) + \phi_1 (2\nu z - \nu^2 + \nu) + \phi . \nu(\nu - 1) = f,$$

$z \neq o, \nu$, donde $f = f(z)$, $\phi = \phi(z)$, $\phi_1 = \phi'(z)$ $\phi_2 = \phi''(z)$, $z \in c$ y ν es arbitrario. Se da una solución particular de esta ecuación

$$\phi = \left((f_{1-\nu} \cdot \frac{z-\nu}{z})_{-1} \cdot \frac{1}{(z-\nu)^2} \right)_{\nu-2}$$

si f_ν existe y $f_\nu \neq 0$, $f_\nu = f_\nu(z)$ significa el differintegral de orden arbitrario ν, de la función f(z)

1. INTRODUCTION

The concept of differintegral (fractional calculus) of complex order ν, which is a generalization of the ordinary n-th derivative and n-times integral for $\nu = n$ (a positive integer) and $\nu = -n$ (a negative integer) respectively, can be introduced in several ways [7,8].

Nishimoto [5,6] defines the differ-integral as follows : Let f(z) is an analytic function and it has no branch points inside and on $C (C = \{C^-, C^+\}$, C^- is an integral curve along the cut joining two points $z(x,y)$ and $-\infty + iy$, and C^+ is an integral curve along the cut joining two points z and $\infty + iz$),

$$f_\nu = {}_c f_\nu(z) = \frac{\Gamma(\nu+1)}{2\pi i} \int_c \frac{f(\xi) d\xi}{(\xi - z)^{\nu+1}}, \nu \notin Z^- \quad \text{and}$$

$$f_{-n} = \lim_{\nu \to -n} f_\nu (n \in Z^+),$$

where $\xi \neq z$, $-\pi \leq \arg(\xi - z) \leq \pi$ for C^- and $0 \leq \arg(\xi - z) \leq 2\pi$ for C^+, then $f_\nu (\nu > 0)$ is the fractional derivative of order ν and $f_\nu (\nu < 0)$ is the fractional integral of order $-\nu$, if f_ν exists.

Recently, there has been a considerable interest in the applications of fractional calculus. Oldham and Spanier [7] have treated several applications to the problems of Chemistry. Kalla and Ross [3], have obtained functional relations and summation of series by invoking operators of fractional integration. Kalla and Al-Saqabi [2] have also treated such problems. Al-Bassam [1] and Lowndes [4] have used differintegrals for obtaining solutions of differential equations.

In the present paper, we invoke differintegrals to solve a non-homogeneous linear ordinary differential equation of Fuchs type. Corresponding homogeneous case is also considered.

2. NON-HOMOGENEOUS DIFFERENTIAL EQUATION

Theorem 1. If f_ν exists and $f_\nu \neq 0$, then the differential equation of Fuchs type

$$\phi_2 \cdot (z^2-\nu z) + \phi_1 \cdot (2\nu z-\nu^2+\nu) + \phi \cdot \nu(\nu-1) = f \quad (z\neq 0, \nu) \tag{1}$$

has a particular solution of the form

$$\phi = \left(\left(f_{1-\nu} \cdot \frac{z-\nu}{z} \right)_{-1} \cdot \frac{1}{(z-\nu)^2} \right)_{\nu-2} \tag{2}$$

for arbitrary ν, where $\phi = \phi(z)$, $z \in C$ and $f = f(z)$ is known.

Proof. Putting

$$\phi = w_{\nu-1} \tag{3}$$

then we have

$$\phi_1 = w_\nu , \tag{4}$$

and

$$\phi_2 = w_{\nu+1} , \tag{5}$$

where $w = w(z)$.

Substituting (3), (4) and (5) into (1), we obtain

$$w_{\nu+1} \cdot (z^2-\nu z) + w_\nu \cdot (2\nu z-\nu^2+\nu) + w_{\nu-1} \cdot \nu(\nu-1) = f , \tag{6}$$

that is,

$$(w_1 \cdot z^2)_\nu - \nu (w_2 \cdot z)_{\nu-1} = f \tag{7}$$

since [5]

$$(w_1 \cdot z^2)_\nu = \sum_{n=o}^{2} \frac{\Gamma(\nu+1)}{\Gamma(\nu-n+1)\Gamma(n+1)} (w_1)_{\nu-n} \cdot (z^2)_n \tag{8}$$

and

$$(w_2 \cdot z)_{\nu-1} = \sum_{n=o}^{1} \frac{\Gamma(\nu)}{\Gamma(\nu-n)\Gamma(n+1)} (w_2)_{\nu-1-n} \cdot (z)_n \tag{9}$$

Making a differintegration of order $(1-\nu)$ of the equation (7), we have then

$$(w_1 \cdot z^2)_1 - \nu (w_2 \cdot z) = f_{1-\nu} \tag{10}$$

hence

$$w_2 + w_1 \cdot \frac{2}{z-\nu} = f_{1-\nu} \cdot \frac{1}{z^2-\nu z} . \tag{11}$$

Multiplying (11) by $(z-\nu)^2$, we obtain

$$w_2 \cdot (z-\nu)^2 + w_1 \cdot 2(z-\nu) = f_{1-\nu} \cdot \frac{z-\nu}{z} , \tag{12}$$

that is

$$(w_1 \cdot (z-\nu)^2)_1 = f_{1-\nu} \cdot \frac{z-\nu}{z} . \tag{13}$$

Consequently we obtain

$$w_1 \cdot (z-\nu)^2 = \left(f_{1-\nu} \cdot \frac{z-\nu}{z} \right)_{-1} \tag{14}$$

hence

$$w = \left(\left(f_{1-\nu} \cdot \frac{z-\nu}{z} \right)_{-1} \cdot \frac{1}{(z-\nu)^2} \right)_{-1} , \tag{15}$$

from (13).

Substituting (15) into (3), we have then

$$\phi = w_{\nu-1} = \left(\left(f_{1-\nu} \cdot \frac{z-\nu}{z} \right)_{-1} \cdot \frac{1}{(z-\nu)^2} \right)_{\nu-2} \tag{16}$$

as a particular solution to the differential equation (1) for arbitrary ν, if f_ν exists and $f_\nu \neq 0$.

Inversely, substituting (16) into the left hand side of the equation (1), yields

$$\phi_2 . (z^2 - \nu z) + \phi_1 . (2\nu z - \nu^2 + \nu) + \phi . \nu(\nu - 1)$$

$$= w_{\nu+1} . (z^2 - \nu z) + w_\nu . (2\nu z - \nu^2 + \nu) + w_{\nu-1} . \nu(\nu - 1) \quad (17)$$

$$= (w_1 . z^2)_\nu - \nu(w_2 . z)_{\nu-1} \quad (18)$$

$$= \left((w_1 . z^2)_1\right)_{\nu-1} - \nu(w_2 . z)_{\nu-1} \quad (19)$$

$$= (w_2 . z^2 + w_1 . 2z - \nu w_2 . z)_{\nu-1} \quad \text{(using (15))} \quad (20)$$

$$= (f_{1-\nu} . \frac{z-\nu}{z} . \frac{1}{(z-\nu)^2} . z^2 + (f_{1-\nu} . \frac{z-\nu}{z})_{-1} . \frac{\nu - 2}{(z-\nu)^3} . z^2 + (f_{1-\nu} . \frac{z-\nu}{z})_{-1} . \frac{1}{(z-\nu)^2} . 2z$$

$$- f_{1-\nu} . \frac{z-\nu}{z} . \frac{1}{(z-\nu)^2} . \nu z - (f_{1-\nu} . \frac{z-\nu}{z})_{-1} . \frac{-2}{(z-\nu)^3} . \nu z)_{\nu-1} \quad (21)$$

$$= (f_{1-\nu})_{\nu-1} \quad (22)$$

$$= f , \quad (23)$$

if f_ν exists and $f_\nu \neq 0$

3. HOMOGENEOUS DIFFERENTIAL EQUATION

Theorem 2. Differential equation of Fuchs type

$$\phi_2 . (z^2 - \nu z) + \phi_1 . (2\nu z - \nu^2 + \nu) + \phi . \nu(\nu - 1) = 0 \quad (z \neq 0, \nu) \quad (24)$$

has a solution

$$\phi = -k\left((z-\nu)^{-1}\right)_{\nu-1} \quad (25)$$

where k is an arbitrary constant of the integration, $\phi = \phi(z)$ and $z \in C$.

Proof. Putting

$$\phi = w_{\nu-1} \quad (26)$$

and substituting this into (24), we have then

$$w_{\nu+1} . (z^2 - \nu z) + w_\nu . (2\nu z - \nu^2 + \nu) + w_{\nu-1} . \nu(\nu - 1) = 0 \quad (27)$$

that is,

$$(w_1 . z^2)_\nu - \nu(w_2 . z)_{\nu-1} = 0 \quad (28)$$

hence

$$(w_2 . z^2 + w_1 . 2z - w_2 . \nu z)_{\nu-1} = 0. \quad (29)$$

Consequently we obtain

$$w_2 . z^2 + w_1 2z - w_2 . \nu z = 0 \quad (30)$$

from (29), hence

$$w_2 . (z^2 - \nu z) + w_1 . 2z = 0 , \quad (31)$$

that is, we have

$$\frac{w_2}{w_1} = \frac{-2}{z-\nu} \quad (32)$$

Integrating both side of (32) with respect to z, we obtain

$$\int \frac{w_2}{w_1} dz = -2 \int \frac{1}{z-\nu} dz + \log k \qquad (k \neq 0)$$

that is,

$$w_1 = k. \frac{1}{(z-\nu)^2} \qquad (33)$$

where k is an arbitrary constant of the integration.

Therefore we obtain

$$w = k\left(\frac{1}{(z-\nu)^2}\right)_{-1} = -k\,(z-\nu)^{-1}, \qquad (34)$$

finally.

Substituting (34) into (26), we have then

$$\phi = w_{\nu-1} = -k((z-\nu)^{-1})_{\nu-1}, \qquad (35)$$

as the solution of the differential equation (24).

Inversely, substituting (35) into the left hand side of (24), yields

$$\phi_2.(z^2-\nu z) + \phi_1.(2\nu z - \nu^2+\nu) + \phi.\nu(\nu-1)$$

$$= w_{\nu+1}.(z^2-\nu z) + w_\nu.(2\nu z-\nu^2+\nu) + w_{\nu-1}.\nu(\nu-1) \qquad (36)$$

$$= (w_1.z^2)_\nu - \nu(w_2.z)_{\nu-1} \qquad (37)$$

$$= (w_2.z^2 + w_1.2z - w_2.\nu z)_{\nu-1} \qquad (38)$$

$$= \left(-2k\frac{1}{(z-\nu)^3}.z^2+k.\frac{1}{(z-\nu)^2}.2z + 2k\,.\,\frac{1}{(z-\nu)^3}.\nu z\right)_{\nu-1} \qquad (39)$$

$$= (0)_{\nu-1} \qquad (40)$$

$$= 0 \qquad (41)$$

Theorem 3. If $f_\nu(\neq 0)$ exists, then the fractional differ-integrated function

$$\phi = \left((f_{1-\nu}\frac{z-\nu}{z})_{-1}.\frac{1}{(z-\nu)^2}\right)_{\nu-2} - k\,(\frac{1}{z-\nu})_{\nu-1} \qquad (42)$$

satisfies equation (1), where k is an arbitrary constant of the integration, $\phi=\phi(z)$ and $z \in C$.

Proof. It is clear by the Theorems 1 and 2.

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