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Nuevas Propiedades de la función Di-Bessel de Exton

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Resumen

En este trabajo se estudian nuevas propiedades de la función Di-Bessel de Exton A_μ. Esta función se define en forma similar a la función de Bessel de Primera clase y orden μ:J_μ. Como consecuencia, se demuestran propiedades del tipo de las conocidas para esta última función. Se obtienen entonces, reglas operacionales, ecuaciones diferenciales, relaciones de recurrencia, transformaciones integrales, representaciones integrales y desarrollos asintóticos.

New Properties of Di-Bessel function of Exton

In this paper we study certain new properties of the Exton's Di-Bessel function A_μ. The function A_μ is defined likeness to the Bessel function J_μ of the first kind and order μ . Then, properties like the ones which are verified by J_μ can be obtained for A_μ. Operational rules, differential equations, recurrence relations, integral transformations, integral representations and asymptotic expansions are thus obtained.

Key Words: Di-Bessel function, Recurrence formulae, Integral Representations, Asymptotic Behaviour.

1. Introduction

Exton [9] studied an extension of the Bessel function of the first kind an order μ by defining what he called a Di-Bessel function. In particular he defined the function $A_\mu(x)$:

$$A_\mu(x)=\sum_{m=0}^{\infty}\frac{(-1)^m\,(x/2)^{\mu+2m}}{(m!)^2\,(\Gamma(\mu+1+m))^2} \qquad (1.1)$$

This function $A_\mu(x)$, for x a complex variable and μ a fixed complex number is analytic on the x plane except possibyly for branch points at $x = 0$ and $x = \infty$. When μ is an integer number we have the following generating function of the Di-Bessel function of integer order (Exton, 9)

$$I_0(2\sqrt{xt})\,J_0(-2\sqrt{x/t})=\sum_{m=-\infty}^{\infty} t^n A_n(x) \qquad (1.2)$$

where J_o is the Bessel function of the first kind and order 0, and I_o denotes the modified Bessel function of the first kind and order 0.

Exton obtained a number of properties of this function including an orthogonal property. He showed that the differential equation satisfied by the Di-Bessel function is self-adjoint, being that some infinite integrals with the A_μ function were studied too.

The function $A_\mu(x)$ was generalized by A. Kumar Agarwal [3] by defining a **n**-ple Bessel function $A_{\mu,n}(x)$ which reduces for n = 1 and n = 2 to the Bessel and Di-Bessel functions, respectively. More recently, Sarabia [14] proves two recurrence relations for the function $A_\mu(x)$. He obtains an integral representation and also studies the oscillatory character of $A_\mu(x)$.

The object of this paper is to investigate $A_\mu(x)$ and study some of its new properties. Then, as the Di-Bessel function appears as a particular case of the hyper-Bessel functions (Delerue, [4]; Dimovski-Kiryakova, [5]; Adamchik, [1] a good number of properties of the function A_μ can be proved. In particular, we obtain in this paper some differential equations which are verified for modified versions of the Di-Bessel function; we study two new recurrence relations for A_μ; special definite integrals for A_μ are also obtained; an integral representation of the Di-Bessel function is likewise found. Lastly we present the asymptotic expansions of the A_μ function.

The present paper will be continued in a new work where we will define a new integral transformation of the Hankel type which contains the Di-Bessel function A_μ in its kernel. Then inversion formulae, convergence theorems, and spaces of test functions where this transformation will be a linear and continuous mapping will be studied.

2. Differential Equations

On investigating differential equations associated with the Di-Bessel function, we begin with the operational equation:

$$\delta^2 \ (\delta+\mu)^2 \left[E_\mu(\lambda x)\right] = \lambda x \ E_\mu \ (\lambda x) \qquad (2.1)$$

where $\delta = x \dfrac{d}{dx}$ and $x > 0, \ \lambda > 0$

which is satisfied by the function:

$$E_\mu(\lambda x) = (-\lambda x)^{-\mu/2} A_\mu(2i\sqrt{\lambda x})$$

(A.K. Agarwal, [3], p. 142).

But, being that:

$\delta \ [x^\alpha f(x)] = x^\alpha \ (\delta + \alpha) \ f(x)$, for f a regular function and:

$$\delta \left[A_\mu(2i\sqrt{x})\right] = 2^{-1} \left[\delta(A_\mu)\right] \ (2i\sqrt{x})$$

we will find that:

$$\delta^2 \ (\delta+\mu)^2 \ (-x)^{-\mu/2} A_\mu(2i\sqrt{x}) =$$
$$= (-1)^{-\mu/2} x^{-\mu/2} (\delta-\mu/2)(\delta-\mu/2)$$
$$(\delta+\mu/2)(\delta+\mu/2) \ (A_\mu(2i\sqrt{x}) =$$

$$(-x)^{-\mu/2} \ \frac{1}{.16} \ (\delta-\mu)^2 \ (\delta+\mu)^2 \left[A_\mu(u)\right] \ (2i\sqrt{x}) =$$
$$(-x)^{-\mu/2} \ x \ A_\mu(2i\sqrt{x})$$

In this way we have proved that the function $A_\mu(x)$ satisfies the operational equation:

$$(\delta-\mu)^2 \ (\delta+\mu)^2 \left[A_\mu(x)\right] = -4 \ x^2 \ A_\mu(x) \qquad (2.2)$$

When we choose the function $B_{\alpha,\beta,\mu}(x) = x^\alpha A_\mu(Kx^\beta)$ for a constant K, K>0, we will find:

$$A_\mu(y) = (y/K)^{-\alpha/\beta} B_{\alpha,\beta,\mu}(y/K)^{1/\beta} \ ; \ y>0$$

and being that:

$$\delta \left[f\left((y/K)^{1/\beta}\right)\right] = (\beta^{-1}\delta(f)\left((y/K)^{1/\beta}\right)$$

for f a regular function; we will obtain:

$$(\delta-\mu)^2 \ (\delta+\mu)^2 \left[(y/K)^{-\alpha/\beta} B_{\alpha,\beta,\mu}\left((y/K)^{1/\beta}\right)\right] =$$
$$- 4y^2 \left[(y/K)^{-\alpha/\beta} B_{\alpha,\beta,\mu}\left((y/K)^{1/\beta}\right)\right]$$

and so we can find the operational relation:

$$(\delta-\mu\beta-\alpha)^2 \ (\delta+\mu\beta-\alpha)^2 \left[B_{\alpha,\beta,\mu}(y)\right] =$$
$$-y^{2\beta} \ 4 \ K^2 \ B_{\alpha,\beta,\mu}(y) \ \beta^4 \qquad (2.3)$$

In particular, when we choose $\alpha = 1/2$ and $\beta = 1$, we obtain:

$$(\delta-\mu-1/2)^2 \ (\delta+\mu-1/2)^2 \ B_{1/2,1,\mu}(y) =$$
$$-y^2 \ 4 \ K^2 \ B_{1/2,1,\mu}(y) \qquad (2.3)$$

and as:

$$(\delta+\gamma)(\delta+\rho) = \delta^2 + (\gamma+\rho)\delta + \gamma\rho, \text{ for } \gamma,\rho \ \mathbb{R}$$

thus:

$$[\delta^4 - 2\delta^3 - (2\mu^2-3/2)\delta^2 + 2(\mu-1/2)(\mu+1/2)\delta +$$
$$+ (\mu+1/2)^2 \ (\mu-1/2)^2]\left[\sqrt{x}A_\mu(Kx)\right] =$$
$$-x^2 \left[4 \ K^2 \ A_\mu \ (Kx)\right] \sqrt{x}$$

i.e.:

$$\left[x^2\frac{d^2}{dx^2} - (\mu^2-1/4)\right]\left[x^2\frac{d^2}{dx^2} - (\mu^2-1/4)\right]$$

$$[\sqrt{x}\,A_\mu\ (Kx)\,] = -\ x^2\ 4\ K\sqrt{x}\,A_\mu\,(x)$$

Furthermore, if $S_\mu = \frac{d^2}{dx^2}\cdot\frac{4\mu^2-1}{4x^2}$ it denotes the Bessel's classical differential operator, we have found the following formula (2.4):

$$\left[S_\mu\ X^2\ S_\mu\ \sqrt{x}\,\right]\ [A_\mu\ (Kx)\,] = -\ 4\ K^2\ (\sqrt{x}\,A_\mu\ (Kx)\ )$$

3. Recurrence Formulae For $A_\mu(x)$

In this section we obtain certain new recurrence relations for the Di-Bessel function $A_\mu(x)$. In Sarabia 14 we can find the following recurrence formulae for this function:

1. $\frac{d}{dx}\ x\ \frac{d}{dx}\ (x^\mu\ A_\mu\,(x)\ ) = 2\ x^\mu\ A_{\mu-1}\ (x)$
2. $\frac{d}{dx}\ x\ \frac{d}{dx}\ (x^{-\mu}\ A_\mu\ (x)\ ) = -2\ x^{-\mu}\ A_{\mu+1}\ (x)$
3. $2_\mu\ A_\mu'(x) = A_{\mu-1}(x) + A_{\mu+1}(x)$
4. $\mu^2\ A_\mu(x) + (1-2\mu)\,x\ A_\mu'(x) + x^2\ A_\mu''(x) = -\ 2x\ A_{\mu+1}(x)$
5. $\mu^2\ A_\mu(x) + (1+2\mu)\,x\ A_\mu'(x) + x^2\ A_\mu''(x) = 2\ x\ A_{\mu-1}(x)$
6. $(\mu-1)\,x^2 A_{\mu+2}(x) + 2(\mu-1)(\mu+1)(2\mu+1)\,xA_{\mu+1}(x) + 2_\mu(\ 2\mu^2(\mu-1)(\mu+1) + x^2)\,A_\mu(x) + 2(\mu-1)(\mu+1)\cdot(1-2\mu)\,xA_{\mu-1}(x) + (\mu+1)\,x\ A_{\mu-2}(x) = 0$

Additionally, if we define the differential operators N_μ and M_μ by:

$$N_\mu = x^{\mu+1/2}\ \frac{d}{dx}\ x\ \frac{d}{dx}\,x^{-\mu-1/2}$$

$$M_\mu = x^{-\mu-1/2}\ \frac{d}{dx}\ x\ \frac{d}{dx}\,x^{\mu+1/2}$$

we find that:

$$x^2 M_\mu N_\mu = x^2\ (x^{\mu-1/2}\ \frac{d}{dx}\ x\ \frac{d}{dx}\ x^{\mu+1/2})\ (x^{\mu+1/2}\ \frac{d}{dx}\ x\ \frac{d}{dx}\ x^{-x-1/2}) = S_\mu\ X^2\ S_\mu$$

and then:

$$N_\mu\left[\sqrt{x}\,A_\mu\,(x)\right] = -2\ \sqrt{x}\,A_{\mu+1}\,(x)$$

$$M_\mu\left[\sqrt{x}\,A_{\mu+1}\,(x)\right] = 2\ \sqrt{x}\,A_\mu\,(x)$$

Also, if R_μ denotes the Riemann-Liouville operator (Ederlyi, [8]) given in (3.1):

$$R_\mu f(x) = \Gamma(\mu)^{-1}\int_0^x f(t)\ (x-t)^{\mu-1}\,dt, \tag{3.1}$$

for a regular function f,$\mu > 0$, $x > 0$,

we know that:

$$\frac{x^{\mu+n}}{\Gamma(\mu+n)} = \frac{1}{\Gamma(n+1)}\ R_\mu\,[x^n]\ n\in\mathbb{N},\ \text{and}\ \mu > 0$$

Then, being as:

$$\left[(x/2)^\mu\ A_\mu(x)\right]\ (\sqrt{x}) = \sum_{n=0}^{\infty}\frac{(-1)^n\ (x/2)^{\mu+n}}{(n!)^2\ [\Gamma(\mu+n+1)]^2}\ (1/2)^{\mu+1}$$

we have:

$$\left[(x/2)^\mu\ A_\mu(x)\right]\ (\sqrt{x}) = \sum_{n=0}^{\infty}\ \infty\ \frac{R_\mu\left[(-1)^n\ (x/2)^n\right]}{(n!)^3\ \Gamma(\mu+n+1)}\,(1/2)^{2\mu+n}$$

$$= R_\mu\left[(x/2)^{-\mu}\sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^{n+\mu}}{(n!^3)\ \ \Gamma(\mu+n+1)}\ (1/2)^{2\mu+n}(x)\right] =$$

$$= R_\mu\left[x^{-\mu}\sum_{n=0}^{\infty}\frac{R_\mu[(-1)^n (x/2)^n]}{(n!)^2}\,(1/2)^{2\mu+n}\right]\ (x) =$$

$$R_\mu\left[x^{-\mu}R_\mu\sum_{n=0}^{\infty}\frac{(-1)^n (x/2)^n}{(n!)^4}\,(1/2)^n\ \ 1/2^{2\mu}\right](x) =$$

$$= \left[R_\mu\ x^{-\mu}\ R_\mu\right][A_0\ (\sqrt{x})\ ]\ \ (x)\ \ 2^{-2\mu}$$

Then we will have obtained the general formulae that follows:

$$A_\mu(x) = (2x)^\mu\ R_\mu\ x^{-\mu}\ R_\mu\left[A_0(\sqrt{x})\right]\ (x^2) \tag{3.2}$$

4. Infinite Integrals

Consider, first of all, the integral

$$I = \int_0^\infty e^{-ax}\ x^{v-1}\ A_\mu(bx)\ dx \qquad Re\ a>0,\ Re(v+\mu)>0 \tag{4.1}$$

Re a > 0, $Re(v + \mu) > 0$

In Exton [9], p. 860 it has been found that "I" can be expressed in the form:

$$a^{-v}\,(b/2a)^\mu\frac{\Gamma(\mu+v)}{[\ \Gamma(\mu+1)]^2}\ \ {}_2F_3\,(\,(v+\mu)/2,\ (v+\mu+1)/2;\mu+1,\mu+1,1;-b^2/a^2) \tag{4.2}$$

Sarabia, [14], p. 72 obtained an other defined integral for $A_\mu(x)$; but it is not of very great interest for usm. We are interested in the application of some integral transformations to $A_\mu(x)$. So, in (4.1)-(4.2) we can find the Laplace transformation of A_μ. Then this transformation $L(A_\mu(x))(a)$ is given in (4.3):

$$L(A_\mu(x))(a) = 2^{-\mu}a^{-\mu-1}\sum_{n=0}^{\infty}\frac{\Gamma(\mu+1+2n)}{(n!)^2[\Gamma(\mu+1+n)]^2}(-1/a^2)^n \tag{4.3}$$

Re $a > 0$

Moreover, we know the hyper-Bessel functions defined by Delerue, [4] in 1953 in the form:

$$J^{(n)}_{\lambda_1,\lambda_2,\ldots\lambda_n}(x) = \frac{x}{n+1}^{\lambda_1+\lambda_2\ldots\lambda_n}\sum_{n=0}^{\infty}\frac{[-x/(n+1)^{n+1}]^r}{r!\,\Gamma(\lambda_1+1+r)\ldots\Gamma(\lambda_n+1+n)} \tag{4.4}$$

$\lambda_1, \lambda_2 \ldots \lambda_n \in R$, $n \in N$, $x > 0$

and being that $A_\mu(x)$ coincides with the function:

$$A_\mu(x) = J^{(3)}_{0,\mu,\mu}(2\sqrt{2x}$$

we find (Delerue, [4], p. 233):

$$L\left[x^{(n\lambda_1-\lambda_2-\ldots\lambda_n)/(n+1)}\,J^{(n)}_{\lambda_1,\ldots\lambda_n}(n+1)^{n+1}\sqrt{x}\right](a) = = a^{(n\lambda_1-\lambda_2-\ldots\lambda_n)/(n+1)\,(n)}J_{\lambda_1,\lambda_2,\ldots\lambda_n}(n/\sqrt[n]{a}) \tag{4.6}$$

and so:

$$L\left[x^{-\mu/2}A_\mu(\sqrt{x}/4)\right](a) = a^{\mu/3}J^{(2)}_{\mu,\mu}(3a^{-1/3}) \tag{4.7}$$

The Mellin transformation $M(A_\mu)(s)$ of A_μ will be given in the form:

$$M(A_\mu(x)).(s) = \int_0^{\infty} x^{s-1}A_\mu(x)\,dx \tag{4.8}$$

and we can obtain an explicit expression of (4.8) when the integral representation of A_μ is known which will be studied in the paragraph 5...

5. Integral Representations of the Di-Bessel Function

We shall next examine various representations of the Di-Bessel function by a system of definite integrals and contour type integrals. Firstly we find the following integral representation obtained by Sarabia, [14], p. 67:

$$A_\mu(x) = \frac{x^\mu}{2\pi i\,\Gamma(\mu+1)\,2^\mu}\int_C e^s s^{-(\mu+1)}\,{}_0F_2(-;\mu+1;-x^2/4s)\,ds \tag{5.1}$$

where C denotes Hankel's contour.

When μ is a nonnegative integer $\mu = n \in N$, the expression of (5.1) will stay in the form (Exton, [9], p. 857):

$$A_n(x) = \frac{1}{2\pi i}\int_{-\infty}^{(0)^+} s^{-n-1}\,I_0(2\sqrt{xs})\,J_0(-2\sqrt{xs})\,ds. \tag{5.2}$$

where I_0 and J_0 denote the classical Bessel functions of order 0.

We now take into account Hankel's well-known integral representation of Bessel functions, and then:

$$\pi[\Gamma(\mu+1/2)]^2\,2^\mu\sum_{n=0}^{\infty}\frac{(-1)^n(x/2)^{\mu+2n}}{n!)^2[\Gamma(\mu+n+1)]^2} =$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n}x^{\mu+2n}}{[(2n)!]^2}\left[\frac{\Gamma(n+1/2)\,\Gamma(\mu+1/2)}{\Gamma(n+\mu+1)}\right]^2 =$$

$$= \sum_{n=0}^{\infty}\frac{2^{2n}x^{\mu+2n}(-1)^n}{(2n!)^2}\int_0^1 t^{2n}(t^2-1)^{\mu-1/2}dt\int_0^1\tau^{2n}(\tau^2-1)^{\mu-1/2}d\tau =$$

$$(2n!)^2$$

$$= \sum_{n=0}^{\infty}\frac{x^{\mu+n}i^n}{(n!)^2}\int_{-1}^{1} t^{n\ n}(t^2-1)^{\mu-1/2}dt\int_{-1}^{1}\tau^n(\tau^2-1)^{\mu-1/2}d\tau$$

Consider the function obtained by interchanging the signs of summation and integration on the right; it is:

$$C_0(-2ixt\tau)(t^2-1)^{\mu-1/2}(\tau^2-1)^{\mu-1/2}$$

where:

$$C_0(z) = \sum_{n=0}^{\infty}\frac{(-z)}{[n!]^2}$$

denotes the Bessel-Clifford function of the first kind and order 0 (Hayek, [11])

This is an analytic of x for all values of x, and when expanded in ascending powers of x by MacLaurin's theorem, the coefficients may be obtained by differentiating with regard to x under the double integral sing and making x zero after diferentiation. Hence:

$$x^{\mu}\int_{-1}^{1}\int_{-1}^{1} C_0(-2ixt\tau)\,(t^2-1)^{\mu-1/2}(\tau^2-1)^{\mu-1/2}\,dt\,d\tau =$$

$$= \pi\,[\,\Gamma(\mu+1/2)]^2\,2^{\mu}\sum_{n=0}^{\infty}\frac{(-1)^n\,(x/2)^{\mu+2n}}{(n!)^2\,(\Gamma(\mu+1+n))^2};$$

i.e., the function $A_\mu(x)$ admits the following integral representation (5.3):

$$A_\mu(x) = \frac{(x/2)^{\mu}}{\pi\,[\,\Gamma(\mu+1/2)]^2}\int_{-1}^{1}\int_{-1}^{1}$$

$$C_0(-2ixt\tau)\,(t^2-1)^{\mu-1/2}(\tau^2-1)^{\mu-1/2}dt\,d\tau \tag{5.3}$$

Furthermore, by the formula (4,6) we obtain:

$$L\left[x^{\mu/2}J^{(3)}_{\mu,\mu,0}(4\sqrt[4]{x})\right](a) = a^{-2\mu/3}J^{(2)}_{\mu,0}(3/\sqrt[-3]{a}$$

and then, L^{-1} being the inverse Laplace transformation; we find:

$$x^{\mu/2}\,J^{(3)}_{\mu,\mu,0}\,(4\sqrt[4]{x}) =$$

$$L^{-1}\left[a^{-2\mu/3}\,J^{(2)}_{\mu,0}\,(3-\sqrt[-3]{a})\right]\,(a\to x) =$$

$$=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ax}\,a-2\mu/3$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n\,3^n\,a^{-n/3}}{(n!)^2\,\Gamma(\mu+n+1)}\,da$$

for a real number c ε R.

So, if we take Hankel's well-known generalisation of the second Eulerian integral:

$$\frac{1}{\Gamma(\mu+m+1)} = \frac{1}{2\pi i}\int_{-\infty}^{(0^+)} t^{\mu-m-1}\,e^{t}\,dt$$

in which the phase of t increases from $-\pi$ to π as t describes the contour, we could carry out calculations analogous to the previous ones and in that way obtain:

$$x^{\mu/2}J^{(3)}_{\mu,\mu,0}(4\sqrt[4]{x} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{ax}a^{-2\mu/3}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{(0^+)} C_0(3/\sqrt[-3]{a})\,e^{t}t^{-\mu-1}\,dt\,da$$

$$Re\ a > 0$$

i.e.; A_μ will allow the new integral representation:

$$A_\mu(x) = x^{-\mu/2}\frac{1}{2\pi i}\int_{-\infty}^{(0^+)} e^{t}\,t^{-\mu-1}\,L^{-1}$$

$$\left[a^{-2\mu/3}C_0(3a^{-1/3}t)\right](x^2/4)\,dt \tag{5.4}$$

Lastly, we can use the formula (5.3) to obtain the Mellin transformation of $A_\mu(x)$

$$M(A_\mu(x))(s) = \int_0^{\infty} x^{s-1}\left[\int_{-1}^{1}(t^2-1)^{\mu-1/2}\right.$$

$$\left.\left[\int_{-1}^{1}(\tau^2-1)^{\mu-1/2}C_0(-2ixt\tau)\,d\tau\right]dt\right]dx$$

$$.\,(x/2)^{\mu}\,\pi^{-1}\,\Gamma(\mu+1/2)^{-2},$$

and interchanging the integration symbols, we shall obtain:

$$M(A_\mu(x))(s) = \int_{-1}^{1}(t^2-1)^{\mu-1/2}\left[\int_{-1}^{1}(\tau^2-1)^{\mu-1/2}\right.$$

$$\left.\frac{2^{-\mu}}{\pi\,[\Gamma(\mu+1/2)]^2}\cdot\left[\int_0 x^{\mu+s-1}J_0(-2\sqrt{2ixt\tau})\,dx\right]dt\right]d\tau$$

$$= \frac{2^{-\mu+1}\,2^{2\mu+2s-1}}{\pi\,\Gamma(\mu+1/2)^2}\frac{\Gamma(\mu+s)}{\Gamma(1-\mu-s)}$$

$$\left[\int_{-1}^{1}(t^2-1)^{\mu-1/2}\,t^{-\mu-s}dt\right]^2 2^{-3(\mu+s)}i^{-(\mu+s)} =$$

for 3/2 > Re s > 0, according to Ditkin-Proudnikov, [7], p. 359

So:

$$M(A_\mu(x))(s) = \frac{2^{-2\mu-s}i^{-\mu-s}}{\pi\,[\Gamma(\mu+1/2)]^2}\frac{\Gamma(\mu+s)}{\Gamma(1-\mu-s)} \tag{5.5}$$

$$\beta(\mu+1/2,1/2-\mu/2-s/2)\,(1+(1)^{\mu+s})/2$$

Then, we shall obtain:

$$M(A_\mu(x))(s) = \pi^{-1}2^{-2-2\mu-s}\,(-1)^{-(\mu+s)/2}\,(1+(-1)^{\mu+s})^2$$

$$\frac{\Gamma(\mu+s)}{\Gamma(1-\mu-s)}\frac{\Gamma((1-\mu-s)/2)}{\Gamma(1+\mu/2-s/2)}$$

$$\cdot\frac{\Gamma(1/2-\mu/2-s/2)}{\Gamma(1+\mu/2-s/2)}$$

for 3/2 > Re s > 0 and $\mu > -1/2$.

Moreover, the following relations are known:

$$[1+(-1)^{\mu-s}]^2 = 2(1+(-1)^{\mu+s})$$

and

$$(-1)^{-(\mu+s)/2}\,[1+(-1)^{\mu+s}]^2 = 4\cos(\mu+s)\pi/2$$

then:

$$M(A_\mu(x))(s) = \pi^{-1}\,2^{-s-2\mu}\cos(\mu+s)\pi/2\;\frac{\Gamma(\mu+s)}{\Gamma(1-\mu-s)}\left[\frac{\Gamma(1/2-\mu/2-s/2)}{\Gamma(1+\mu/2-s/2)}\right]^2$$

but, by using the well known formulae:

$$\sin(\pi z)\,\Gamma(z)\,\Gamma(1-z) = \pi$$

and

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\,\Gamma(z)\,\Gamma(z+1/2)$$

we can operate on (5.5) until we obtain:

$$M(A_\mu(x))(s) = \pi^{-1}\,2^{\mu+s-1}\sin\pi(\mu+s)/2\left[\frac{\Gamma(\mu/2+s/2)}{\Gamma(1+\mu/2-s/2)}\right]^2 \quad (5.6)$$

6. The Asymptotic Behaviour of A_μ

In Mathai - Saxena [12], p. 43 we find that:

$G^{m,n}_{p,q}\left[z/^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots b_q}\right]$ being Meijer's **G**-function which was defined by Meijer in 1936 by means of a finite series of the generalized hypergeometric function; then the generalized hypergeometric function ${}_pF_q(a_1, a_2,\ldots, a_p; b_1, b_2,\ldots, b_q; z)$ admits the following representation through the function **G**; i. e.:

$${}_pF_q(a_1,a_2,\ldots,a_p;b_1,b_2,\ldots b_q;z) = \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}\,G^{1,p}_{p,q+1}\left[z/^{1-a_1,\ldots,1-a_p}_{0,1-b_1,\ldots,1-b_q}\right]$$

when certain conditions arel satisfied. Then, our function $A_\mu(x)$ can be expressed in the form:

$$[z^{-\mu}A_\mu(z)](2\sqrt{z}) = 2^{-\mu}F_3(-;1,\mu+1,\mu+1;-z) = 2^{-\mu}\,\Gamma(\mu+1)^2\,G^{1,0}_{0,4}\left[z/\overline{0,0,-\mu,-\mu}\right] \quad (6.1)$$

Then, it is posible to use the theorems about the asymptotic expansions of Meijer's function in Mathai-Saxena [12]; ϕ 1.8. to obtain that:

" there exist constants: $M_1, M_2,\ldots, M_n,\ldots$ and:

$$G^{1,0}_{0,4}\left[z/\overline{0,0,-\mu,-\mu}\right] \approx \sum_{i,-i}(-2\pi i)^{-3}\,e^{2\mu\pi i}\exp{-4(ze^{i\pi 3})^{1/4}}\,.\,(ze)^{-(4\mu+3)/8}\left[\frac{(2\pi)^{3/2}}{2}+\frac{M_1}{(ze^{i\pi 3})^{1/4}}+\frac{M_2}{(ze^{1}\pi_3)^{1/2}}\ldots\right].,$$

In this way, we can find the following asymptotic expansion of the Di-Bessel function A_μ:

$$\underset{\substack{z\to\infty\\ \arg z=0}}{A_\mu(z)} \approx \Gamma(\mu+1)^2(-1)^{(4\mu+3)/8}(2\pi)^{-3}\,2^{7/4}\sin 2\mu\pi\,e^{-2\sqrt{2zi}}z^{-3/4}\left[\frac{(2\pi)^{3/2}}{2}+\frac{M_1}{z^{1/2}}+\frac{M_2}{z}+\ldots\right] \quad (6.2)$$

where $M'_1, M'_2,\ldots, M'_n,\ldots$ are appropiate constants

The asymptotic behaviour of A_μ can be obtained from (6.2) or seeing that, according to Mathai-Saxena, [12], p. 307, the growth of $G^{m,o}_{p,q}$ in $z\to\infty$ is given in:

$$G^{m,0}_{p,q}\left[z/^{a_1,a_2,\ldots,a_p}_{b_1,b_2,\ldots,b_q}\right] = O\left[\exp[(p-q)z^{1/(q-p)}]z^{\beta}\right] \quad (6.3)$$

where:

$$\beta = \frac{1}{q-p}\left[\frac{p-q+1}{2}+\sum_{1=1}^{q}bi-\sum_{i=1}^{p}a_i\right]$$

$$|\arg z| < (q-p+1)\pi$$

Thus, we shall find that:

$$\underset{x\to\infty}{A_\mu(x)} = O\left[e^{-2\sqrt{2x}}x^{-3/4}\right] \quad (6.4)$$

In the same form, it would easily be seen that:

$$\underset{x\to 0}{A_\mu(x)} = O[x^{\mu}] \quad (6.5)$$

References

1. ADAMCHIK, V.S. "Hyper-Bessel solutions of a differential equation" Diff. Uravneniya, 22, (N°4), (1986), 697-700

2. AGARWAL, A.K. "A basic Di-Bessel function", Ganita, 34, (1983), 80-88

3. AGARWAL, A.K. "A generalization of Di-Bessel function of Exton", Indian J. pure appl. Math., 15(2), (1984), 139-148

4. DELERUE, P. "Sur le calcul symbolique á n variables et les fonctions hyperbesséliennes", Annales Soc. Sci. Bruxelles, 67, N°3, (1953), 229-274

5. DIMOVSKI, I.H. - KIRYAKOVA, V.S."Generalized Poisson transmutations and corresponding representations of hyper-Bessel functions", C.R. Acad. Bulgare Sci., 39, N°10, (1986), 29-32

6. DIMOVSKI, I.H.- KIRYAKOVA, V.S."Generalized Poisson representation of hypergeometric functions ${}_pF_q$, $p < q$ using fractional integrals", Mathematics and Education in Mathematics, (1987), April 6-10, 205-212

7. DITKIN, V. - PROUDNIKOV, A."Transformations intégrales et calcul opérationnel", Ed. Mir, Moscou, (1978)

8. EDERLYI, A.- MAGNUS - OBERHETTINGER - TRICOMI "Tables of integral transforms, vols. 1 and 2", McGraw Hill, New York, (1974)

9. EXTON, H. "The Di-Bessel Function", J. Pure appl. Math., July (1980), 856, 862

10. GONZALEZ, J.M.- TRUJILLO, J.J."La transformada de Laplace y los operadores fraccionarios de Liouville-Weyl sobre ciertos espacios del tipo S" Actas VII Grupo Matemático de Expressao Latina, Coimbra, (1985)

11. HAYEK, N. "Estudio de la ecuación diferencial $xy'' + (\mu+1)y' + y = 0$ y de sus aplicaciones", Collectanea Mathematica (Vol. XVIII), Barcelona, (1966-67)

12. MATHAI, A.M.- SAXENA, R.K. "Generalizad Hypergeometric Functions with Applications in Statistics and Physical Sciences", Lecture Notes in Mathematics, N°348, Springer-Verlag, Berlin, (1973)

13. MENDEZ PEREZ, J.M."La transformación integral de Hankel-Clifford", Secretariado de Publicaciones de la Universidad de La Laguna, (1981).

14. SARABIA, J."On the Exton (Di-Bessel) function", Rev. Técn. Fac. Ingr. Univ. Zulia, 10, N°1, (1987), 65-73

15. WATSON, G.N."A treatise on the Theory of Bessel functions", Cambridge University Press, Second Edition, London, (1966)

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