Rev. Téc. Ing., Univ. Zulia
Vol. 14, No. 1, Edición Especial, 1991

MIKLÓS MIKOLÁS
Department of Mathematics
Technical University of Budapest
H-1125 Budapest, Diósárok út 12/b
Hungary

ON THE EXTENSION OF THE BASIC LAW OF LAMINAR FLOWS BY MEANS OF ELLIPTIC INTEGRALS

ABSTRACT

A few years ago, the author succeeded in extending the main law of laminar flows to certain more general forms of hoop cross sections, namely by using geometrical and analytical considerations of new type. The present paper deals with the case of elliptical sections for which the theory of elliptic integrals allows of a quite simple integral representation and series expansion for the so-called volume stream.

RESUMEN

Hace unos pocos años el autor logró extender la ley principal del flujo laminar a una cierta forma más general de secciones transversales circulares, a saber usando consideraciones geométricas y analíticas de un nuevo tipo. El presente trabajo trata el caso de secciones elípticas para las cuales la teoría de integrales elípticas permite una representación integral completamente simple y un desarrollo en serie para el así llamado flujo de volumen.

I. INTRODUCTION

As it is well-known, the *Hagen-Poiseuille law* on laminar flow of Newtonian fluids (1839-40) which has become familiar shortly after its finding on varuous filds of physics, chemistry and biology, recently was built also into the engineering. (Cf. e.g. [3], [6], [13]). However its application possibilites are very limited by the fact that this relation holds, strictly speaking, only in the case of circular or annular cross sections. In any other case, the classical deduction of the law (establishing the condition of equilibrium for the totality of forces acting to an infinitesimal streamline, and hence conclusion by integration to the corresponding speed distribution and volume stream, respectively cf. e.g. [10]) yields merely approximative results. The situation remains the same if we start from and "analogy of Greenhill" (1881) and assume that the velocity ν satisfies everywhere in the flow domain the Poisson equation $\Delta\nu \approx$ constant with the boundary condition $\nu = 0$. (See e.g. [2], 179-182; [3], 95-98; [10], 660-663; [11], 62-63.)

In the practice, it has been used until recent past another idea: to try the reduction of flows with more complicated cross sections to the case of circular ones on the basis of the so-called hydraulic diameter. Unfortunately, some investigations in the seventies showed hat this way may result essential errors when the occurring pressure drop is calculated. (Cf. [15]). So since then certain approximate analytic methods have been preferred again on the research field at issue. (E.g. [5] and [16].)

In this article, a method of such type which arose during the mathematical foundation of a new engine insulation technology (see [12]) is developed further for the important particular case of elliptic ducts.

II. A SPEED DISTRIBUTION FORMULA FOR ELLIPTICAL DUCTS

Let us study a cylindrical pipe of length l, containing so-called Newtonian fluid which flows stationarily with (scalar) velocity ν. Suppose that the cross section of the pipe is limited by two concentric wllipses whose common centre lies on the axis of the duct and such that the inner ellipse rises from the outer one by a contraction in the ratio 1: ε; denote the half longer axis of the outer ellipse by α, the half shorter axis of the same ellipse by b. If we assume still that the stream lines are parallel with the axis fo the duct, then *Newton's law* yields the shearing tension τ along some "fluid stratum" at a disctance r from the axis in the form

$$\tau = \eta \frac{d\nu}{dr}, \qquad (1)$$

where the factor of proportionality η is called "dynamic viscosity" of the fluid.

Take on the initial cross section of the pipe a Cartesian rectangular coordinate system xy such that its origin O lies on the axis for the duct; let us facten to the system a third axis z, namely in the direction of the flow. If the system xy is supplemented also to a planar polar system $r\varphi$ with the same origin O, then the equation of the outer ellipse of the cross section in the system $r\varphi$ will be:

$$r = ab(a^2\sin^2\varphi + b^2\cos^2\varphi)^{-1/2} \quad , \qquad (2)$$

while the inner ellipse in characterized by

$$r = \varepsilon . ab(a^2\sin^2\varphi + b^2\cos^2\varphi)^{-1/2} \quad , \qquad (3)$$

with $0 < \varepsilon < 1$. We remark at the same time: if the duct of the flow is crossed by a plane through the z-axis, belonging to a polar angle φ, then - R denoting the right-hand expression in (2) - we obtain a rectangle of breadth $(1 - \varepsilon)R$ and of length l. For the sake of brevity, we want to use in the sequel not only the notation

$$R = R(\varphi) = ab/\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi} \; , \qquad (4)$$

but also we write throughout R' for $dR/d\varphi$

Now, our setting of objective is to deduce a new expression for the shearing tension τ and hence a representation for the velocity ν, as functions of r and φ. To that end, we have to start with the stationarity condition of the flow in case of an arbitrary volume element inside the steam duct. As usual, we take the restrictions that the pressure p is constant on each cross section of the duct, furthermore that it is a linearly decreasing function of z. Note that the second specification implies for the difference $\Delta p = p(z + \Delta z) - p(z)$ (<0) the formula:

$$\frac{\Delta p}{\Delta z} = -\frac{p_1 - p_2}{l} \; , \qquad (5)$$

where p_1, p_2 mean the initial (maximal) and the last (minimal) pressure value, respectively.

Theorem 1. On the above conditions, the velocity of the flow in elliptical duct satisfies the following distribution law:

$$\nu = \frac{p_1 - p_2}{4\eta l}\left[1 + \left(\frac{R'}{R}\right)^2\right]^{-1/2}\left(R^2 - r^2 - R^2\frac{\varepsilon^2 - 1}{\log\varepsilon}\log\frac{R}{r}\right), \qquad (6)$$

where R has the meaning (4), $0 < \varepsilon < 1$ and $\varepsilon R < r < R$.

Proof. Consider a region ABCD in the plane xy, which is bounded by the ellipses $r = \vartheta R$ ($\varepsilon < \vartheta < 1$), $r + \Delta r = (\vartheta + \Delta\vartheta)R$ and the straight lines characterized by the polar angles φ, $\varphi + \Delta\varphi$. Let fall perpendiculars to the boundary points of ABCD, and cut the bundle of these perpendiculars by two planes going through the points with coordinates z, $z + \Delta z$ of the duct axis, further being parallel with the plane xy. (Cf. Figures 1.-2.) The condition of equilibrium for a volume elemnt of the type just mentioned in the coordinate system xyz may be written:

$$\Delta p. \text{ (area of ABCD)} \approx \Delta Z \; .$$
$$\left[\text{(length of } \overline{CD}).\ \tau_{(\vartheta+\Delta\vartheta)R} - \text{(length of } \overline{AB}).\tau_{\vartheta R}\right]. \qquad (7)$$

Here $\tau_{\varphi R}$, $\tau_{(\varphi+D\varphi)}$ denote the value of the shearing tension on those parts of the volume element which are perpendicular to $\overline{AB}$ and $\overline{CD}$,respectively. (Observe that these tensions are approximately constant whenever the volume element is "small"; further the concept of shearing tension implies that it does not effect on the set of boundary points corresponding to $\overline{AB}$ and $\overline{BC}$.)

The measures occurring in the above formula can be obtained by well-known propositions of differential geometry; we have namely

$$\text{length of } \overline{AB} = \vartheta\Delta s, \qquad \text{length of } \overline{CD} = (\vartheta + \Delta\vartheta)\Delta s,$$
$$\text{area of ABCD} = \Delta\vartheta(2\vartheta + \Delta\vartheta)\Delta I,$$

where

$$\Delta s = \int_{\varphi}^{\varphi+\Delta\varphi} \sqrt{[R(\Phi)]^2 + [R'(\Phi)]^2}d\Phi, \qquad (8)$$

$$\Delta I = \frac{1}{2}\int_{\varphi}^{\varphi+\Delta\varphi} [R(\Phi)]^2\, d\Phi. \qquad (9)$$

Thus (7) is equivalent to the formual ($\Delta\vartheta \neq 0$):

$$(\Delta\vartheta)^{-1}[(\vartheta + \Delta\vartheta)\tau_{(\vartheta+\Delta\vartheta)R} - \vartheta\tau_{\vartheta R}] \approx (\vartheta + \Delta\vartheta)\frac{\Delta p}{\Delta z} \cdot \frac{\Delta I}{\Delta s} \; , \qquad (10)$$

which by (5) for $\Delta\vartheta \to 0$, $\Delta\varphi \to 0$, $\Delta s \to 0$ implies

$$\frac{\partial(\vartheta\tau_{\vartheta R})}{\partial\vartheta} = -2\vartheta\,\frac{p_1 - p_2}{l}\,\frac{dI}{ds} \; ,$$

i.e. owing to (8)-(9) the differential relation

$$\frac{\partial(\vartheta\tau_{\vartheta R})}{\partial\vartheta} = -\vartheta\frac{p_1-p_2}{l}\frac{R^2}{\sqrt{R^2+R'^2}}. \quad (11)$$

Integrating with respect to ϑ and writing simply $\tau_r=\tau$, we get therefore

$$\tau = -\frac{p_1-p_2}{2l}\frac{rR}{\sqrt{R^2+R'^2}} + M_{\varepsilon,\varphi}\frac{R}{r}, \quad (12)$$

where for all values of φ

$$\lim_{\varepsilon\to 0+} M_{\varepsilon,\varphi} = 0 \quad (13)$$

holds. But on the basis of (1), (12) takes the form:

$$\frac{\partial\nu}{\partial r} = -\frac{p_1-p_2}{2\eta l}\frac{rR}{\sqrt{R^2+R'^2}} + \frac{M_{\varepsilon,\varphi}}{\eta} + \frac{M_{\varepsilon,\varphi}}{\eta}\frac{R}{r}$$

or after another integration with respect to r:

$$\nu = -\frac{p_1-p_2}{4\eta l}r^2\left[1+\left(\frac{R'}{R}\right)^2\right]^{-1/2} + \frac{M_{\varepsilon,\varphi}}{\eta}R\log r + \Omega_{\varepsilon,\varphi}.$$

If we realize still the boundary conditions

$$\nu_{r=R} = \nu_{r=\varepsilon R} = 0, \quad (15)$$

the dependence of the integration parameters M, Ω on ε and φ can be specified easily. In fact, the suitable substitutions in (14) show that

$$M_{\varepsilon,\varphi} = \frac{p_1-p_2}{4l}R\left[1+\left(\frac{R'}{R}\right)^2\right]^{-1/2}\frac{\varepsilon^2-\varphi}{\log\varepsilon},$$

$$\Omega_{\varepsilon,\varphi} = \frac{p_1-p_2}{4l}R^2\left[1+\left(\frac{R'}{R}\right)^2\right]^{-1/2}\left[1-(\log R)\frac{\varepsilon^2-1}{\log\varepsilon}\right].$$

Putting these expressions into (14), the assertion (6) follows immediatly.

* * *

It is to be streesed that our result becomes in case of an annulus i.e. for $R = R_1$ = constant, $\varepsilon R = R_2$ = constant a well-known exact formula of flow (cf. e.g. [11], p.63):

$$\nu = \frac{p_1-p_2}{4\eta l}\left[R_1^2 - r^2 + \frac{R_1^2-R_2^2}{\log(R_1/R_2)}\log\frac{r}{R_1}\right], \quad (16)$$

where $0 < R_2 < r < R_1$

III. GENERALIZATION OF THE HAGEN-POISEUILLE LAW

Let us denote by V the so-called *valume steam*, i.e. the valume of the fluid flowing through some cross section of the duct during the unit of time. According to as classical formula of vector analysis, we have the double integral representation:

$$\dot{V} = \iint_{(\sigma)} \nu\, dx\, dy, \quad (17)$$

where σ is the cross section of the duct and $\nu = \nu(x,y)$ means the velocity of flow. Applying the polar transformation $x = r\cos\varphi$, $y = r\sin\varphi$ and using the above notations, the integral (17) becomes

$$\int_0^{2\pi}\left[\int_{\varepsilon R}^{R}\nu(r,\varphi)r\,dr\right]d\varphi.$$

Consequently, on the basis of (4) and (6) we get:

$$\dot{V} = \frac{p_1-p_2}{16\eta l}\left[1-\varepsilon^4+\frac{(1-\varepsilon^2)^2}{\log\varepsilon}\right]\int_0^{2\pi}R^4\left[1+\left(\frac{R'}{R}\right)^2\right]^{-1/2}d\varphi \quad (18)$$

The latter integral may also be written in the form:

$$4(ab)^4\int_0^{\pi/2}\frac{d\varphi}{T_{a,b}(\varphi)\sqrt{[T_{a,b}(\varphi)]^2+[U_c(\varphi)]^2}}, \quad (19)$$

where $c = (a^2-b^2)^{1/2}$ denotes the *linear excentricity* of the outer boundary ellipse of the cross section at issue, furthermore

$$T_{a,b}(\varphi) = a^2\sin^2\varphi + b^2\cos^2\varphi, \quad U_c(\varphi) = c^2\sin\varphi\cos\varphi.$$

We remark that (19) belongs by terms of the *theory of elliptic functions and integrals* (cf. e.g. [4] or [14] and [9], section 21.6) to the class of complete elliptic integrals of the third kind, because it can be transformed by the substitution $x = \text{tg}\,\varphi$ into the type

$$\text{const.}\int_0^{\infty}\frac{x^2+1}{(a^2x^2+b^2)\sqrt{(a^2x^2+b^2)^2+c^2x^2}}dx. \quad (20)$$

In such a way, one has the possibility of estimating (19) means of certain elementary integrals of rational functions.

Nevertheless, there is another, more elegant and advantageous method for the study of $\dot{V}$, namely the use of the connection of the integral (19) with

$$K(k) = \int_0^{\pi/2} \frac{d\psi}{(1-k^2\sin\psi)^{1/2}} \qquad (|k|<1), \qquad (21)$$

i.e. the so-called *complete elliptic integral of the first kind.* We refer primarily to the fact that for arbitrary real numbers k, λ with $k^2 + \lambda^2 = 1$, the transformation formula

$$\int_0^{\pi/2} \frac{d\psi}{[1-(1-\lambda)\sin^2\psi]\sqrt{1-k^2\sin^2\psi}} = \frac{1}{\lambda} K\left(\frac{1-\lambda}{1+\lambda}\right) \qquad (22)$$

holds. (See [1], 599-600; [7], 68-69; [8], 39-41.)

Thus applying to (19) the substitution $\varphi = \frac{\pi}{2} - \psi$ and putting in (22) $\lambda = b^2/a^2$, $k^2 = (a^4-b^4)/a^4$, we obtain

$$\int_0^{\pi/2} \frac{d\psi}{(a^2-c^2\sin^2\psi)\sqrt{a^4-(a^4-b^4)\sin^2\psi}} = \frac{1}{a^2b^2} K\left(\frac{c^2}{a^2+b^2}\right) \qquad (23)$$

and hence by the above it follows:

Theorem 2. With notations of Theorem 1, the volume sream of a Newtonian fluid in case of an elliptical duct can be expressed as a simple elliptic integral, namely

$$\dot{V} = \frac{(p_1-p_2)}{4\eta l}\left[1 - \varepsilon^4 + \frac{(1-\varepsilon^2)^2}{\log\varepsilon}\right] K\left(\frac{c^2}{a^2+b^2}\right) =$$

$$= \frac{a^2b^2\pi(p_1-p_2)}{8\eta l}\left[1 - \varepsilon^4 + \frac{(1-\varepsilon^{\cdot})^{-}}{\log\varepsilon}\right] \times \qquad (24)$$

$$\times \left[1 + \left(\frac{1}{2}\right)^2\left(\frac{c^2}{a^2+b^2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2\left(\frac{c^2}{a^2+b^2}\right)^4 + \dots\right].$$

It is apparent that (24) reduces for $a=b=R_1=$ constant, $c=0^\circ$ (i.e. in case of an *annulus*), with notation $\varepsilon R_1 = R_2$ to a formula known from the literature:

$$\dot{V} = \frac{\pi(p_1-p_2)}{8\eta l}\left[R_1^4 - R_2^4 - \frac{(R_1^2 - R_2^2)^2}{\log(R_1/R_2}\right] \qquad (R_1 > R_2). \qquad (25)$$

In particular for $R_2 \to 0+$ (case of a *full circle*), with notation $R_1 = \rho$, (25) becomes

$$\dot{V} = \frac{\pi(p_1 - p_2)}{8\eta l}\rho^4, \qquad (26)$$

which is the classical *law of Hagen and Poiseuille.* (Cf. e.g. [10], 662-663 and [11], 63.)

Finally let us notice that the method presented here for extension of (25)-(26) could be developed further quite recently by the author also to a wide class of centrally symmetrical cross sections.

REFERENCES

[1] ABRAMOWITZ, M.; STEGUN, I.A.: Handbook of Mathematival Functions (7 edition). U.S. National Bureau of Standards, Washington, D.C., 1968.

[2] BATCHELOR, G.K.: Fluid Dynamics. Cambridge University Press, 1967.

[3] BECKER, E.: Technische Strömungslehre (2.Auflage). Teubner-Verlag, Stuttgart, 1970.

[4] BYRD, P.F.; FRIEDMAN, M.D.: Handbook of Elliptic Integrals. Springer-Verlag, Berlin, 1954.

[5] CORNISH, R.J.: Flow in a Pipe of Rectangular Cross Section. Proceedings of the Royal Society, London, 120 A(1928), 691-700.

[6] EIRICH, F.R.: Rheology Theory and Applications I-III. Academic Press, New York, 1956.

[7] GRÖBNER, W.;HOFREITER, N.: Integraltafel I. (2. Auflage). Springer-Verlag, Wien-Innsbruck, 1957.

[8] GRÖBNER, W.; HOFREITER, N.: Integraltafel II. Springer-Verlag, Wien-Innsbruck, 1950.

[9] KORN, G.A.; KORN, T.W.: Mathematical Handbook for Scientists and Engineers (2 edition). Mc Graw-Hill, New York, 1970.

[10] LAMB, H.: Hydrodynamik (2.Auflage). Teubner-Verlag, Leipzig-Berlin, 1931.

[11] LANDAU, L.D.; LIFSCHITZ, E.M.: Lehrbuch der theoretischen Physik VI, Akademie-Verlag, Berlin, 1966. (Translation.).

[12] MIKOLAS, M.; BARDOCZ, L.: Neuere Untersuchungen im Problemenkreis des Hagen-Poiseuilleschen Strömungsgesetzes. Acta Physica Academiae Scientiarum Hungaricae, Budapest, 47 (1980), 335-343.

[13] SCHENKEL, G.: KUNSTSTOFF-EXTRUDERTECHNIK (2.Auflage)-Verlag, München, 1963.

[14] TRICOMI, F.G.: Elliptische Funktiomem. Akademie-Verlag, Leipzig, 1948.

[15] WHITE, F.M.: Viscous Fuid Flow. Mc Graw-Hill, New York, 1974.

[16] ZARLING, J.P.: An Analysis of Laminar Flow and Pressure Drop in Complex-Shaped Ducts. Transactions of the American Society of Mechanical Engineers, Journal of Fluids Engineering, 98(I)/4, 702-706.

Recibido el 07 de Diciembre de 1990