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UNIFORM ASYMPTOTIC SOLUTIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH DOUBLE POLE AND TURNING POINT

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ABSTRACT

Uniform asymptotic expansions are derived for the solutions to the differential equation

$$\frac{d^2w}{dx^2} = \{ u^2 f(x,\alpha) + g(x,\alpha) \} W$$

for large positive values of u. The real variable x ranges over the (possibly unbounded) interval I_x and α is a bounded real parameter. In I_x the function $f(x,\alpha)$ has a simple zero and a double pole, each depending continuously on α, which coincide for some value of α. The expansions are in terms of Bessel functions and are uniform with respect to x and α. Strict error bounds are provided for these expansions.

RESUMEN

Se deriva desarrollos asintóticos para las soluciones de la ecuación diferencial

$$\frac{d^2W}{dx^2} = \{ u^2 f(x,\alpha) + g(x,\alpha) \} w$$

para valores positivos grandes de u. La variable real x recorre sobre el intervalo I_x (posiblemente no-acotado) y α es un parámetro real acotado. En I_x la función f(x,a) tiene un cero simple y un polo doble, cada uno depende continuamente sobre α, los cuales coinciden para algunos valores de α. Las expansiones están dadas en términos de la función de Bessel y son uniformes con respecto a x y α. Se provee acotaciones estrictas para estas expansiones.

1. INTRODUCTION

In this paper we investigate the differential equation

$$\frac{d^2W}{dx^2} = \{u^2 f(x,\alpha) + g(x,\alpha)\} W \qquad (1.1)$$

where u is a large positive parameter and α is a bounded real parameter which ranger over the interval $[A_1,A_2]$. The independent variable x ranges over the finite or infinite interval I_x. The function $f(x,\alpha)$ is to have only one turning point (zero) in I_x at $x = x$ and one double pole at $x = x_0$, which are both continous functions of α. As α tends to a critical value α_0 in $[A_1,A_2]$, the turning point coincides with the double pole. The function $g(x,\alpha)$ is assumed to be small in absolute value compared to $u^2|f(x,\alpha)|$ except near x_1 where it is small compared to $|f(x,\alpha)/(x-x_1)|$. Furthermore, we assume that $(x-x_0)^2 g(x,\alpha)$ is analitic in I_x.

We shall construct asymptotic expansions for the solutions of (1.1) that are uniform with respect to x in I and α in $[A_1,A_2]$. Strict error bounds will be provided for these expansions. This problem is one on the list of open questions which appeared in the survey [1] by Olver. Although the method we use applies as well to complex variables, we restrict the analysis to the real case since this covers most of the physical applications and sharper error bounds can be derived.

To our knowledge, the only problem dealing with coalescing turning points or singularities that has been treated successfully is the case of two coalescing simple turning points discussed by Olver [2]. The asymptotic theory for a differential equation with one turning point and one regular singular point has been worked out by Thorne [3], but the present calculation differs from his in two respects. First, the asymptotic expansions are uniform with respect to α in $[A_1,A_2]$, including the critical value α_0, whereas Thorne's expansions are valid for fixed $\alpha \neq \alpha_0$. Secondly, error bounds are provided in our case.

Application of the present theory include the associated Legendre equation, Whittaker's equation, and the hypergeometric equation. The necessity of such result arises in physical problems dealing with the solution to the wave equation in prolate spheroidal coordinates for large wave number [4]. Several of these applications will be discussed in a subsequent paper.

2. PRELIMINARY TRANSFORMATIONS

We begin by applying the Liouville transformation

$$W = (dx/d\xi)^{-1/2}w \qquad (2.1)$$

to equation (1.1), where the relation between x and ξ is tobe specified. Then (1.1) becomes

$$d^2W/d\xi^2 = \left(u^2x^2f(x,\alpha) + x^2g(x,\alpha) + x^{1/2}\frac{d^2}{d\xi^2}x^{-1/2}\right)W, \qquad (2.2)$$

where the dot denotes differentiation with respect to ξ. The properties of $f(x,\alpha)$ given in section 1 lead us to define ξ by the first order differential equation

$$x^2f(x,\alpha) = b^2\frac{a-\xi}{\xi^2}, \qquad (2.3)$$

with the contidions that

$$x = x_1 \text{ corresponds to } \xi = a \qquad (2.4)$$

$$x = x_0 \text{ corresponds to } \xi = 0 \qquad (2.5)$$

where a and b are functions of α to be determined.

With the relation (2.3), eq. (2.2) can be reexpressed

$$\frac{d^2W}{d\xi^2} = \left\{u^2b^2\frac{a-\xi}{\xi^2} + \frac{\rho}{\xi^2} + \frac{1}{\xi}\psi(\xi,\alpha)\right\}W, \qquad (2.6)$$

where

$$\psi(\xi,\alpha) = \xi\left[-\frac{\rho}{\xi^2} + x^2g(x,\alpha) + x^{1/2}\frac{d^2}{d\xi^2}x^{-1/2}\right], \qquad (2.7)$$

and ρ is a constant to be determined. If the term $\frac{1}{\xi}\psi(\xi,\alpha)$ is small compared to

$$u^2b^2\frac{a-\xi}{\xi^2} + \frac{\rho}{\xi^2},$$

then by neglecting it we reduce (2.7) to Bessel's equation

$$\frac{d^2W}{d\xi^2} = \left[u^2b^2\frac{a-\xi}{\xi^2} + \frac{\rho}{\xi^2}\right]W,$$

with the solutions $|\xi|^{1/2}C_\nu(2ub|\xi|^{1/2})$, where C_ν denotes any cylinder functions of order ν, and

$$\nu^2 = 1 + 4\rho + 4ab^2u^2. \qquad (2.9)$$

DETERMINATION OF ξ, a AND b

Now we write the function $f(x,\alpha)$ in the form

$$f(x,\alpha) = (x_1-x)(x-x_0)^{-2}p(x,\alpha) \qquad (2.10)$$

and assume that $p(x,\alpha)$ satisfies the following conditions:

(i) $p(x,\alpha)$ is positive, analytic, and has no zeros in I.

(ii) $\frac{\partial}{\partial x}p(x,\alpha)$ and $\frac{\partial^2}{\partial x^2}p(x,\alpha)$ are continuous functions of x and α in I_x and $[A_1,A_2]$, respectively.

(iii) $\frac{\partial^3}{\partial x^3}p(x,\alpha)$ is bounded near the point $x = x_0$

Without loss of generality we can assume that $x_0 \leq x_1$. We determine the relation between x and ξ by integrating (2.3):

$$\int_x^{x_1}\frac{(x_1-t)^{1/2}}{t-x_0}p^{1/2}(t)dt = b\int_\xi^a\frac{(a-\tau)^{1/2}}{\tau}d\tau \quad (x \leq x_1), \qquad (2.11)$$

$$\int_{x_1}^{x}\frac{(t-x_1)^{1/2}}{t-x_0}p^{1/2}(t)\,dt = b\int_a^\xi\frac{(\tau-a)^{1/2}}{\tau}d\tau \quad (x \geq x_1), \qquad (2.12)$$

With this choice of the limite of integration ξ is an increasing function of x. We denote the image of I_x by

$$I_\xi = (\delta,\gamma).$$

The unknowns a and b can now be determined. From (2.11) we have

$$\int_{x_0}^{x} \frac{(x_1-t)^{1/2}}{t-x_0} p^{1/2}(t)dt = b\int_0^{\xi} \frac{(a-\tau)^{1/2}}{\tau} d\tau \qquad (2.13)$$

In view of the condition (2.5) the integrands in (2.13) can be expanded about the points $x = x_0$ and $\xi = 0$, respectevely; we find

$$\frac{(a-\tau)^{1/2}}{\tau} = \frac{\sqrt{a}}{\tau}\left[1 - \frac{\tau}{2a} - \frac{1}{8a^2}\tau^2 + O(\tau^3)\right].$$

On the order hand, because of conditions (i) and (iii) on p(x,α) we can apply Taylor's theorem to expand $p^{1/2}(t)$ in powers of $(t-x_0)$. After integrating these expressions term by term we have

$$(x_1-x_0)^{1/2} p^{1/2}(x_0)\left[\ln(x-x_0) + \left\{\frac{p'(x_0)}{2p(x_0)} - \frac{1}{2(x_1-x_0)}\right\}(x-x_0)\right.$$

$$+\left\{\frac{p''(x_0)}{4p(x_0)} - \frac{p'^2(x_0)}{8p^2(x_0)} - \frac{p'(x_0)}{4p(x_0)(x_1-x_0)} - \frac{1}{8(x_1-x_0)^2}\right\}$$

$$(x-x_0)^2 \qquad (2.14)$$

$$\left. + O\left\{(x-x_0)^3\right\}\right] = b\sqrt{a}\left[\ln\xi - \frac{\xi}{2a} - \frac{\xi}{8a^2} + O(\xi^3)\right]$$

Note that when $x \to x_0$, both sides of (2.14) have logarithmic singularities at the origin; these singularities cancel if

$$ba^{1/2} = (x_1 - x_0)^{1/2} p^{1/2}(x_0). \qquad (2.15)$$

This is our first equation relating a and b A second equation is found by evaluating (2.13) at $x = x_1$ and $\xi = a$.

The next step in this analysis is to prove that under the given assumptions $\psi(x,\alpha)$ is continuous in the interval

$$I_\xi = (\delta,\gamma) \text{ for all values of } \alpha \text{ in } (A_1,A_2).$$

3. CONTINUITY OF $\psi(\xi,\alpha)$

The function $\psi(\xi,\alpha)$ can be written as

$$\psi(\xi,\alpha) = \xi\left[-\frac{\rho}{\xi^2} + \dot{x}^2 g(x,\alpha) - \frac{1}{2}\left\{x,\xi\right\}\right], \qquad (3.1)$$

where the Schwarzian derivative is given by

$$\left\{x,\xi\right\} = \dddot{x}/\dot{x} - (3/2)(\ddot{x}/\dot{x})^2. \qquad (3.2)$$

By making use of the relations (2.3) and (3.2) we obtain

$$\psi(\xi,\alpha) = -\frac{\rho}{\xi} + \frac{5\xi}{16(a-\xi)^2} + \frac{1}{4(a-\xi)} - \frac{1}{4\xi} + \frac{b^2(a-\xi)(x-x_0)^2}{\xi(x_1-x)p(x)}$$

$$\left\{ g(x,\alpha) - \left(\frac{5p'^2(x)}{16p^2(x)} - \frac{p''(x)}{4p(x)} + \frac{1}{8}\frac{p'(x)}{p(x)}\left(\frac{1}{x_1-x} + \frac{2}{x-x_0}\right)\right.\right.$$

$$\left.\left. - \frac{1}{4(x_1-x)(x-x_0)} - \frac{5}{16(x_1-x)^2} + \frac{1}{4(x-x_0)}\right\} \qquad (3.3)$$

The constant ρ must be defined as

$$\rho = \lim_{x\to x} (x-x_0)^2 g(x,\alpha). \qquad (3.4)$$

which can be seen from (3.3) and the requirement that $\psi(\xi,\alpha)$ be continous at $\xi = 0$. Therefore we shall prove the continuity of the Schwarzian derivative (3.2). For this purpose,, we consider separtely the three $\xi - \alpha$ regions:

$$\xi = 0\ ,\ \text{all}\ \ \alpha \neq \alpha \qquad (I)$$

$$\xi \leq a\ ,\ \text{all}\ \ \alpha \neq \alpha \qquad (II)$$

$$\xi \geq a\ ,\ \text{all}\ \ \alpha \neq \alpha_0 \qquad (III)$$

The remaining portion of the ξ-α domainwill not be discussed, since continuity there can be proven by following the steps for the analogous case of two turning points [2].

REGION (I)

Consider the case where ξ approaches 0 from the right. Determining the relation between x and ξ near the points $x = x_0$ and $\xi = 0$ by solving (2.14) iteratively gives.

$$x(\xi,\alpha) - x_0 = \xi + c_1\xi^2 + c_2\xi^3 + O(\xi^4), \quad (3.5)$$

where

$$c_1 = -\frac{p'(x_0)}{p(x_0)} + \frac{1}{2(x_1-x_0)} - \frac{1}{2a} \quad (3.6)$$

$$c_2 = -\frac{1}{16a^2} + \frac{3}{2}C_1^2 + \frac{C_1}{2a} - \frac{1}{2} \quad (3.7)$$

$$\left\{\frac{p''(x_0)}{4p(x_0)} - \frac{p'^2(x_0)}{8p^2(x_0)} - \frac{p'(x_0)}{4p(x_0)(x_1-x_0)} - \frac{1}{8(x_1-x_0)^2}\right\}$$

The next step is to show that the O-term in (3.5) holds uniformly for

$$0 < \xi \le \delta_\xi < a, \qquad a \ne \alpha_0 \quad (3.8)$$

or equivalently $0 < x-x_0 \le \delta_x < x_1 - x_0$, where δ_ξ and δ_x are small non-zero numbers. We first determine the behavior of c_1 and c_2 near the critical value, i.e. as $\alpha \to \alpha_0$ or $x_1 \to x_0$. The behavior of a as $x_1 \to x_0$ can be found from (2.13) by expanding $p^{1/2}(x)$ about x_0 and setting the upper limits on the integrals to $x = x_1$ and $\xi = a$. The result is

$$a = (x_1-x_0)\left[1 + \frac{1}{3}\frac{p'(x_0)}{p(x_0)}(x_1-x_0) + \right.$$

$$\left.\frac{1}{15}\left\{\frac{p''(x_0)}{p(x_0)} + \frac{1}{3}\frac{p'^2(x_0)}{p^2(x_0)}\right\}(x-x_0)^2 + O\left\{(x_1-x_0)^3\right\}\right]$$

Then by substituting (3.9) into (3.6) and (3.7) we find that all the terms with negative powers of a cancel leaving

$$c_1 = -\frac{1}{3}\frac{p'(x_0)}{p(x_0)} - \frac{1}{30}$$

$$\left\{-\frac{p''(x_0)}{p(x_0)} + \frac{4p'^2(x_0)}{3p^2(x_0)}\right\}(x_1-x_0) + O(x_1-x_0)^2 \quad (3.10)$$

and

$$c_2 = \frac{11}{45}\frac{p'^2(x_0)}{p^2(x_0)} - \frac{p''(x_0)}{10p(x_0)} + O((x_1-x_0)). \quad (3.11)$$

These expansions show that the coefficients are continous as $\alpha \to \alpha_0$. Consider again (2.13), which we rewrite in the form

$$\int_0^{x-x_0}\frac{(x_1-x_0-t)^{1/2}}{t}p^{1/2}(t+x_0)\,dt = b\int_0^{\xi}\frac{(a-\tau)^{1/2}}{\tau}d\tau \quad (3.12)$$

Denote the error term in (3.5) by η and let

$$\hat{\xi} = \xi + c_1 + \xi^2 + c_2\xi^3 \quad (3.13)$$

Therefore we can express (3.5) in the form

$$x(\xi,\alpha) - x_0 = \xi + \eta\,. \quad (3.14)$$

and (3.12) as

$$I_1 + I_2 = J_1\,, \quad (3.15)$$

where

$$I_1 = \int_{\hat{\xi}}^{\hat{\xi}+\eta}\frac{(x_1-x_0-t)^{1/2}}{t}p^{1/2}(t+x_0)dt \quad (3.16)$$

$$I_2 = \int_0^{\hat{\xi}}\frac{(x_1-x_0-t)^{1/2}}{t}p^{1/2}(t+x_0)dt \quad (3.17)$$

and

$$J_1 = b\int_0^{\xi} \frac{(a-\tau)^{1/2}}{\tau}\, d\tau \qquad (3.18)$$

In the integral I_2 we make the substitution $t = v+c_1v^2 + c_2v^3$, so that $t=0$, $t=\hat{\xi}$ correspond to $v=0$, $v=\xi$, respectively. If δ_ξ is sufficiently small, then $dt/dv = 1+2c_1v + 3c_2v^2$ is positve, which means that the correspondence between t and v is one-to-one. Expanding the integrand about $v=0$ we find

$$I_2 = p^{1/2}(x_0)(x_1-x_0)^{1/2}\int_0^{\xi}\frac{1}{v}\Bigg[1 + \left\{c_1 + \frac{p'(x_0)}{2p(x_0)} - \frac{1}{2(x_1-x_0)}\right\}v$$

$$+\left\{2c_2 - c_1^2 + \frac{p'(x_0)}{p(x_0)} + \frac{p''(x_0)}{4p(x_0)} - \frac{p'^2(x_0)}{8p^2(x_0)} - \frac{1}{(x_1-x_0)}c_1\right.$$

$$\left. - \frac{1}{4(x_1-x_0)}\frac{p'(x_0)}{p(x_0)} - \frac{1}{8(x_1-x_0)}\right\}v^2 + O(v^3)\Bigg]\, dv\,. \qquad (3.19)$$

By using formulas (3.6) and (3.7) for c_1 and c_2, we find that (3.14) reduces to

$$I_2 = p^{1/2}(x_0)(x_1-x_0)^{1/2}$$

$$\int_0^{\xi}\frac{1}{v}\left\{1 - \frac{1}{2a}v - \frac{1}{8a^2}v^2 + O(v^3)\right\} dv. \qquad (3.20)$$

Similarly, expanding the integrand of J_1 about $\tau = 0$ yields

$$J_1 = ba^{1/2}\int_0^{\xi}\frac{1}{\tau}\left\{1 - \frac{1}{2a}\tau - \frac{1}{8a^2}\tau^2 + O(\tau^3)\right\} d\tau \qquad (3.21)$$

The remainder terms in (3.20) and (3.21) hold uniformly in the intervals (3.8). The combination of (3.15), (3.20) and (3.21) leads to the estimate

$$J_1 - I_2 = p^{1/2}(x_0)(x_1-x_0)^{1/2}\int_0^{\xi} O(\xi^2)\, d\xi =$$

$$p^{1/2}(x_0)(x_1-x_0)^{1/2}O(\xi^3) \qquad (3.22)$$

On the other hand I_1 can be estimated by applying the mean value theorem to the integral (3.16), which gives.

$$I_1 = (x_1-x_0-\tilde{t})^{1/2}\, p^{1/2}(\tilde{t}+x_0)\,\ln\left|\frac{\xi+\eta}{\hat{\xi}}\right| \qquad (3.23)$$

where $0 < \tilde{t} \le \delta_x$. The logarithmic term can be simplified by using the inequality

$$\ln(x/y) \ge 2(x-y)/(x+y) \qquad (0 < y \le x) \qquad (3.24)$$

Therefore, $\ln\left|\dfrac{\xi+\eta}{\xi}\right| \ge K(\eta/\xi)$, and

$$|I_1| \ge K(x_1-x_0-\tilde{t})^{1/2}\, p^{1/2}(\tilde{t}+x_0)\,\eta/\xi\,, \qquad (3.25)$$

But, from (3.22) we have

$$|I_1| = (x_1-x_0)^{1/2}\, p^{1/2}(x_0)O(\xi^3)\,. \qquad (3.26)$$

Thus, $\eta = O(\xi^4)$ uniformly in the intervals (3.8), which means that the remainder terms in the expansion

$$x(\xi,\alpha) - x_0 = \xi + c_1\xi^2 + c_2\xi^3 + O(\xi^4) \qquad (3.27)$$

hols uniformy for $0 < \xi < \delta_\xi < a$, $\alpha \neq \alpha_0$ To find the expansions of the derivaties as $\xi \to 0^+$, we use the relation

$$x^2 = \frac{b^2(x_1-x)(a-\xi)}{\xi^2(x-x_0)^2 p(x)} \qquad (3.28)$$

substitute (3.27) for x and solve for $\dot{x}$.This gives,

$$\dot{x} = 1 + 2c_1\xi + 3c_2\xi^2 + O(\xi^3) ,$$
$$\ddot{x} = 2c_1 + 3c_2\xi + O(\xi^2),$$
$$\dddot{x} = 3c_2 + O(\xi).$$

All the O-terms above hold uniformly in the intervals (3.8); hence x, $\dot{x}$, $\ddot{x}$, and $\dddot{x}$ are continuoua in these intervals, which implies the continuity of the Schwarzian derivative under these circumstances. A similar analysis works for the case $\xi \to 0^-$, $\alpha \neq \alpha_0$. This completes the proof that $\psi(\xi,\alpha)$ is continuous for $\xi = 0$ and $\alpha \neq \alpha_0$.

REGION (II)

In the case that ξ approaches a from the left, we expand the integrands in (2.11) around the points ξ = a and $x = x_1$, integrate term by term and solve by iteration to get

$$x_1 - x(\xi,\alpha) = d_0(a-\xi) + d_1(a-\xi)^2 + d_2(a-\xi)^3 + O((a-\xi)^4) \qquad (3.29)$$

where

$$0 < a - \xi \le \delta_\xi < a,\ \alpha \neq \alpha_0 \qquad (3.30)$$

(or equivalently $0 < x_1 - x \le \bar{\delta}_x < x_1 - x_0$))

$$d_0 = \left[\frac{b}{a} \frac{x_1 - x}{p^{1/2}(x_1)} \right]^{3/2}, \qquad (3.31a)$$

$$d_1 = \frac{2}{5a} d_0 - \frac{2}{5} \left\{ - \frac{p'(x_1)}{2p(x_1)} + \frac{1}{x_1 - x_0} \right\} d_0^2 \qquad (3.31b)$$

$$d_2 = \frac{2}{7a^2} d_0 - \frac{1}{4}\frac{d_1^2}{d_0} - \left\{ - \frac{p'(x_1)}{2p(x_1)} + \frac{1}{x_1-x_0} \right\} d_0 d_1$$

$$- \frac{2}{7} \left\{ \frac{p''(x_1)}{4p(x_1)} - \frac{p'^2(x_1)}{8p^2(x_1)} - \frac{p'(x_1)}{2p(x_1)(x_1-x_0)} + \frac{1}{(x_1-x_0)^2} \right\} d_0^3 \qquad (3.31c)$$

Substituting the expansion (3.9) for a as $\alpha \to \alpha_0$ into (3.31) we find that all the terms containing negative powers of (x_1-x_0) cancel, which means that the coefficients d_0, d_1, d_2 are continous as $\alpha \to \alpha_0$. To prove the uniformity of the term in (3.29) in the intervals (3.30) we denote the error terms in (3.29) by $\bar{\eta}$ and
let

$$\bar{\xi} = d_0(a-\xi) + d_1(a-\xi)^2 + d_2(a-\xi)^3 . \qquad (3.32)$$

Hence

$$x_1 - x = \bar{\xi} + \bar{\eta} . \qquad (3.33)$$

Now, equation (2.11) can be written as

$$\int_0^{x_1-x} \frac{t^{1/2}}{x_1-x_0-t} p^{1/2}(x_1-t)\, dt = b \int_0^{a-\xi} \frac{\tau^{1/2}}{\alpha-\tau} d\tau \qquad (3.34)$$

or in the form

$$I_3 + I_4 = J_2 \qquad (3.35)$$

where

$$I_3 = \int_{\bar{\xi}}^{\bar{\xi}+\bar{\eta}} \frac{t^{1/2}}{x_1-x_0-t} p^{1/2}(x_1-t)\, dt \qquad (3.36)$$

$$I_4 = \int_0^{\bar{\xi}} \frac{t^{1/2}}{x_1 - x_0 - t} \; p^{1/2} \; (x_1 - t) \; dt \qquad (3.37)$$

and

$$J_2 = b \int_0^{a-\xi} \frac{\tau^{1/2}}{a-\tau} \; d\tau. \qquad (3.38)$$

In I we introduce the new variable $\bar{v}$ by $t = d_0\bar{v} + d_1\bar{v}^2 + d_2\bar{v}^3$. The correspondence between t and $\bar{v}$ is one - to - one since $dt/d\bar{v} > 0$ for sufficiently small $\bar{\delta}$. Expanding the integrand about $\bar{v} = 0$ gives

$$I_4 = \frac{p^{1/2}(x_1)d_0^{3/2}}{x_1 - x_0} \int_0^{a-\xi} \bar{v}^{1/2} \Bigg[1 +$$

$$\left\{ \frac{5}{2}\frac{d_1}{d_0} + \frac{d_0}{x_1 - x_0} + \frac{p'(x_1)}{2p(x_1)} d_0 \right\} \bar{v} \; + \left\{ \frac{7}{8}\frac{d_1^2}{d_0^2} + \frac{7}{2}\frac{d_2}{d_0} + \right.$$

$$\frac{7}{2}\frac{d_1}{x_1 - x_0} + \frac{d_0^2}{(x_1 - x_0)^2} + \frac{p'(x_1)}{2p(x_1)} \left(\frac{d_0}{x_1 - x_0} + \frac{7}{2} d_1 \right)$$

$$\left. + \left(\frac{p''(x_1)}{4p(x_1)} - \frac{p'^2(x_1)}{8p^2(x_1)} \right) d_0^2 \right\} \bar{v}^2 + O(\bar{v}^3) \Bigg] \; d\bar{v} \; . \qquad (3.39)$$

By substituting the values for d_0, d_1, d_2 from (3.31) into (3.39) this reduces to

$$I_4 = (b/a) \int_0^{a-\xi} \bar{v}^{1/2} \left\{ 1 + \frac{1}{a}\bar{v} + \frac{1}{a^2}\bar{v}^2 + O(\bar{v}^3) \right\} d\bar{v} \qquad (3.40)$$

Expanding the integrand in J_2 about $\tau = 0$ yields

$$J_2 = (b/a) \int_0^{a-\xi} \tau^{1/2} \left\{ 1 + \frac{1}{a}\tau + \frac{1}{a^2}\tau^2 + O(\tau^3) \right\} d\tau \; (3.41)$$

By combining (3.40) and (3.41) we have the estimate

$$J_2 - I_4 = (b/a) \int_0^{a-\xi} O(\tau^{7/2}) \; d\tau \; = \; \frac{b}{a} \; O\left\{(a-\xi)^{9/2}\right\} \qquad (3.42)$$

The O-terms in (3.40), (3.41), and(3.42) hold uniformly in the intervals (3.30). By applying the mean value theorem to the integral I_3 we get

$$I_3 = \frac{p^{1/2}(x_1 - \bar{t})}{x_1 - x_0 - \bar{t}} \int_{\bar{\xi}}^{\bar{\xi}+\bar{\eta}} t^{1/2} \; dt = \qquad (3.43)$$

$$\frac{2p^{1/2}(x_1 - \bar{t})}{3(x_1 - x_0 - \bar{t})} \; \bar{\xi}^{3/2} \; [(1 + \bar{\eta}/\bar{\xi})^{3/2} - 1]$$

where $0 < \bar{t} \leq \bar{\delta}_x$. Since $\bar{\xi} + \bar{\eta} = x_1 - x > 0$

then $\bar{\eta}/\bar{\xi} \in (-1, \infty)$.

But in this range the inequality

$$\left| \frac{\bar{\eta}}{\bar{\xi}} \Big/ \left\{ \left(1 + \frac{\bar{\eta}}{\bar{\xi}}\right)^{3/2} - 1 \right\} \right| \leq C \text{ is valid,}$$

where C is an assignable possitive constant. Thus, •

$$|I_3| \geq \frac{2}{3} \; \frac{p^{1/2}(x_1 - \bar{t})}{x_1 - x_0 - \bar{t}} \; \frac{\bar{\xi}^{1/2}\bar{\eta}}{C} \qquad (3.44)$$

But, from (3.2) we have

$$|I_3| = (b/a)O\left\{(a-\xi)^{9/2}\right\}$$

Since $\bar{\xi} = O(a-\xi)$ we obtain the estimate

$$\bar{\eta} = O\left\{(a-\xi)^4\right\}$$

uniformly in the intervals (3.30).

REGION (III)

In the case that ξ approaches a from the right we start with eq. (2.12) and through the same analysis as in the previous case we obtain the expansion

$$x(\xi,\alpha)-x_1 = d_0(\xi-a) - d_1(\xi-a)^2 + d_2(\xi-a)^3 + O\left\{(\xi-a)^4\right\} \tag{3.45}$$

where d_0, d_1 and d_2 are defined in (3.31). The O-terms holds uniformly in the intervals

$$0 \le \xi - a \le \bar{\delta}_\xi \text{ for all } \quad (\text{or } 0 \le x - x_1 \le \bar{\delta}_x). \tag{3.46}$$

In particular, for the critical value

$$\alpha = \alpha_0, \text{i.e } x_1 = x_0$$

$$x(\xi,\alpha) - x_1 = \xi + \frac{1}{3}\frac{p'(x_0)}{p(x_0)}\xi^2 + O(\xi^3). \tag{3.47}$$

In cases (II) and (III) the dervatives of $x(\xi,\alpha)$ near $\xi = a$ are obtained by substituting (3.29) or (3.45) into (3.28). The results are

$$\dot{x} = d_0 - 2d_1(\xi-a) + 3d_2(\xi-a)^2 + O\left\{(\xi-a)^3\right\}$$

$$\ddot{x} = -2d_1 + 6d_2(\xi-a) + O\left\{(\xi-a)^2\right\}$$

$$\dddot{x} = 6d_2 + O\left\{(\xi-a)\right\}$$

which are uniformly valid for all α. Therefore, the Schwarzian derivative is continous at $\xi = a$. This completes the proof of the continuity of $\psi(\xi,\alpha)$ for $\xi \in I_\xi$ and $\alpha \in [A_1,A_2]$.

4. ASYMPTOTIC EXPANSIONS

The equation

$$\frac{d^2W}{d\xi^2} = \left\{ u^2b^2\frac{a-\xi}{\xi^2} + \frac{\rho}{\xi^2} + \frac{\psi(\xi,\alpha)}{\xi}\right\} W \tag{4.1}$$

has a regular singular point in

$$I_\xi = (\delta,\gamma) \text{ at } \xi = 0$$

Since our concern is in constructing only real solutions, we consider the intervals $(0,\gamma)$ and $(\delta,0)$ separately. For $(0,\gamma)$ we take as two linearly independent solutions for the comparison equation (2.8), the Bessel functions

$$\xi^{1/2} J_\nu(2ub\xi^{1/2}) \text{ and } \xi^{1/2} Y_\nu(2ub\xi^{1/2}),$$

where

$$\nu^2 = 1 + 4\rho + 4ab^2u^2 .$$

For $(\delta,0)$ we take the modified Bessel functions

$$(-\xi)^{1/2} I_\nu\left\{(2ub(-\xi)^{1/2}\right\}$$

and

$$(-\xi)^{1/2} K_\nu\left\{2ub\,(-\xi)^{1/2}\right\}$$

As a solution to equation (4.1) for positive ξ we try the series

$$W = \xi^{1/2} C_\nu(2ub\xi^{1/2})\sum_{s=0}^{\infty}\frac{A_s(\xi)}{(2ub)^{2s}}$$
$$\frac{\xi}{2ub} C_{\nu+1}(2ub\xi^{1/2})\sum_{s=0}^{\infty}\frac{B_s(\xi)}{(2ub)^{2s}} \tag{4.2}$$

where C_ν denotes J_ν or Y_ν. By differentiating (4.2) twice and making use of Bessel function properties [5], then substituting the result in (4.1) and equating like powers of u, we find that $A_s(\xi)$ and $B_s(\xi)$ satisfy the equations

$$A_0(\xi) = \text{constant} \quad (4.3)$$

$$\xi A''_s(\xi) + (\nu+1)A'_s(\xi) - \psi(\xi,\alpha)\, A_s(\xi) - \xi B'_s(\xi) - (1/2)B_s(\xi) = 0 \quad (4.4)$$

$$A'_{s+1}(\xi) + \xi B''_s(\xi) + (1-\nu)B'_s(\xi) - \psi(\xi,\alpha)B_s(\xi) = 0 \quad (4.5)$$

Without loss of generality, the constant in (4.3) can be taken to be unity. Integration of (4.4) and (4.5) yield:

$$B_s(\xi) = -A'_s(\xi) + \xi^{-1/2}\int_0^{\xi}\left[\psi(\tau,\alpha)A_s(\tau)-(\nu+1/2)A'(\tau)\right]\tau^{1/2}\,d\tau \quad (4.6)$$

and

$$A_{s+1}(\xi) = \nu B_s(\xi) - \xi B'_s(\xi) + \int \psi(\xi,\alpha)\, B_s(\xi)d\xi. \quad (4.7)$$

These relations determine $A_s(\xi)$ and $B_s(\xi)$ successively, apart from an arbitrary constant of integration in (4.7). The lower limit in (4.6) is chosen to be $\xi = 0$ so that $B_s(\xi)$ is continuous at this point. The continuity and *differentiability of* the coefficients $A_s(\xi)$ and $B_s(\xi)$ in the interval is and immediate consequence of Olver's lemma [6, p.410]. For $\xi \in (\delta,0)$ we have the same series solution, where C now denotes either I_ν or $e^{i\nu\pi}K_\nu$ and ξ is replaced by $-\xi$. These coefficients satisfy relations (4.4) and (4.5) with

$$B_s(\xi) = -A_s(\xi) + (-\xi)^{-1/2}\int_{\xi}^{0}\left[\psi(\tau,\alpha)A_s(\tau)-(\nu+1/2)A_s(\tau)\right](-\tau)^{1/2}\,d\tau, \quad (4.8)$$

corresponding to (4.6).

Asymptotic expansions for the solutions of (4.1) are obtained by truncating the series (4.2), so we have

$$W_p(\xi) = |\xi|^{1/2} C_\nu(2ub|\xi|^{1/2}) \sum_{s=0}^{p} \frac{A_s(\xi)}{(2ub)^{2s}} - \frac{|\xi|}{2ub} C_{\nu+1}(2ub|\xi|^{1/2}) \sum_{s=0}^{p-1} \frac{B_s(\xi)}{(2ub)^{2s}} + \varepsilon_p(\xi,u)$$

The function $\varepsilon_p(\xi,u)$ represents the error associated with the respective asymptotic expansions. From analytical and computational standpoints it is necessary to have realistic bounds for the functions $\varepsilon(\xi,u)$. Most frequently, the procedure used to derive them is to find a differential equation they *satisfy. Then by using the method of variation of* parameters this equation is converted into an integral equation to which one can apply Olver's theorem to bound the solutions [2,6]. The continuity of $\psi(\xi,\alpha)$ is a necessary requirement in the error analysis and this property has been verified in section 3. Because of the similarity of the comparison equation (2.8) and that in the theory for a simple pole [6], in the sense that both lead to Bessel expansions, the error analysis is similar in these two cases. We therefore omit the details and summarize the results in the following section.

5. MAIN THEOREMS

THEOREM 1.

For $\xi \in (0,\gamma)$ the differential equation

$$\frac{d^2W}{d\xi^2} = \left\{ u^2b^2\,\frac{a-\xi}{\xi^2} + \frac{\rho}{\xi^2} + \frac{\psi(\xi,\alpha)}{\xi} \right\} W \quad (5.1)$$

where a,b, ρ *and* $\psi(\xi,\alpha)$ *are defined in section 2, has* solutions given by

$$W_{p,1}(\xi,u) = \xi^{1/2} J_\nu(2ub\xi^{1/2}) \sum_{s=0}^{p} \frac{A_s(\xi)}{(2ub)^{2s}} - \frac{\xi}{2ub} J_{\nu+1}(2ub\xi^{1/2}) \sum_{s=0}^{p-1} \frac{B_s(\xi)}{(2ub)^{2s}} + \varepsilon_{p,1}(\xi,u) \quad (5.2)$$

$$W_{p,2}(\xi,u)= \xi^{1/2}\, Y_\nu(2ub\xi^{1/2}) \sum_{s=0}^{p} \frac{A_s(\xi)}{(2ub)^{2s}} - \frac{\xi}{2ub}\, Y_{\nu+1}(2ub\xi^{1/2})$$

$$\sum_{s=0}^{p-1} \frac{B_s(\xi)}{(2ub)^{2s}} + \varepsilon_{p,2}(\xi,u) \qquad (5.3)$$

where

$$\nu^2 = 1 + 4\rho + 4ab^2 u^2$$

and

$$|\xi_{p,1}(\xi,u)| \le \frac{\lambda_3(\nu)}{(2ub)^{2p+1}}\, \xi^{1/2}\, E_\nu^{-1}.$$

$$(2ub\xi^{1/2}) M_\nu(2ub\xi^{1/2})\; V_{0,\xi}(\tau^{1/2} B_p(\tau)).$$

$$\exp\left\{ \frac{\lambda_2(\nu)}{2ub} V_{0,\xi}\, (\tau^{1/2}\, B_0(\tau))\right\} \qquad (5.4)$$

$$\left| \frac{\partial}{\partial\xi}\, \varepsilon_{p,1}\, (\xi,u) \right| \le \frac{\lambda_3(\nu)}{2(2ub)^{2p}}\, E_{\nu+1}$$

$$(2ub\xi^{1/2}) E_\nu^{-2}\, (2ub\xi^{1/2})\, M_{\nu+1}(2ub\xi^{1/2}) .$$

$$V_{0,\xi}(\tau^{1/2} B_p(\tau)) \exp\left\{ \frac{\lambda_2(\nu)}{2ub}\, V_{0,\xi}(\tau^{1/2} B_0(\tau))\right\} +$$

$$\frac{\nu+1}{2\xi}\, |\varepsilon_{p,1}\, (\varepsilon,u)|. \qquad |\, \varepsilon_{p,2}(\xi,u)| \le \qquad (5.5)$$

$$\frac{\lambda_4(\nu)}{(2ub)^{2p+1}}\, \xi^{1/2} E_\nu(2ub\xi^{1/2}) M_\nu(2ub\xi^{1/2}) V_{\xi,\gamma}(\tau^{1/2} B_p(\tau)).$$

$$\left\{ \exp \frac{\lambda_2(\nu)}{2ub}\, V_{\xi,\gamma}\, (\tau^{1/2} B_0(\tau)) \right\}, \qquad (5.6)$$

$$\left| \frac{\partial}{\partial\xi}\, \varepsilon_{p,2}(\xi,u) \right| \le \frac{\lambda_4(\nu)}{(2ub)^{2p}}\, E_{\nu+1}$$

$$(2ub\xi^{1/2})\, M_{\nu+1}(2ub\xi^{1/2}) V_{\xi,\gamma}(\tau^{1/2} B_\rho(\tau)).$$

$$\exp\left\{ \frac{\lambda_2(\nu)}{2ub}\, V_{\xi,\gamma}(\tau^{1/2}\, B_0(\tau))\right\} + \frac{\nu+1}{2\xi}\, |\varepsilon_{p,2}(\xi,u)|.$$

The coefficients A_s and B_s are given in (4.6) and (4.7). The functions E_ν and M_ν are the auxiliary functions for Bessel functions and are discussed in [6]:

$$E_\nu(x) =$$

$$\left\{ - Y_\nu(x)/J_\nu(x)\right\}^{1/2} (0 < x < X_\nu),\; E_\nu(x) = 1\; (x \ge X_\nu)$$

(5.8)

where X_ν denotes the smallest positive root of the equation

$$J_\nu(x) + Y_\nu(x) = 0 \qquad (5.9)$$

and

$$M_\nu(x) = \begin{cases} (2|Y_\nu(x)|\, J_\nu(x))^{1/2} & 0 < x \le X_\nu \\ (J_\nu^2(x) + Y_\nu^2(x))^{1/2} & x \ge X_\nu \end{cases} \qquad (5.10)$$

The functional $V_{a,b}(f(x))$ is the total variation of f, i.e.

$$V_{a,b}(f) = \int_a^b |f'(x)|\, dx.$$

The factors λ_2, λ_3, λ_4 are defined by [6]

$$\lambda_2(\nu) = \text{Sup}\ \{\pi\ x\ M_\nu^2(x)\} \qquad (5.11a)$$

$$\lambda_3(\nu) = \text{Sup}\ \{\pi\ x\ E_\nu(x)\ M_\nu(x)\ J_\nu(x)\} \qquad (5.11b)$$

$$\lambda_4(\nu) = \text{Sup}\ \{\pi\ x\ E_\nu^{-1}(x)\ M_\nu(x)|Y_\nu(x)|\} \qquad (5.11c)$$

for $x \in (0,\infty)$.

THEOREM 2.

For $\xi \in (\delta,0)$ the differential equation (5.1) has solutions

$W_{p,3}(\xi,u)$ and $W_{p,4}(\xi,u)$ given by

$$W_{p,3}(\xi,u) = |\xi|^{1/2}\ I_\nu(2ub|\xi|^{1/2}) \sum_{s=0}^{p} \frac{A_s(\xi)}{(2ub)^{2s}} \qquad (5.12)$$

$$+ \frac{|\xi|}{2ub}\ I_{\nu+1}(2ub|\xi|^{1/2}) \sum_{s=0}^{p-1} \frac{B_s(\xi)}{(2ub)^{2s}} + \varepsilon_{p,3}(\xi,u).$$

$$W_{p,4}(\xi,u) = |\xi|^{1/2}\ K_\nu(2ub|\xi|^{1/2}) \sum_{s=0}^{p} \frac{A_s(\xi)}{(2ub)^{2s}}$$

$$- \frac{|\xi|}{2ub}\ K_{\nu+1}(2ub|\xi|^{1/2}) \sum_{s=0}^{p-1} \frac{B_s(\xi)}{(2ub)^{2s}} + \varepsilon_{p,4}(\xi,u), \qquad (5.13)$$

where

$$|\varepsilon_{p,3}(\xi,u)| \le \frac{\lambda_1(\nu)}{(2ub)^{2p+1}}\ |\xi|^{1/2}$$

$$I_\nu(2ub|\xi|^{1/2})\ V_{\xi,0}(|\tau|^{1/2}\ B_p(\tau)).$$

$$\cdot \exp\left\{\frac{\lambda_1(\nu)}{2ub}\ V_{\xi,0}\ (|\tau|^{1/2}\ B_0(\tau))\right\} \qquad (5.14)$$

$$\left|\frac{\partial}{\partial\xi}\ \varepsilon_{p,3}(\xi,u)\right| \le \frac{\lambda_1(\nu)}{(2ub)^{2p}}\ \frac{K_{\nu+1}(2ub|\xi|^{1/2})I_\nu(2ub|\xi|^{1/2})}{K_\nu(2ub|\xi|^{1/2})}$$

$$\cdot V_{\xi,0}(|\tau|^{1/2}\ B_p(\tau))\exp\left\{\frac{\lambda_1(\nu)}{2ub}\ V_{\xi,0}(|\tau|)^{1/2}\ B_0(\tau))\right\}$$

$$+ \left|\frac{\nu+1}{2\xi}\ \varepsilon_{p,3}(\xi,u)\right|, \qquad (5.15)$$

$$|\varepsilon_{p,4}(\xi,u)| \le \frac{\lambda_1(\nu)}{(2ub)^{2p+1}}\ |\xi|^{1/2}\ K_\nu(2ub|\xi|^{1/2})$$

$$V_{\delta,\xi}\ (|\tau|^{1/2}\ B_p(\tau)).\ \exp\left\{\frac{\lambda_1(\nu)}{2ub}\ V_{\delta,\xi}\ (|\tau|^{1/2}\ B_0(\tau))\right\} \qquad (5.16)$$

$$\left|\frac{\partial}{\partial\xi}\ \varepsilon_{p,4}\ (\xi,u)\right| \le \frac{\lambda_1(\nu)}{(2ub)^{2p}}$$

$$K_{\nu+1}\ (2ub|\xi|^{1/2})V_{\delta,\xi}(|\tau|^{1/2}B_p(\tau))$$

$$\cdot \exp\left\{\frac{\lambda_1(\nu)}{2ub}\ V_{\delta,\xi}\ (|\tau|^{1/2}\ B_0(\tau))\right\} + \left|\frac{\nu+1}{2\xi}\ \varepsilon_{p,4}\ (\xi,u)\right| \qquad (5.17)$$

where

$$\lambda_1(\nu) = \text{Sup}\left\{2x\ I_\nu(x)K_\nu(x)\right\}\ ,\quad x \in (0,\infty)\ . \qquad (5.18)$$

6. UNIFORMITY OF THE EXPANSIONS

The uniformity of our asymptotic expansions for $W_{p,j}(\xi,u)$ will be proven by showing that $\varepsilon_{p,j}(\xi,u)$ is uniformly bounded for $\xi\epsilon(\delta,\gamma)$ as u becomes infinite. It has been shown [6] that the factors $\lambda_1(\nu)$ are bounded. All the total variations appearing in the bounds of $\varepsilon_{p,j}(\xi,u)$ are bounded due to the continuity and differentiability of the functions $B_0(\tau)$ and $B_s(\xi)$ for ξ e (δ,γ). In the case $\delta = -\infty$ or $\gamma = \infty$ a necessary contidion for the validity of the theorem is the convergence of $|\xi|^{1/2}B_0(\xi)$ and $|\xi|^{1/2}$ $B_p(\xi)$ at infinity. Consequently, for large u we have uniformly from (5.4) that

$$|\varepsilon_{p,1}(\xi,u)| = \xi^{1/2}\, E_\nu^{-1}\,(2ub\xi^{1/2})M_\nu(2ub\xi^{1/2})O(\frac{1}{u^{2p+1}}) \tag{6.1}$$

Writing $2ub\xi^{1/2} = y$ and making use of the definitions of the functions E and M, we write

$$E_\nu^{-1}\,(y)M_\nu(y) = U_\nu(y)J_\nu(y) + S_\nu(y)\,J_{\nu+1}(y)\,, \tag{6.2}$$

where

$$U_\nu(y) = 2^{1/2}\,,\ S_\nu(y) = 0 \qquad (0 \le y \le X_\nu)$$

and

$$U_\nu(y) = - \frac{\pi}{2}\, y\, Y_{\nu+1}(y)M_\nu(y)\ ,\ S_\nu(y) = \frac{\pi}{2}\, Y_\nu(y)M_\nu(y)\ ,\ (y \ge X_\nu). \tag{6.3}$$

For y large, we know that [5]

$$J_\nu(y) = (2/\pi y)^{1/2}\,\{\cos(y-\pi\nu/2-\pi/4) + O(1/y)\}$$

$$Y_\nu(y) = (2/\pi y)^{1/2}\{\sin(y-\pi\nu/2-\pi/4) + O(1/y)\}. \tag{6.4}$$

Hence, for $y \;\epsilon\; (0,\infty)$, $|U_\nu(y)|$ and $|S_\nu(y)|$ are bounded. Therefore, the right hand side of (4.2) with $C_\nu = J_\nu$ provides a uniform asymptotic expansions of $W_{p,1}(\xi,u)$ to 2p+1 terms. Similarly, from (5.6) we have

$$|\varepsilon_{p,2}(\xi,u)| = \xi^{1/2}\; E_\nu(2ub\xi^{1/2})M_\nu(2ub\xi^{1/2})O(-\frac{1}{u^{2p+1}})$$

Writing

$$E_\nu(y)M_\nu(y) = \bar{U}_\nu(y)Y_\nu(y) + y\bar{S}(y)Y_{\nu+1}(y)\,, \tag{6.5}$$

where

$$\bar{U}_\nu(y) = 2^{1/2}\,,\ \bar{S}_\nu(y) = 0 \qquad (0 \le y \le X_\nu) \tag{6.6}$$

$$\bar{U}_\nu(y) = \frac{\pi y}{2}\, J_{\nu+1}(y)M_\nu(y),\ \bar{S}_\nu(y)$$

$$= -\frac{\pi}{2}\, J_\nu(y)M_\nu(y) \qquad (y \le X_\nu)$$

We note that $|\bar{U}_\nu(y)|$ and $|\bar{S}_\nu(y)|$ are bounded on $(0,\infty)$; therefore the right hand side of (4.2) with $C_\nu = Y_\nu$ provides a uniform asymptotic expansion of $W_{p,2}(\xi,u)$ to 2p+1 terms. Consider now the error functions $\varepsilon_{p,3}(\xi,u)$ and $\varepsilon_{p,4}(\xi,u)$ for $\xi\ \epsilon(\delta,0)$ and u approching infinity. The factor $\lambda_1(\nu)$ is finite [6]. Hence,

$$\varepsilon_{p,3}(\xi,u) = |\xi|^{1/2}\, I_\nu(2ub|\xi|^{1/2})O(\frac{1}{u^{2p+1}}) \tag{6.8}$$

$$\varepsilon_{p,4}(\xi,u) = |\xi|^{1/2}\, K_\nu(2ub|\xi|^{1/2})O(\frac{1}{u^{2p+1}}) \tag{6.9}$$

uniformly as $u \to \infty$. Therefore the right hand side of (4.2) with $C_\nu = I_\nu$ and $C_\nu = K_\nu$ provides a uniform asymptotic expansions for $W_{p,3}(\xi,u)$ and $W_{p,4}(\xi,u)$, respectively, to 2p+1 terms.

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