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UNA GENERALIZACION DE LA INTEGRAL DE HUBBELL

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RESUMEN

En este trabajo estudiamos una integral generalizada de radiación de la forma

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ \alpha,\beta,\gamma \end{bmatrix} = \frac{\sigma a}{4\pi}\int_0^b x^{\lambda}\,(x^2+p)^{-\alpha}\left(1-\frac{x^2}{b^2}\right)^{\mu} \times {}_2F_1\left(\alpha,\beta;\gamma;-\frac{a^2}{x^2+p}\right) dx$$

donde $p > 0$, $0 < a \le b < \infty$
$\mu > -1$, $-1 < \lambda < 2\alpha-2\mu-1$
y ${}_2F_1(\alpha,\beta;\gamma;t)$ es la función hipergeométrica de Gauss.

Casos particulares de esta integral han sido considerados por varios autores, incluyendo a Hubbell, Kalla, Al-Saqabi y Conde, y aparecen en problemas de radiación . Para esta integral generalizada derivamos su representación en términos de la función F_2 de Appell y obtenemos la transformada de Laplace del producto de dos funciones hipergeométricas confluentes $_1F_1$. También se presenta su desarrollo en serie y se mencionan algunos casos particulares.

2. ABSTRACT

In this work we consider a generalized integral radiation of the form

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ \alpha,\beta,\gamma \end{bmatrix} = \frac{\sigma a}{4\pi}\int_0^b x^{\lambda}\,(x^2+p)^{-\alpha}\left(1-\frac{x^2}{b^2}\right)^{\mu} \times {}_2F_1\left(\alpha,\beta;\gamma;-\frac{a^2}{x^2+p}\right) dx$$

where $p > 0$, $0 < a \le b < \infty$
$\mu > -1$, $-1 < \lambda < 2\alpha-2\mu-1$
and ${}_2F_1(\alpha,\beta;\gamma;t)$ is the Gauss hypergeometric function.

Particular cases of this integral have been considered by various authors, including Hubbell, Kalla, Al-Saqabi and Conde, and appear in radiation - field problems . For this generalized integral we derive its representation in terms of the Appell's function F_2 , and obtain the Laplace transform of product of two confluent hypergeometric functions $_1F_1$. Also its series expansion is presented and some particular cases are mentioned.

3. INTRODUCCION

La respuesta I(a,b) de un detector de radiación unidireccional a una altura h directamente sobre una esquina de una fuente rectangular plana isotrópica (placa) de longitud l ,ancho w y distribución uniforme σ puede ser expresada como [1]

$$I(a,b) = \frac{\sigma}{4\pi}\int_0^b \text{arc tg}\left(\frac{a}{\sqrt{x^2+1}}\right)\frac{dx}{\sqrt{x^2+1}} \qquad (1)$$

$a > 0$, $b > 0$

donde $a = w / h$, $b = l / h$

Este trabajo trata una generalización de (1) , definida por la integral,

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ \alpha,\beta,\gamma \end{bmatrix} = \frac{\sigma a}{4\pi}\int_0^b x^{\lambda}\,(x^2+p)^{-\alpha}\left(1-\frac{x^2}{b^2}\right)^{\mu} \times {}_2F_1\left(\alpha,\beta;\gamma;-\frac{a^2}{x^2+p}\right) dx \qquad (2)$$

con $p > 0$, $0 < a \le b < \infty$
$\mu > -1$, $-1 < \lambda < 2\alpha-2\mu-1$

donde ${}_2F_1(\alpha,\beta;\gamma;t)$ es la función hipergeométrica de Gauss.

De (2) obtenemos como casos particulares, usando la identidad:

$$\text{arc tg } z = z \; {}_2F_1(1,1/2;3/2;\; -z^2) \qquad (3)$$

los siguientes resultados:

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ 1,1/2,3/2 \end{bmatrix} = \frac{\sigma}{4\pi}\int_0^b x^\lambda \left(1-\frac{x^2}{b^2}\right)^\mu \text{arc tg}\left(\frac{a}{\sqrt{x^2+p}}\right) \times \frac{dx}{\sqrt{x^2+p}} \qquad (4)$$

y

$$I\begin{bmatrix} a,b,1,0,0 \\ 1,1/2,3/2 \end{bmatrix} = \frac{\sigma}{4\pi}\int_0^b \text{arc tg}\left(\frac{a}{\sqrt{x^2+1}}\right) \times \frac{dx}{\sqrt{x^2+1}} = I(a,b) \qquad (5)$$

Seleccionando valores apropiados para los parámetros α,β y γ,(2) reduce a diferentes integrales con potenciales aplicaciones en problemas de campo de radiación de fuente,protector y detector de configuración específica. Tales resultados son además útiles en iluminación y en ingeniería de intercambio de calor [1,2].

La transformación simple de la función hipergeométrica:

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha}\; {}_2F_1\left(\alpha,\gamma-\beta;\gamma;\; \frac{z}{z-1}\right) \qquad (6)$$

con $|\arg(1-z)| < \pi$

produce el resultado :

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ \alpha,\beta,\gamma \end{bmatrix} = \frac{\sigma a}{4\pi}\int_0^b x^\lambda (x^2+p+a^2)^{-\alpha} \times \left(1-\frac{x^2}{b^2}\right)^\mu {}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{a^2}{x^2+p+a^2}\right) dx \qquad (7)$$

y como

$$\text{arc sen } z = z \; {}_2F_1(1/2,1/2;3/2;z^2) \qquad (8)$$

tenemos

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ 1/2,1,3/2 \end{bmatrix}^* = \frac{\sigma}{4\pi}\int_0^b x^\lambda \left(1-\frac{x^2}{b^2}\right)^\mu \times \text{arc sen}\left(\frac{a}{\sqrt{x^2+p+a^2}}\right) dx \qquad (9)$$

Podemos expresar (7) en términos de las integrales elípticas completas de primera y segunda clase,

$$K(z) = \frac{\pi}{2}\; {}_2F_1(1/2,1/2;1;z^2) \qquad |\arg(1\pm z)| < \pi \qquad (10)$$

$$E(z) = \frac{\pi}{2}\; {}_2F_1(-1/2,1/2;1;z^2) \qquad |\arg(1\pm z)| < \pi \qquad (11)$$

en la forma siguiente:

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ 1/2,1/2,1 \end{bmatrix} = \frac{\sigma a}{2\pi^2}\int_0^b x^\lambda (x^2+p+a^2)^{-1/2} \times \left(1-\frac{x^2}{b^2}\right)^\mu K\left(\frac{a}{\sqrt{x^2+p+a^2}}\right) dx \qquad (12)$$

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ -1/2,1/2,1 \end{bmatrix} = \frac{\sigma a}{2\pi^2}\int_0^b x^\lambda (x^2+p+a^2)^{1/2} \times \left(1-\frac{x^2}{b^2}\right)^\mu E\left(\frac{a}{\sqrt{x^2+p+a^2}}\right) dx \qquad (13)$$

Si hacemos $\alpha = -n$, $\gamma = \alpha+1$ y reemplazamos β por $n+\alpha+\beta+1$ en (2), obtenemos:

$$I\begin{bmatrix} a,b,p,\lambda,\mu \\ -n,n+\alpha+\beta+1,\alpha+1 \end{bmatrix} = \frac{\sigma a}{4\pi}\frac{n!}{(\alpha+1)_n}\int_0^b x^\lambda (x^2+p)^n \times \left(1-\frac{x^2}{b^2}\right)^\mu P_n^{(\alpha,\beta)}\left(\frac{x^2+p+2a^2}{x^2+p}\right) dx \qquad (14)$$

donde $P_n^{(\alpha,\beta)}(x)$ son los Polinomios de Jacobi definidos por:

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\; {}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\; \frac{1-x}{2}\right) \qquad (15)$$

Escogiendo valores particulares de α y β, podemos expresar (14) en términos de otros polinomios ortogonales, tales como Gegenbauer, Legendre, Chebyshev, etc.

4 - $I\begin{bmatrix} a,b,p,\lambda,\mu \\ \alpha,\beta,\gamma \end{bmatrix}$ en términos de la Función F_2 de Appell

Sustituyendo en (2) a ${}_2F_1(\alpha,\beta;\gamma;z)$ por su representación integral tenemos:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\,\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^b\!\!\int_0^1 x^{\lambda}(1-t)^{\gamma-\beta-1} \times$$

$$\left(1-\frac{x^2}{b^2}\right)^{\mu} t^{\beta-1}\,(x^2+p+a^2t)^{-\alpha}\;dt\,dx$$

haciendo el cambio $x=b\sqrt{\tau}$ y usando la representación integral de la función F_2 de Appell [4]:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a\; b^{\lambda+1}}{8\pi\; p^{\alpha}}\;\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{\lambda}{2}+\frac{3}{2}\right)} \times$$

$$F_2\left(\alpha,\beta,\frac{\lambda}{2}+\frac{1}{2};\gamma,\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \qquad (16)$$

Tomando $\mu=0$ en (16) tenemos:

$$I\begin{bmatrix}a,b,p,\lambda,0\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a\; b^{\lambda+1}}{4\pi\; p^{\alpha}}\;\frac{1}{(\lambda+1)} \times$$

$$F_2\left(\alpha,\beta,\frac{\lambda}{2}+\frac{1}{2};\gamma,\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \qquad (17)$$

que es un resultado dado por Kalla [5].

5.- $I\begin{bmatrix}a,b,p,\lambda,\mu\\ 1,\beta,\gamma\end{bmatrix}$ como una Transformada de Laplace respecto al parámetro p

De (2) tenemos que:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ 1,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\int_0^b x^{\lambda}\,(x^2+p)^{-1} \times$$

$$\left(1-\frac{x^2}{b^2}\right)^{\mu} {}_2F_1\left(1,\beta;\gamma;-\frac{a^2}{x^2+p}\right)dx$$

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ 1,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\,\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^b x^{\lambda}\,(x^2+p)^{-1} \times$$

$$\left(1-\frac{x^2}{b^2}\right)^{\mu}\left\{\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left(1+\frac{a^2}{x^2+p}\,t\right)^{-1} \times\right.$$

$$\left.dt\right\} dx = \frac{\sigma a}{4\pi}\,\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^b x^{\lambda}\left(1-\frac{x^2}{b^2}\right)^{\mu} \times$$

$$\left\{\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\left[\int_0^{\infty} e^{-u(x^2+p+ta^2)}du\right]dt\right\}dx$$

intercambiando el orden de integración, en base a la convergencia absoluta de las integrales involucradas,

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ 1,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\,\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^{\infty} e^{-up}\left\{\int_0^b x^{\lambda} \times\right.$$

$$\left(1-\frac{x^2}{b^2}\right)^{\mu} e^{-ux^2}\left[\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}\,e^{-ua^2t}\,dt\right]$$

$$\left.dx\right\}du$$

haciendo el cambio $x=b\tau^{1/2}$ y de la definición de la función hipergeométrica confluente nos queda:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ 1,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{8\pi}\;\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{\lambda}{2}+\frac{3}{2}\right)}\,b^{\lambda+1} \times$$

$$\int_0^{\infty} e^{-up}\;{}_1F_1(\beta;\gamma;-ua^2) \times$$

$${}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right)du \qquad (18)$$

Comparando (16) y (18) obtenemos que:

$$L\left[{}_1F_1(\beta;\gamma;-ua^2) \times {}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right);p\right] = \frac{1}{p} \times$$

$$F_2\left(1,\beta,\frac{\lambda}{2}+\frac{1}{2};\gamma,\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \qquad (19$$

6.- DESARROLLO EN SERIE.-

De (2), expresando a ${}_2F_1(\alpha,\beta;\gamma;t)$ como una serie e intercambiando el orden de la integral y la suma en virtud de la convergencia absoluta, tenemos:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\;k!}\,(-a^2)^k \times$$

$$\int_0^b x^{\lambda}\,(x^2+p)^{-\alpha-k}\left(1-\frac{x^2}{b^2}\right)^{\mu}dx$$

haciendo el cambio $x=b\sqrt{\tau}$ y usando la representación integral de la función hipergeométrica de Gauss,

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{8\pi}\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{\lambda}{2}+\frac{3}{2}\right)}\frac{b^{\lambda+1}}{p^{\alpha}} \times$$

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,k!}\left(-\frac{a^2}{p}\right)^k \times$$

$${}_2F_1\left(\alpha+k,\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{b^2}{p}\right) \qquad (20)$$

Si en (20) hacemos $\mu=0$ queda:

$$I\begin{bmatrix}a,b,p,\lambda,0\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\frac{b^{\lambda+1}}{(\lambda+1)\,p^{\alpha}}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,k!} \times$$

$$\left(-\frac{a^2}{p}\right)^k {}_2F_1\left(\alpha+k,\frac{\lambda}{2}+\frac{1}{2};\frac{\lambda}{2}+\frac{3}{2};-\frac{b^2}{p}\right)$$

$$I\begin{bmatrix}a,b,p,\lambda,0\\ \alpha,\beta,\gamma\end{bmatrix} = H\begin{bmatrix}a,b,p,\lambda\\ \alpha,\beta,\gamma\end{bmatrix}$$

que es un resultado dado por Kalla [5].

Por lo tanto :

$$I\begin{bmatrix}a,b,p,0,0\\ 1,\beta,\gamma\end{bmatrix} =$$

$$H\begin{bmatrix}a,b,p,0\\ 1,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi\sqrt{p}}\left(\text{arc tg}\,\frac{b}{\sqrt{p}}\right) +$$

$$\frac{\sigma ab}{4\pi}\sum_{k=1}^{\infty}\frac{(\beta)_k}{(\gamma)_k\,p}\left(-\frac{a^2}{p}\right)^k {}_2F_1\left(1+k,\frac{1}{2};\frac{3}{2};-\frac{b^2}{p}\right)$$

y

$$I\begin{bmatrix}a,b,1,0,0\\ 1,1/2,3/2\end{bmatrix} =$$

$$H\begin{bmatrix}a,b,1,0\\ 1,1/2,3/2\end{bmatrix} = I(a,b) = \frac{\sigma a}{4\pi}\,\text{arc tg}\,b +$$

$$\frac{\sigma ab}{4\pi}\sum_{k=1}^{\infty}\frac{(-a^2)^k}{2k+1}\,{}_2F_1\left(1+k,\frac{1}{2};\frac{3}{2};-b^2\right)$$

Usando la transformación [6 , pg. 249]

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)\,\Gamma(b-a)}{\Gamma(c-a)\,\Gamma(b)}(-z)^{-a} \times$$

$${}_2F_1\left(a,a+1-c;a+1-b;\frac{1}{z}\right) + \frac{\Gamma(c)\,\Gamma(a-b)}{\Gamma(c-b)\,\Gamma(a)}(-z)^{-b}$$

$${}_2F_1\left(b+1-c,b;b+1-a;\frac{1}{z}\right)$$

tenemos:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} =$$

$$\frac{\sigma a}{8\pi}\,b^{\lambda+1}\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k\,(-a^2)^k}{(\gamma)_k\;k!\;p^{\alpha+k}} \times$$

$$\left\{\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}-\alpha-k\right)\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{\lambda}{2}+\frac{3}{2}-\alpha-k\right)}\left(\frac{b^2}{p}\right)^{-\alpha-k} \times\right.$$

$${}_2F_1\left(\alpha+k,\alpha+k-\mu-\frac{\lambda}{2}-\frac{1}{2};\alpha+k-\frac{\lambda}{2}+\frac{1}{2};-\frac{p}{b^2}\right)$$

$$+\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma\left(\alpha+k-\frac{\lambda}{2}-\frac{1}{2}\right)}{\Gamma(\alpha+k)}\left(\frac{b^2}{p}\right)^{-\lambda/2-1/2}$$

$$\left.\times\;{}_2F_1\left(-\mu,\frac{\lambda}{2}+\frac{1}{2};\frac{\lambda}{2}+\frac{3}{2}-\alpha-k;-\frac{p}{b^2}\right)\right\} \qquad (21)$$

Haciendo $\mu = 0$ en (21) obtenemos:

$$I\begin{bmatrix}a,b,p,\lambda,0\\ \alpha,\beta,\gamma\end{bmatrix} = H\begin{bmatrix}a,b,p,\lambda\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\,b^{\lambda+1} \times$$

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k\,(-a^2)^k}{(\gamma)_k\;k!\;p^{\alpha+k}} \times$$

$$\left\{\frac{1}{(\lambda+1-2\alpha-2k)}\left(\frac{b^2}{p}\right)^{-\alpha-k} \times\right.$$

$${}_2F_1\left(\alpha+k,\alpha+k-\frac{\lambda}{2}-\frac{1}{2};\alpha+k-\frac{\lambda}{2}+\frac{1}{2};-\frac{p}{b^2}\right) +$$

$$\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma\left(\alpha+k-\frac{\lambda}{2}-\frac{1}{2}\right)}{2\,\Gamma(\alpha+k)} \times$$

$$\left.\left(\frac{b^2}{p}\right)^{-\lambda/2-1/2}\right\}$$

el cual fue dado por Kalla [5] .

De (2) tenemos que:

$$\lim_{b\to\infty} I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = \lim_{b\to\infty} H\begin{bmatrix}a,b,p,\lambda\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi} \times$$

$$\frac{\Gamma\left(\frac{\lambda}{2}+\frac{3}{2}\right)\Gamma\left(\alpha-\frac{\lambda}{2}-\frac{1}{2}\right)}{\Gamma(\alpha)\,(\lambda+1)\,p^{\alpha-\lambda/2-1/2}} \times$$

$${}_2F_1\left(\beta,\alpha-\frac{\lambda}{2}-\frac{1}{2};\gamma;-\frac{a^2}{p}\right)$$

De (20) y usando la relación [6 , pg. 247]

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{-\beta}\,{}_2F_1\left(\gamma-\alpha,\beta;\gamma;\frac{z}{z-1}\right),$$

$|\arg(1-z)| < \pi$

obtenemos:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{8\pi}\,\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)\Gamma(\mu+1)}{\Gamma\left(\mu+\frac{\lambda}{2}+\frac{3}{2}\right)}\,\frac{b^{\lambda+1}}{p^{\alpha-\lambda/2-1/2}}$$

$$\times\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,k!}\left(-\frac{a^2}{p}\right)^k (p+b^2)^{-\lambda/2-1/2}\times$$

$${}_2F_1\left(\mu+\frac{\lambda}{2}+\frac{3}{2}-\alpha-k,\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};\frac{b^2}{b^2+p}\right)$$

y para $\mu = 0$

$$I\begin{bmatrix}a,b,p,\lambda,0\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\,\frac{b^{\lambda+1}}{(\lambda+1)\,p^{\alpha-\lambda/2-1/2}}\times$$

$$\sum_{k=0}^{\infty}\frac{(\alpha)_k\,(\beta)_k}{(\gamma)_k\,k!}\left(-\frac{a^2}{p}\right)^k (p+b^2)^{-\lambda/2-1/2}\times$$

$${}_2F_1\left(\frac{\lambda}{2}+\frac{3}{2}-\alpha-k,\frac{\lambda}{2}+\frac{1}{2};\frac{\lambda}{2}+\frac{3}{2};\frac{b^2}{b^2+p}\right)$$

De (2) y usando la relación [6 , pg. 248]

$${}_2F_1(\alpha,\beta;\gamma;z) = (1-z)^{\gamma-\alpha-\beta}\,{}_2F_1(\gamma-\alpha,\gamma-\beta;\gamma;z),$$

$|\arg(1-z)| < \pi$

obtenemos que :

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = \frac{\sigma a}{4\pi}\int_0^b x^{\lambda}\,(x^2+p)^{\beta-\gamma}\left(1-\frac{x^2}{b^2}\right)^{\mu}\times$$

$$(x^2+p+a^2)^{\gamma-\alpha-\beta}\,{}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma;-\frac{a^2}{x^2+p}\right)dx$$

Usando el desarrollo del binomio para

$(x^2+p+a^2)^{\gamma-\alpha-\beta}$ obtenemos:

$$I\begin{bmatrix}a,b,p,\lambda,\mu\\ \alpha,\beta,\gamma\end{bmatrix} = a^{2\gamma-2\alpha-2\beta}\, I\begin{bmatrix}a,b,p,\lambda,\mu\\ \gamma-\beta,\gamma-\alpha,\gamma\end{bmatrix}+$$

$$\frac{\sigma}{4\pi}\,a^{1+2\gamma-2\alpha-2\beta}\sum_{k=1}^{\infty}\binom{\gamma-\alpha-\beta}{k}a^{-2k}\int_0^b x^{\lambda}\,(x^2+p)^{\beta-\gamma+k}$$

$$\times\left(1-\frac{x^2}{b^2}\right)^{\mu}{}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma;-\frac{a^2}{x^2+p}\right)dx,\quad p < a^2-b^2 \qquad (22)$$

$$I\begin{bmatrix}a,b,p,\lambda,0\\ \alpha,\beta,\gamma\end{bmatrix} =$$

$$H\begin{bmatrix}a,b,p,\lambda\\ \alpha,\beta,\gamma\end{bmatrix} = a^{2\gamma-2\alpha-2\beta}\, H\begin{bmatrix}a,b,p,\lambda\\ \gamma-\beta,\gamma-\alpha,\gamma\end{bmatrix}+$$

$$\frac{\sigma}{4\pi}\,a^{1+2\gamma-2\alpha-2\beta}\sum_{k=1}^{\infty}\binom{\gamma-\alpha-\beta}{k}a^{-2k}\times$$

$$\int_0^b x^{\lambda}\,(x^2+p)^{\beta-\gamma+k}\times$$

$${}_2F_1\left(\gamma-\alpha,\gamma-\beta;\gamma;-\frac{a^2}{x^2+p}\right)dx,\quad p < a^2-b^2$$

el cual fue dado por Kalla [5].

7.- CASOS ESPECIALES .-

En esta sección mencionamos algunos casos particulares de los resultados generales aquí establecidos.

De (19) con $\gamma=\beta+1$ y del resultado

$${}_1F_1(a;a+1;-x) = a\,x^{-a}\,\gamma(a,x) \qquad (23)$$

tenemos :

$$L\left[u^{-\beta}\,\gamma(\beta;ua^2)\;{}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right);p\right] = \frac{a^{2\beta}}{p\beta}\times$$

$$F_2\left(1,\beta,\ \frac{\lambda}{2}+\frac{1}{2}\ ;\beta+1,\mu+\frac{\lambda}{2}+\frac{3}{2}\ ;-\frac{a^2}{p}\ ,-\frac{b^2}{p}\right) \tag{24}$$

De (24) con $\mu = 0$ y de (23) :

$$L\left[u^{-\beta-\lambda/2-1/2}\ \gamma(\beta;ua^2)\ \gamma\left(\frac{\lambda}{2}+\frac{1}{2};ub^2\right)\ ;p\right]$$

$$=\frac{2\ a^{2\beta}\ b^{\lambda+1}}{(\lambda+1)\ p\ \beta}\ \times\ F_2\left(1,\beta,\frac{\lambda}{2}+\frac{1}{2};\beta+1,\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \tag{25}$$

Si en (25) tomamos $\lambda=0$ y usamos el resultado [7,pg.262]

$$\gamma\left(\frac{1}{2},x^2\right) = \sqrt{\pi}\ \ \mathrm{erf}\ x \tag{26}$$

se tiene:

$$L\left[\text{[illegible]}^{\beta-1/2}\ \gamma(\beta;ua^2)\ \mathrm{erf}(b\sqrt{u})\ ;\ p\right]$$

$$=\frac{2\ a^{2\beta}\ b}{\sqrt{\pi}\ p\ \beta}\ \times$$

$$F_2\left(1,\beta,\frac{1}{2}\ ;\beta+1,\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \tag{27}$$

Haciendo $\beta=1/2$ en (27) y en (24), usando el resultado (26) obtenemos respectivamente:

$$L\left[u^{-1}\ \mathrm{erf}(a\sqrt{u})\ \ \mathrm{erf}(b\sqrt{u})\ ;\ p\right] = \frac{4ab}{\pi\ p}\ \times$$

$$F_2\left(1,\frac{1}{2},\frac{1}{2}\ ;\frac{3}{2},\frac{3}{2}\ ;-\frac{a^2}{p},-\frac{b^2}{p}\right) \tag{28}$$

$$L\left[u^{-1/2}\ \mathrm{erf}(a\sqrt{u})\ \ {}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right);p\right] =$$

$$\frac{2\ a}{\sqrt{\pi}\ p}\ F_2\left(1,\frac{1}{2},\ \frac{\lambda}{2}+\frac{1}{2}\ ;\frac{3}{2},\mu+\frac{\lambda}{2}+\frac{3}{2}\ ;-\frac{a^2}{p}\ ,-\frac{b^2}{p}\right) \tag{29}$$

De (19) con $\mu = 0$ y del resultado (23) :

$$L\left[u^{-\lambda/2-1/2}\ {}_1F_1(\beta;\gamma;-\ ua^2)\ \ \gamma\left(\frac{\lambda}{2}+\frac{1}{2};ub^2\right);p\right]$$

$$=\frac{2\ b^{\lambda+1}}{(\lambda+1)\ p}\ \times$$

$$F_2\left(1,\beta,\ \frac{\lambda}{2}+\frac{1}{2}\ ;\gamma,\ \frac{\lambda}{2}+\frac{3}{2}\ ;-\frac{a^2}{p}\ ,-\frac{b^2}{p}\right) \tag{30}$$

De (30) con $\lambda = 0$ y usando el resultado (26) :

$$L\left[\ u^{-1/2}\ \ {}_1F_1(\beta;\gamma;-\ ua^2)\ \ \mathrm{erf}(b\sqrt{u})\ ;\ p\right]$$

$$=\frac{2}{\sqrt{\pi}}\ \frac{b}{p}\ \times$$

$$F_2\left(1,\beta,\frac{1}{2}\ ;\gamma,\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \tag{31}$$

De (19) con $\gamma=\beta$:

$$L\left[e^{-ua^2}\ {}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-\ ub^2\right)\ ;\ p\right]$$

$$=\frac{1}{(p+a^2)}\ \times$$

$${}_2F_1\left(1,\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{b^2}{p+a^2}\right) \tag{32}$$

donde se usó la identidad :

$$F_2(a,b,b';b,c';x,y) = (1-x)^{-a} \; {}_2F_1\left(a,b';c';\frac{y}{1-x}\right)$$

Haciendo $\mu=0$ en (32) y usando (23) :

$$L\left[u^{-\lambda/2-1/2}\; e^{-ua^2}\; \gamma\left(\frac{\lambda}{2}+\frac{1}{2};ub^2\right);p\right] = \frac{2\; b^{\lambda+1}}{(\lambda+1)\;(p+a^2)} \times$$

$${}_2F_1\left(1,\frac{\lambda}{2}+\frac{1}{2};\frac{\lambda}{2}+\frac{3}{2};-\frac{b^2}{p+a^2}\right) \quad (33)$$

Los resultados (25), (28) y (33) fueron dados por Kalla [5].

De (19) con $\beta=\gamma/2$ y usando el resultado [6,pg.274]

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)}\; e^z \; {}_1F_1\left(\nu+\frac{1}{2};2\nu+1;-2z\right),$$

$|\arg z| < \pi$

obtenemos :

$$L\left[u^{1/2-\gamma/2}\; e^{-ua^2/2}\; I_{\frac{\gamma-1}{2}}\left(\frac{ua^2}{2}\right) \times {}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right);p\right]$$

$$= \frac{2^{1-\gamma}\; a^{\gamma-1}}{\Gamma\left(\frac{\gamma+1}{2}\right)\; p} \times$$

$$F_2\left(1,\frac{\gamma}{2},\frac{\lambda}{2}+\frac{1}{2};\gamma,\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \quad (34)$$

De (34) con $\mu = 0$ y usando (23) :

$$L\left[u^{-\gamma/2-\lambda/2}\; e^{-ua^2/2}\; \gamma\left(\frac{\lambda}{2}+\frac{1}{2};ub^2\right) \times I_{\frac{\gamma-1}{2}}\left(\frac{ua^2}{2}\right);p\right] = \frac{b^{\lambda+1}\; 2^{2-\gamma}}{p\; a^{1-\gamma}\;(\lambda+1)\;\Gamma\left(\frac{\gamma+1}{2}\right)} \times$$

$$F_2\left(1,\frac{\gamma}{2},\frac{\lambda}{2}+\frac{1}{2};\gamma,\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \quad (35)$$

De (19) con $\lambda = 1$, $\mu = -1/2$ y del resultado [6, pg.272]

$$F(z) = z\; {}_1F_1\left(1;\frac{3}{2};-z^2\right)$$

donde F(z) es la integral de probabilidad de argumento imaginario, tenemos:

$$L\left[{}_1F_1(\beta;\gamma;-ua^2)\; u^{-1/2}\; F(b\sqrt{u});p\right] = \frac{b}{p} \times$$

$$F_2\left(1,\beta,1;\gamma,\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right) \quad (36)$$

De (19) se tiene que :

$$\int_0^\infty e^{-up}\; {}_1F_1(\beta;\gamma;-ua^2) \times {}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right)du = \frac{1}{p} \times$$

$$F_2\left(1,\beta,\frac{\lambda}{2}+\frac{1}{2};\gamma,\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p},-\frac{b^2}{p}\right)$$

Usando el resultado [8 ,pg. 366]

$$F_2(a,b,b';c,c';x,y) = (1-y)^{-a} \times F_2\left(a,b,c'-b';c,c';\frac{x}{1-y},\frac{y}{y-1}\right)$$

se tiene:

$$\int_0^\infty e^{-up}\; {}_1F_1(\beta;\gamma;-ua^2) \times {}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right) du =$$

$$\frac{1}{p+b^2}\; F_2\left(1,\beta,\mu+1;\gamma,\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{p+b^2},\frac{b^2}{p+b^2}\right) \quad (37)$$

Tomando lim $p\to 0$ en (37):

$$\int_0^\infty {}_1F_1(\beta;\gamma;-ua^2) \times$$

$$ {}_1F_1\left(\frac{\lambda}{2}+\frac{1}{2};\mu+\frac{\lambda}{2}+\frac{3}{2};-ub^2\right)du$$

$$= \frac{1}{b^2} \times$$

$$F_2\left(1,\beta,\mu+1;\gamma,\mu+\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{b^2},1\right) \qquad (38)$$

Si en (38) tomamos $\mu=0$ y usamos (23)

$$\int_0^\infty u^{-\lambda/2-1/2}\ {}_1F_1(\beta;\gamma;-ua^2)$$

$$\gamma\left(\frac{\lambda}{2}+\frac{1}{2};ub^2\right) du = \frac{2\ b^{\lambda+1}}{(\lambda+1)} \times$$

$$F_2\left(1,\beta,1;\gamma,\frac{\lambda}{2}+\frac{3}{2};-\frac{a^2}{b^2},1\right)$$

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