Rev. Téc. Ing. Univ. Zulia, Vol 15, No. 3, 147 -150 , 1992

Movimiento lento de un cuerpo pequeño en un semi-espacio con deslizamiento sobre la pared

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Resumen

En el presente trabajo se considera el flujo de Stokes actuando sobre un cuerpo S que se mueve en presencia de una superficie acotada Σ cuando el deslizamiento tiene lugar sobre Σ . El estudio ha sido efectuado con la ayuda del tensor de Green el cual provee una ecuación integral de la formulación del problema. La fuerza de arrastre sobre el cuerpo ha sido obtenida cuando el cuerpo es de dimensiones lineales pequeñas y la distancia mínima entre un punto de S y un punto de Σ es grande. Se obtuvo que el corrector sobre la aproximación de primer orden de la fuerza de arrastre no es afectada por la presencia de la superficie acotada. Como una ilustración se encuentra la fuerza de arrastre sobre una espera que se mueve paralela al plano de la pared.

Slow motion of a small body in a half-space with slip at the wall

Abstract

In the present paper the Stokes flow on account of a body S moving in the presence of a boundary surface Σ has been considered when slipping takes place on Σ . The study has been effected by the help of Green's tensor which provides an integral equation formulation of the problem. The drag force on the body has been obtained when the body is of small linear dimensions and the minimum distance between a point of S and a point of Σ is large. It is found that correct upto the first order approximation the drag force remains unaffected by the presence of the boundary surface. As an illustration the drag force on a sphere moving parallel to a plane wall has been found.

Introduction

The study of wall effects on the slow motion of a sphere was initiated by Lorentz [7] . He used the method of reflection which has been extended by several authors and is admirably described in [4]. O'Neill [8] gave the solution to this problem in the form of an infinite series. The Stokes flow on account of a body S moving in the presence of a boundary surface Σ may also be studied by the help of Green's tensor [3] and the results for an infinite region extended to a bounded region [5, 9].

The slow motion is determined by the well known Stokes equations [1]

$$\left.\begin{aligned} \mu\nabla^2 u_j - \frac{\partial p}{\partial x_j} &= 0, \\ \frac{\partial u_j}{\partial x_j} &= 0. \end{aligned}\right] \qquad (1.1)$$

These equation are to be supplemented by the boundary conditions, no slip on the surface S of the body, and the slip condition namely, the velocity of the boundary surface Σ is proportional to the stress force there [2]. Such slow flow problems are of considerable physical and mathematical interest [4,6].

Integral Representation

Let us introduce the Green's tensor $G_{jk}(x;y)$ which satisfies Stokes equation every where in the space bounded by Σ except for x = y, and also the condition

$$G_{jk}(x;y)\,|_{Tangential} = \sigma\ [\text{Tangential component of the corresponding viscous stress}]\ \text{on}\ \Sigma \qquad (2.1)$$

where $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and σ is the slip coefficient.

In unbounded fluid the Green's tensor is the fundamental solution of the Stoke equations termed Stokeslet [1] and is given by

$$u_{jk}(x,y) = \frac{\delta_{jk}}{|x-y|} + \frac{(x_j - y_j)(x_k \dot{y}_k)}{|x y|^3}, \qquad (2.2)$$

$$p_j(x, y) = \frac{2\mu\,(x_j - y_j)}{|x-y|^3}. \qquad (2.3)$$

where δ_{jk} is Kronecker's delta.

The associated stress tensor is

$$t_{jki} = -p_j\,\delta_{ki} + \mu\left(\frac{\partial u_{jk}}{\partial y_i} + \frac{\partial u_{ji}}{\partial y_k}\right)$$

$$= -\frac{6\mu\,(x_i - y_i)(x_j - y_j)(x_k - y_k)}{|x-y|^5}. \qquad (2.4)$$

Now we can split $G_{jk}(x;y)$ in two parts

$$G_{jk}(x;y) = u_{jk}(x, y) + g_{jk}(x;y), \qquad (2.5)$$

where $g_{jk}(x;y)$ satisfies Stokes equations and is regular everywhere within Σ. Also, for the corresponding stress, we write

$$G_{jki}(x;y) = t_{jki}(x, y) + g_{jki}(x;y). \qquad (2.6)$$

Now, in a manner similar to the derivation of generalized Green's integral formula [4,5,6], using (2.1) ant the no slip condition that u_j has a constant value U_j on S, we get

$$u_k(x) = -\frac{1}{8\pi\mu}\int_S G_{jk}(x;y)\,t_{ji}(y)\,n_i(y)\,dS$$
$$+\frac{u_j}{8\pi\mu}\int_S t_{jki}(x, y)\,n_i(y)\,dS$$
$$+\frac{U_j}{8\pi\mu}\int_S g_{jki}(x;y)\,n_i(y)\,dS$$
$$+\frac{1}{8\pi\mu}\int_\Sigma G_{jki}\,u_j(y)\,n_i(y)\,dS \qquad (2.7)$$

where x is a point of the region D bounded by Σ and S, y is a surface point and $n_i(y)$ represents unit normal drawn in D. The second integral in (2.7) vanishes since it represents a double layer potential with constant density [10] and, exploiting the regularity of $g_{jk}(x;y)$, the third integral can be shown to have the zero value. Next, suppose that the characteristic length "a" of the body is small and the minimum distance "h" between a point of S and a point of Σ is large. Now, we know that the slow motion of a body is determined by a distribution of Stokeslets and higher order singularities [1]. Further, since the velocity field due to a Stokeslet is of order 1 / (distance) and due to the other singularities at the most of order 1 / (distance)^2, it is apparent that, when h is large, the motion is primarily determined by the Stokeslets and their image distribution. Exploiting this together with the slip boundary condition on Σ, an order analysis reveals that when a is small and h large the last integral is of $O(1/h^2)$ and so may be omitted. Thus, (2.7) simplifies to

$$u_k(x) = -\frac{1}{8\pi\mu}\int_S G_{jk}(x;y)\,t_{ji}(y)\,n_i(y)\,dS \qquad (2.8)$$

The above representation is of the same type as the single layer formulation [10]. The integral is seen to be a continuous function and so applying the boundary condition $u_k(x) = U_k$ on S, we get the integral equation

$$U_k = -\frac{1}{8\pi\mu}\int_S G_{jk}(x:y)\, f_j(y)dS \qquad (2.9)$$

where now x, y ∈ S, and

$$f_j(y) = t_{jl}(y)\, n_l(y) \qquad (2.10)$$

determines the stress vector.

Drag Force

The drag force on the body is given by

$$F_K = \int_S t_{kl}(y)\, n_l(y)dS = \int_S f_k(y)dS \qquad (3.1)$$

In the case of infinite fluid the corresponding drag force F_k^{∞} may be expressed in terms of the resistance tensor Φ_{kl}; thus

$$F_l^{\infty} = U_k\, \Phi_{kl}^{\infty} \qquad (3.2)$$

Now, using (2.5), equation (2. 9) may be written as

$$U_k = -\frac{1}{8\pi\mu}\int_S u_{jk}(x,y)\, f_j(y)dS - \frac{1}{8\pi\mu}\int_S g_{jk}(x:y)\, f_j(y)dS. \qquad (3.3)$$

If yo locates the centroid of the body then, since it is of small linear dimensions, we have the approximation

$$g_{jk}(x;j) \simeq g_{jk}(y_o, y_o) = g_{jk}^o\ (say) \qquad (3.4)$$

Substituting this in (3.3), we obtain

$$U_k + \frac{F_j\, g_{jk}}{8\pi\mu} = -\frac{1}{8\pi\mu}\int_S u_{jk}(x,y)\, f_j(y)dS \qquad (3.5)$$

Remembering that in unbounded fluid $G_{jk}(x;y)$ reduces to $u_{jk}(x,y)$, we see [5,10] that the equation (3.5) represents the motion of the body in unbounded fluid but with modified velocity $U_k + \frac{F_j}{8\pi\mu} g_{jk}^o$. Since $f_j(y)$ is the stress vector for the case of bounded fluid, we may write by the help of

$$F_l = [U_k + \frac{F_j}{8\pi\mu} g_{jk}^o]\, \Phi_{kl}^{\infty} \qquad (3.6)$$

First order approximation may be obtained by replacing F_j by F_j^{∞} on the right hand side of (3.6); thus

$$F_l = [U_k + \frac{F_j^{\infty}}{8\pi\mu} g_{jk}^o]\, \Phi_{kl}^{\infty} \qquad (3.7)$$

It is seen here that g_{jk}^o is independent of the form of the surface S. The principal axes of the resistance of the body are defined so that, when the body moves parallel to one of them in unbounded fluid, the force is in the direction of the motion. If 1 represents a principal direction, then we can derive from (3.7) the force in that direction as

$$F_1 = F_1^{\infty}[\,1 + \frac{F_1^{\infty}}{8\pi\mu\, U_1} g_{11}^o\,] \qquad (3.8)$$

Illustration

A sphere moving parallel to a plane wall. For a sphere of radius α moving in an infinite mass of fluid parallel to x_1 direction with uniform velocity U, we known that the force is given by

$$F_1^{\infty} = -6\pi\mu\, U\alpha \qquad (4.1)$$

In this case formula (3.8) gives

$$F_1 = -6\pi\mu U\alpha\,(\,1 - \frac{3}{4}\,\alpha\, g_{11}^o). \qquad (4.2)$$

when the wall is taken along the x-axis.

Now the Green's function for the problem is given by the image system of a Stokeslet, in a wall with slip [2]. Thus, we have

$$g_{11}(x,y) = \frac{1}{R} - \frac{(x_1 - y_1)^2}{R^3} - \frac{6y_3 (x_1 - y_1)^2 (x_3 + y_3)}{R^5}$$

$$- \frac{2y_3 (x_3 + y_3)}{R^3} - 2y_3^2 \left[-\frac{1}{R^3} + \frac{3(x_1 - y_1)^2}{R^5}\right]$$

$$+4\sigma \left\{ \frac{3(x_1-y_1)(x_3+y_3)}{R^5} + 3y_3 \frac{\partial}{\partial y_3} \frac{(x_1-y_1)^2 (x_3+y_3)}{R^5} \right.$$

$$- \frac{\partial}{\partial y_3} \frac{(x_3 + y_3)}{R^3} + y_3 \left[\frac{1}{R^3} - \frac{3(x_1 - y_1)^2}{R^5}\right] \qquad (4.3)$$

$$\left. + y_3^2 \frac{\partial}{\partial y_3} \left[\frac{1}{R^3} - \frac{3(x_1 - y_1)^2}{R^5}\right] \right\},$$

where $R = [(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2]$ and σ is the slip coefficient.

Neglecting terms of the $O(\frac{1}{h^2})$ the centroid may be taken at (O,O,h) and then we find that

$$g_{11}^{o} = -\frac{3}{4h} \qquad (4.4)$$

Substituting the above value in (4.2), we get

$$F_1 = -6\pi\mu U\alpha \left(1 + \frac{9}{16}\frac{\alpha}{h}\right) \qquad (4.5)$$

Thus, we see that with the neglect of terms of $O(1/h^2)$, the slip has no effect on the drag. It may also be noted that since g_{11}^{o} is independent of the shape the result (3.8) may be employed to write down the drag on any particle provided the value of the drag for its motion in unbounded fluid is known.

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Recibido el 12 de Diciembre de 1989
En forma revisada el 16 de Junio de 1992.