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Teoremas abelianos para la transformación índice $_2F_1$

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Resumen.

En este articulo establecemos algunos teoremas Abelianos para la $_2F_1$-transformación índice,

$$F(\tau) = \int_0^\infty f(t)\, {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau ; \mu + 1 ; - t\right) t^{\alpha}\, dt$$

(α, μ parámetros complejos, τ real).

Palabras claves: Teoremas Abelianos, $_2F_1$ -transformación indice

Abelian theorems for the index $_2F_1$-transform

Abstract.

In this paper we establish some Abelian theorems for the Index $_2F_1$-transform,

$$F(\tau) = \int_0^\infty f(t)\, {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau ; \mu + 1 ; - t\right) t^{\alpha}\, dt$$

(α, μ complex parameters, τ real)

Key words: Abelian, Theorems, Index $_2F_1$- Transform.

1. Introduction.

The Index $_2F_1$-transformation introduced in [3] (see also [9],[10]) of a real valued function f is defined by

$$F(\tau) = \int_0^\infty f(t) \cdot {}_2F_1\left(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau ; \mu + 1 ; - t\right) t^{\alpha}\, dt \quad (1.1)$$

where α, μ are complex parameters ant τ real. ${}_2F_1(\mu + \frac{1}{2} + i\tau, \mu + \frac{1}{2} - i\tau : \mu + 1 ; - t)$ represents the Gauss hypergeometric function defined for $|t| < 1$, which can be defined by analytic continuation for $|t| \geq 1$ (see [2]).

The main classics properties of this transform are summarized as follows (see [3]):

a) If f(t) es locally integrable in $(0,\infty)$ and

$$f(t) = 0\,(t^{\beta}),\ \beta \in R,\ t \to 0$$

$$f(t) = 0\,(t^{\lambda}),\ \lambda \in R,\ t \to +\infty$$

then (1.1) converges absolutely if and only if $\beta + \mathrm{Re}\,\alpha > -1$ and $\lambda + \mathrm{Re}\,(\alpha - \mu) < -\frac{1}{2}$ and (see [4]).

b) If $f \in M^{-1}_{c,\gamma}(L)$[1] with 2 sgn c + sgn $\gamma + \mathrm{Re}(\mu - 2\alpha)) \geq 0$, $\mathrm{Re}\,\alpha > 0$, $\frac{1}{8} < \mathrm{Re}(\mu - \alpha) < \frac{1}{4}$, Re $(\frac{\mu}{2} - \alpha) < -\frac{1}{2}$, Re $\mu > 0$ and F (τ) is defined by (1.1), then

$$f(t) = \frac{t^{\mu - \alpha}}{\pi\,\Gamma(\mu + 1)^2} \int_0^\infty \tau\, sh\, \pi\, \tau\, \Gamma(\mu + \frac{1}{2} + i\,\tau)\Gamma(\mu + \frac{1}{2} - i\,\tau)$$

$$\cdot {}_2F_1\,(\frac{1}{2} + i\,\tau, \frac{1}{2} - i\,\tau; \mu{+}1; -t)\, F(\tau)\, d\tau \qquad (1.2)$$

c) Parseval's relation: if F(τ) and G(τ) are the $_2F_1$-transforms g(t) respectively, one has:

$$\frac{1}{\pi\Gamma(\mu{+}1)^2} \int_0^\infty \tau\, sh\, \pi\, \tau\, \Gamma(\mu + \frac{1}{2} + i\,\tau)\Gamma(\mu + \frac{1}{2} - i\,\tau).$$

$$F(\tau)\; G(\tau)\; d\tau = \int_0^\infty t^{\,2\alpha - \mu}\,(t{+}1)^{-\mu} f(t)\; g(t)\; dt$$

The aim of the present paper is to obtain some Abelian theorems for the index $_2F_1$-transform. For this purpose, we follow the technique developed in several integral transforms (see [6]) by making use of the asymptotic behavior of the kernel function.

2. Abelian theorems.

Proposition 1. Let μ be a complex parameter with Re μ > O. Then one has:

$$|\,{}_2F_1\,(\,\mu + \frac{1}{2} + i\tau,\, \mu + \frac{1}{2} - i\,\tau;\, \mu + 1;\, -t\,)\,|$$

$$\leq C\,[t\,(t + 1)]^{-\frac{1}{2} - \mathrm{Re}\frac{\mu}{2}}\, \tau^{-\frac{1}{2} - \mathrm{Re}\mu} \qquad (2.1)$$

as $\tau \to \infty$

Proof:

Starting from the asymptotic estimation (cf. [7], p. 231, (24); see also [5])

$$P^{-\mu}_{-1/2 + i\tau}(ch\,\xi) \sim \frac{\tau^{-1/2 - \mu}}{\sqrt{\pi}(e^{2\xi} - 1)^{1/2}}$$

$$\left(e^{(\frac{1}{2} + i\tau)\xi} + e^{i(\mu + \frac{1}{2})\pi}\, e^{(\frac{1}{2} - i\tau)\xi}\right)\tau \Rightarrow \infty$$

and taking ch ξ = 2t + 1, we can write:

$$|\; P_{-1/2 + i\tau}^{-\mu}(2t{+}1)\,| \leq C_1\, [t(t{+}1)]^{-1/2}\, \tau^{-1/2 - \mathrm{Re}\mu}$$

Therefore from the relation between $_2F_1\,(\,\mu + \frac{1}{2} + i\,\tau,\, \mu + \frac{1}{2} - i\,\tau : \mu + 1 : -t\,)$ and $P^{\mu}_{1/2 + i\tau}(2t{+}1)$ (cf [2], (7), p 122), it follows

$$\left|\,{}_2F_1(\mu{+}\frac{1}{2}\; +i\tau, \mu{+}\frac{1}{2}\; -i\tau;\mu{+}1;-t)\,\right|$$

$$\leq\; C\;\; [t(t{+}1)]^{-\frac{1}{2} - \mathrm{Re}\frac{\mu}{2}}\;\; \tau^{-\frac{1}{2} - \mathrm{Re}\mu}$$

as $\tau \to \infty$

Theorem 1. Let f(t) be a measurable function on the interval (0,∞) such that

$$t^{\mathrm{Re}(\alpha - \frac{\mu}{2} - \frac{1}{2})}\,(t{+}1)^{-\frac{1}{2} - \mathrm{Re}\frac{\mu}{2}}\, f(t)$$

is Lebesgue integrable on every interval of the form (T,∞), T>O. Assume that

1 $M^{-1}_{c,\gamma}$ (L) is the space of the functions f(x), x ∈ (0,∞), which are represen table in the form $f(x) = \int \phi\,(s)\, x^{-s} ds$
$\phi\,(s) = \varsigma^{-\gamma} e^{-\pi c |\mathrm{Im}\, s|}$. F(s) ∈ L(σ) , where σ = { s ∈ C: Re s = $\frac{1}{2}$ } and c , γ being real numbers such that ?. sgn c + sgn γ ≥ 0 (see [8]).

$$\lim_{t\to 0^+} t^{-\gamma} f(t) = \beta \qquad (2.2)$$

where $\gamma, \beta \in C$ with $-\operatorname{Re}\alpha < \operatorname{Re}\gamma < \operatorname{Re}(\mu-\alpha)$

Then

$$\lim_{\tau\to\infty} [F(\tau) - \beta\, G(\alpha,\mu,\gamma.\tau)] = 0 \qquad (2.3.)$$

where

$$G(\alpha,\mu,\gamma,\tau) = \frac{\Gamma(\mu+1)\Gamma(\mu+\frac{1}{2}-\alpha-\gamma+i\tau)\Gamma(\mu+\frac{1}{2}-\alpha-\gamma-i\tau)}{\Gamma(\mu+1-\alpha-\gamma)\Gamma(\mu+\frac{1}{2}+i\tau)\Gamma(\mu+\frac{1}{2}-i\tau)} \qquad (2$$

and $F(\tau)$ being defined by (1.1).

Proof:

From ([1] p. 336)

$$\int_0^\infty {}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\mu}\,dt = G(\alpha,\mu,\gamma.\tau) \qquad (2$$

for $-\operatorname{Re}\alpha < \operatorname{Re}\gamma < \operatorname{Re}(\mu-\alpha)+\frac{1}{2}$.

Now, in view of (1.1) ant (2.5) we have

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| = \left|\int_0^\infty (t^{-\gamma}f(t) - \beta)\, {}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}dt\right|$$

$$\le \int_0^\infty |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt \le \int_0^\delta |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt +$$

$$\int_\delta^\infty |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt \le \sup_{0<t\le\delta} |t^{-\gamma}f(t) - \beta| \int_0^\delta |{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt +$$

$$\int_\delta^\infty |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt \qquad (2.6)$$

and using the Proposition 1, can be written:

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| \le \sup_{0<t\le\delta} |t^{-\gamma}f(t) - \beta| \int_0^\delta |{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt + C\,\tau^{-\frac{1}{2}-\operatorname{Re}\mu}\int_\delta^\infty |t^{-\gamma}f(t) - \beta|\,[t(t+1)]^{-\frac{1}{2}-\operatorname{Re}\frac{\mu}{2}}\, t^{\operatorname{Re}(\mu+\gamma)}\,dt \qquad (2.7)$$

According to (2.2), for given $\varepsilon > 0$, we can choose $\delta > 0$ such that the first term on the right-hand side of inequality (2.7) is less than ε. Therefore,

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| \le \varepsilon + C\,\tau^{-\frac{1}{2}-\operatorname{Re}\mu}\int_\delta^\infty |t^{-\gamma}f(t) - \beta|\,[t(t+1)]^{-\frac{1}{2}-\operatorname{Re}\frac{\mu}{2}}\, t^{\operatorname{Re}(\mu+\gamma)}dt \qquad (2.8)$$

Since $\operatorname{Re}\gamma < \operatorname{Re}(\mu-\alpha)$, in view of the hypothesis on $f(t)$ the last integral on (2.8) is convergent. Hence as $\tau\to\infty$, the last term on (2.8) tends to zero. Thus:

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| < \varepsilon$$

and therefore:

$$\lim_{\tau\to\infty} [F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)] = 0$$

Theorem 2. Let f(t) be a measurable function on the interval $(0,\infty)$ such that

$$t^{\mathrm{Re}(\alpha-\frac{\mu}{2})-\frac{1}{2}}(t+1)^{\frac{1}{2}-\mathrm{Re}\frac{\mu}{2}} f(t)$$

is Lebesgue integrable on every interval of the form $(0,T)$, $0 < T < \infty$. Assume that there exists a complex number β such that

$$\lim_{t\to\infty} t^{-\gamma} f(t) = \beta \tag{2.9}$$

where $\gamma \in C$ with $-\mathrm{Re}\,\alpha < \mathrm{Re}\,\gamma < \mathrm{Re}\,(\mu - \alpha)$, $\mathrm{Re}\,\gamma > \mathrm{Re}\,(\frac{\mu}{2} - \alpha) - \frac{1}{2}$. Then

$$\lim_{\tau\to\infty} [F(\tau) - \beta\, G(\alpha,\mu,\gamma;\tau)] = 0$$

with $G(\alpha,\mu,\gamma,\tau)$ and $F(\tau)$ defined by (2.4) and (1.1) respectively.

Proof:

From (1.1) and (2.5) one has:

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| \le \int_0^\infty |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt =$$

$$\int_0^T |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt + \int_T^\infty |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt \le \int_0^T |t^{-\gamma}f(t) -$$

$$\beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt + \sup_{t\ge T} |t^{-\gamma}f(t) - \beta| \int_T^\infty |{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt$$

Using the Proposition 1, We have

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| \le C\,\tau^{\frac{1}{2}-\mathrm{Re}\frac{\mu}{2}} \int_0^T |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt + \sup_{t\ge T} |t^{-\gamma}f(t) - \beta| \int_T^\infty |{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt \tag{2.10}$$

and by (2.9), for given $\varepsilon > 0$, the last term on right-hand side of (2.10) is less than $\frac{\varepsilon}{2}$ for an appropriate T. Thus,

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| < C\,\tau^{\frac{1}{2}-\mathrm{Re}\frac{\mu}{2}} \int_0^T |t^{-\gamma}f(t) - \beta|\,|{}_2F_1(\mu+\tfrac{1}{2}+i\tau,\mu+\tfrac{1}{2}-i\tau;\mu+1;-t)t^{\alpha+\gamma}|\,dt + \frac{\varepsilon}{2} \tag{2.11.}$$

Since $\mathrm{Re}\,\gamma > \mathrm{Re}\,(\frac{\mu}{2} - \alpha) - \frac{1}{2}$, in view of the hypothesis on f(t), the last integral converges absolutely and therefore, as $T \to \infty$, the first term on the right hand side of the inequalitg y (2.11.) is less than $\frac{\varepsilon}{2}$, and thus:

$$|F(\tau) - \beta\, G(\alpha,\mu,\gamma,\tau)| < \varepsilon$$

Therefore,

$$\lim_{\tau\to\infty} [F(\tau) - \beta\ G(\alpha,\mu,\gamma,\tau)] = 0$$

References

1. A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, Tables of Integral Transforms, Vol. I McGraw-Hill Book Co., Inc. New York, 1954.
2. A. Erdelyi, W. Magnus, F. Oberhettinger and F. Tricomi, Higher Transcendental Functions, Vol I, McGraw-Hill Book Co. Inc., New York, 1953.
3. N. Hayek, E. R. Negrin, B. González, Una clase transformada índice relacionada con la Olevskii, Actas Jornadas Hispano-Lusas de Matemáticas, Puerto de la Cruz, (Canary Islands), 1989.
4. A. M. Mathay, R. K. Saxena, Generalized Hypergeometric Functions with applications in statistics and physical sciencies, Lec. Not. Math, Springer Verlag, Berlin, 1973.
5. F.W. J. Olver, Asymptotics and special functions, Academic Press, New York 1974.
6. R. N. Pandey, Abelian Theorems for Kontorovic-Lebedev Transformation Indian J. Pure and Appl. Math., 21 (8), 737-739, 1990.
7. L. Robin, Foctions Sphèriques de Legendre et fonctions Sphèroidales Tome II, Gauthier-Villars, París 1958.
8. Vu Kim Tua, O.I. Marichev, S. B. Yakubovich, Composition structure of integral transformations, Soviet. Math. Dokl. Vol 33, N. 1, 166-170, 1986.
9. J. Wimp, A class of integral transforms, Proc. Edinburgh Math, Soc., II, Ser. 14, 33-40, 1964.
10. S. B. Yakubovich, Vu Kim Tuan, O.I. Maravichev, S. L. Kalla, A class of index integral tranforms, Rev. Téc. Ing. Univ., Zulia, Vol. 10 N. 1, Edic. Espec. 105-118, 1987.

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