Rev. Téc. Ing. Univ. Zulia, Vol. 16, No. 1, 27 - 33 , 1993

# Hamiltonian path of $M_k$ - v

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## Abstract

Let $M_k$ be a graph which is obtained by successive substitutions [2] of k vertices of a complete graph $K_n$ ($k \leq n$), by isomorphic copies of the cycle $C_{n-1}$. We prove that "For each pair of distinct vertices x, y adjacent to a vertex v, there is at least a x-y hamiltonian path of $M_k$-v", where v is a vertex in $M_k$.

**Key words:** Hamiltonian path, graph.

# Trayectoria Hamiltoniana de $M_k$ - v

## Resumen

Sea $M_k$ un gráfico el cual se obtiene por sustituciones sucesivas [2] de k vértices de un gráfico completo $K_n$ ($k \leq n$), por copias isomórficas del ciclo $C_{n-1}$. Probamos que "para cada par de vértices distintos x, y adyacentes al vértice v, existe al menos una trayectoria hamiltoniana x-y de $M_k$ - v", donde v es un vértice en $M_k$.

**Palabras claves:** Trayctoria Hamiltoniana, gráfico.

## 1. Introduction

A graph G is a system (V,E) where V=V (G) is a finite nonempty set. The elements of V are called "vertices" and E=E (G) (possibly empty) is a set of pairs (x,y), where x and y are different vertices in V; the members of E are called "edges". For any vertex x denote by $N_x$ the set of neighbors of x.

To simplify the notation, an edge (x,y) is written as xy. A graph G is defined to be Hamiltonian if it has a cycle containing all the vertices of G. A cycle of a graph G contain every vertex of G is called a Hamiltonian cycle; thus a Hamiltonian graph is one that possesses a Hamiltonian cycle. A path in a graph G containing every vertex of G is called Hamiltonian path. Other concepts used in this paper and not defined explicitly can be found in [1].

An operation, introduced in [2], called substitution in performed by replacing a vertex with a graph; a more precise description is the following: Assume that G and K are two graphs with no common vertices. For a vertex v in G and a function s: $N_v \rightarrow V(K)$, we define the **substitution** of the vertex v by the graph K, as the graph M= G (v,s) K such that:

$$V(M) = [\, V(G) \cup V(K)] - \{v\} \qquad (1)$$

$$E(M) = [\, E(G) \cup E(K) \cup \{xs(x) / x \in N_v\}] - \{vx / x \in N_v\}. \qquad (2)$$

Note that the vertex v is the vertex substituted by K in G, under the substitution function s.

Figure 1 shows a diagram of the graph $K(1,5)(v,s)K_5$, where the vertex v of $K(1,5)$ is substituted by $K_5$ through the bijective function $s:N_v \rightarrow V(K)$.

Figure 2 shows a diagram of the graph $G(v,s)K$, where the vertex v of G is substituted by K, through the function s: $N_v \rightarrow V(K)$; note that the function s is not necessarly bijective.

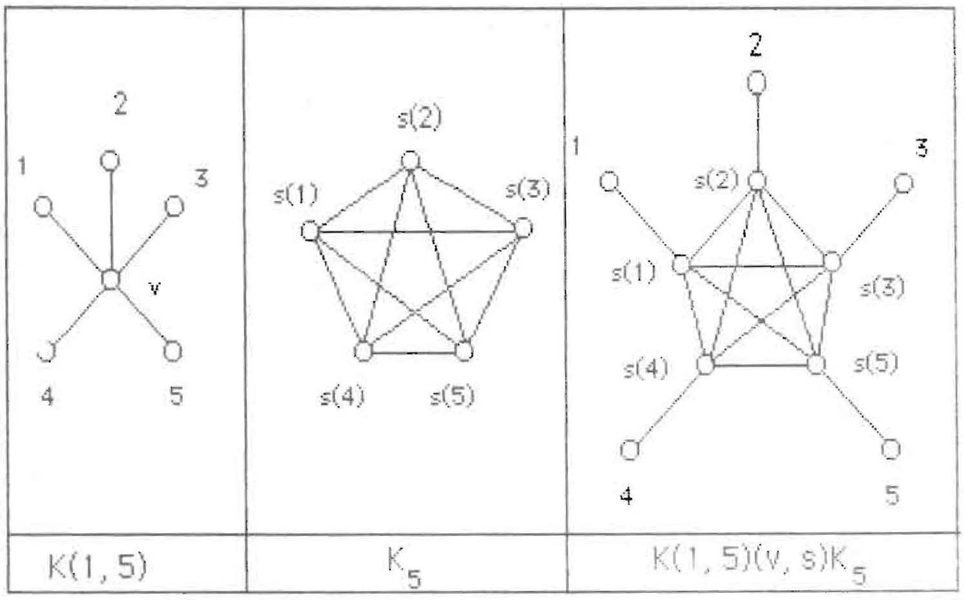


Figure 1

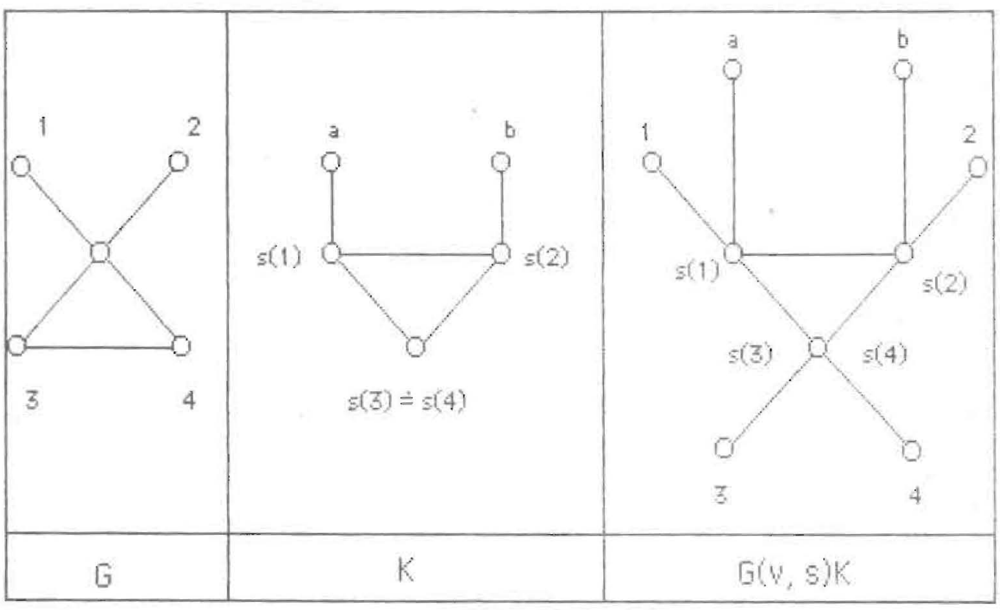


Figure 2

**Remark** : In this paper we will assume that the substitution functions are bijective.

Now let $v_1,...,v_p$ be the vertices of a graph G and $H_1,...H_p$, a sequence of graphs with no common vertices between themselves and with G. We will denote by $M_k = M_{k-1} (v_k, s_k) H_k$, $1 \le k \le p$, the graph which is obtained by substitution of p vertices of G by p graphs $H_i$, $1 \le i \le p$, where $M_0$=G. In other words, $M_1$ denotes a graph obtained by substitution of only one vertex of G, $M_2$ denotes a graph obtained by substitution of only one vertex of $M_1$, and so on; thus recursively we get $M_3,...,M_p$. Note that every vertex substituted must belong to V(G).

In [5] Sabidussi introduced a graph operation by the following construction:

Given a graph G, the graph G' is defined by

i. $V(G') = \{(x,e) \in V(G) \times E(G) / x$ is incident with e$\}$

ii. $E(G') = (x,e)(x',e') \in E(G')$ iff $x = x'$ and $e \neq e'$ or $x \neq x'$ and $e = e'$

The graph G' may be seen to be the result of substituting each vertex v of G by the complete graph $K_{deg(v)}$ through bijective substitution functions. Figura 3 shows a diagram of a graph $M_4$, where the substitution functions $s_i$, $1 \le i \le 4$, are bijective and the graphs $H_i$ are isomorphic to $C_3$. Note that $M_4$ is isomorphic to $(K_4)'$.

In the following we will assume that the substitutions to be considered are formed by replacing the vertices of $K_n$ by isomorphic copies of $C_{n-1}$; if H is a isomorphic copy of $C_{n-1}$, we write $K_n(H)$ to denote a graph obtained by substituting each vertex of the complete graph $K_n$ by H, by means of a bijective substitution and we call $M_k$ a graph obtained by successive substitutions of K vertices of $K_n$, $k \le n$. We say that $s(v_i)$ is the projection of $v_i$ under s, when $v_i s(v_i)$ is an edge of $M_k$ and $v_i$ is a vertex not substituted in $M_k$.

An operation called restitution [4] is performed by replacing a graph by a vertex; a more precise description in the following: Assume that G and K are two graphs with no common vertices. For a vertex v in G and a bijective function s: $N_v \rightarrow V(K)$, we define the restitution of the vertex

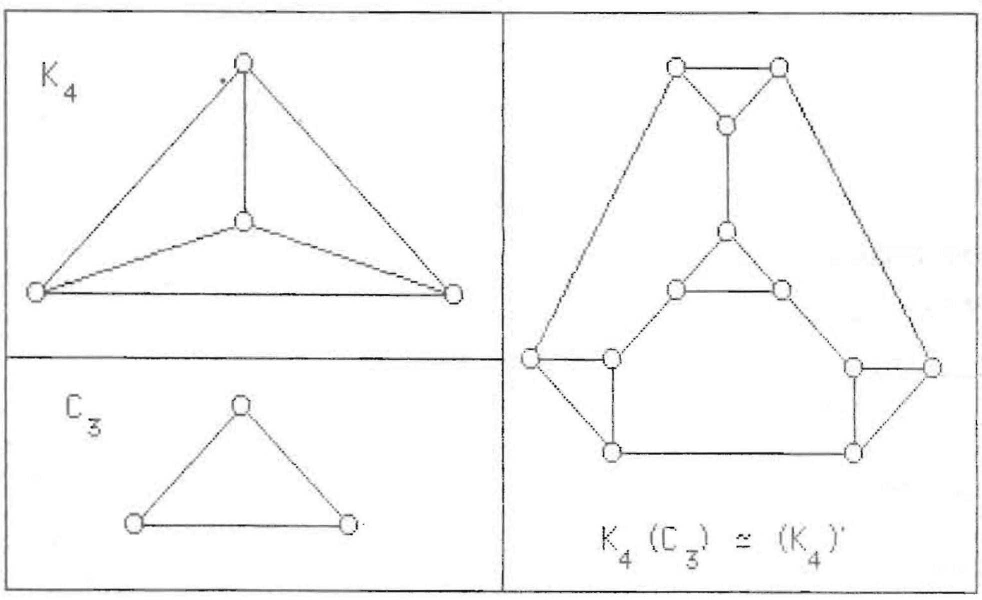


Figure 3

v, by the graph K as the graph $M^{-1} = K(s,v)M$ such that:

i. $V(M^{-1}) = [V(M) - s(N_v)] \cup \{v\}$

ii. $E(M^{-1}) = [E(M) - (E(K) \cup \{xs(x) / x \in N_v\})] \cup \{vx / x \in N_v\}$ and

iii. $M = G(v,s)K$.

Note that the vertex v is the vertex restored by K in G, under the restoring function $s^{-1}$.

Figure 4. shows a diagram of the graph $M^{-1}$, where the vertex v of $M^{-1}$ is restored by $K \subset M$ through a bijective functions s: $N_v \rightarrow V(K)$.

Note that $V(M_k) \supseteq W_k$.

Let uv be an edge of $M_k$. We say that uv is an internal edge [2] if $uv \in E(H_i)$ for any $i \in J_n$. if $uv \notin E(H_i)$ for every $i \in J_n$ we will called by external edge [2]. Note that by no means an internal edge is external and conversely.

**Lemma 2.** A graph $M_k$ does not have internal and external edges at the same time.

**Proof:** By definition of the external and the internal edge.

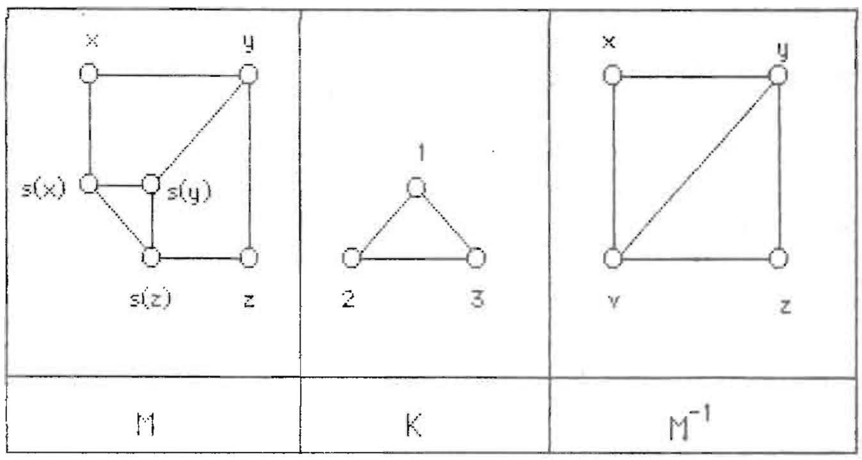


Figure 4

## 2. Resolution of the problem

The following Lemma, can be easily proved.

**Lemma 1:** $M_1 = K_n(v,s)\, C_{n-1}$ is hamiltonian-connected.

Now, we will denote by:

$J_n = \{1, 2, \ldots, n\}$, $W_k = V(H_1) \cup \ldots \cup V(H_k)$, where $H_i = C_{n-1}$, $n \geq 4$ and $i \in J_n$ and $R_k = M_k - W_k$ [1].

**Lemma 3** For every pair the adjacent edge in $M_1$, there is at least a hamiltonian cycle in which are contained.

**Proof:** See [4]

**Theorem 1:** For each pair of distinct vertices u, v projections of $u_i$, $v_i$ ($u_i \neq v_i$) in $W_k$ respectively, there is a u-v path in $M_k$ which contains all vertices of $W_k$, and no others vertices of $M_k$.

**Proof:** See [3]

**Corollary:** 1. The graphs $M_k$ are hamiltonian

2. $K_n$ $(C_{n-1})$ are hamiltonian.

**Theorem 2:** For each pair of distinct vertices x, y adjacent to the vertex v in $M_k$, there is a hamiltonian path x-y of $M_k$-v ($n \geq 4$ and $n \geq k + 3$)

**Proof:** In the proof of the theorem we must distinguish two cases. In the first one we use induction and the other case we use Theorem 1.

Without loss of generality, we may focus the analysis in the cycle $H_1$ or cycle $H_k$ of $M_k$, in all the cases.

We will prove that for each of distinct vertices x,y adjacent to v in $M_k$, there is a hamiltonian path. x-y of $M_k$ -v.

**CASE 1.** (by induction on k, $1 \leq k \leq n$)

If k = 1 the result is true (by the preceding Lemma 1).

Let v be an arbitrary vertex in the cycle $H_k$, therefore $N_v = \{x,y,z\}$, where x, y are in $H_k$ and z is in another cycle $H_l$ or z is a vertex not substituted in $M_k$ (see Figure 5.a or 5.b.).

Assume now that the Theorem is true for $M_1, \ldots, M_{k-1}$.

Let x' be a vertex adjacent to x belonging to $H_k$.

By restitution of a vertex $v_k$ in $M_k$, we obtain a graph $M_{k-1}$, note that x' and z are adjacent to $v_k$ in $M_{k-1}$. By the induction hypothesis, there is a hamiltonian path P' of $M_{k-1}$ - $v_k$ of extremes x'- Z.

Now in $M_k$ - **v**, this path P', together the edge xx' and path x - y of length n-3 which contains n-3 edges of $H_k$ form a hamiltonian path P of $M_k$-v of extremes z-y. The paths x-z and x-y are obtained respectively in a similar way.

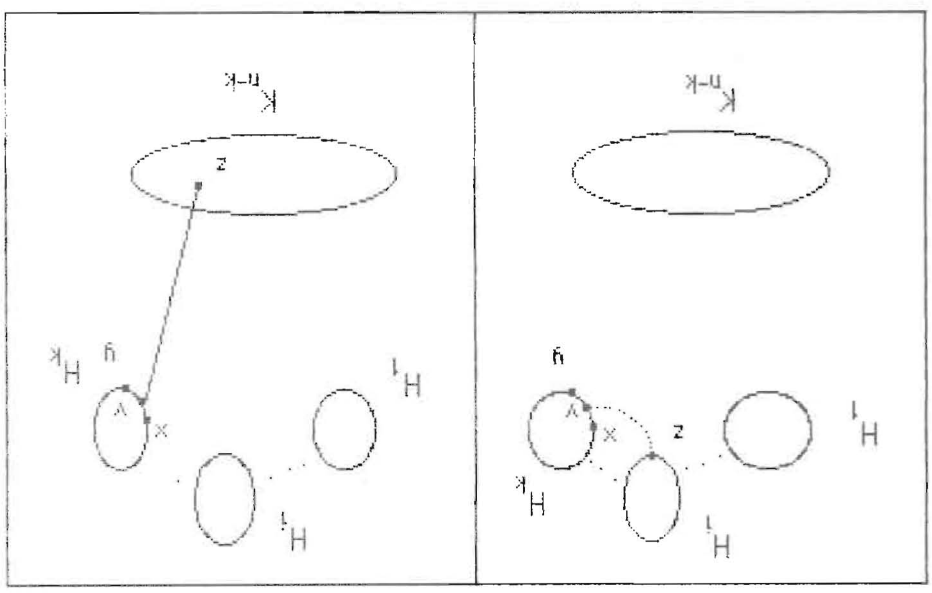

Figure 5.a Figure 5.b

**CASE 2.**

Let v be a vertex not substituted in $M_k$, x and y be vertices adjacent to v; we must distinguish three subcases, but it is necessary to consider only one of them, since by restitution the other subcases are similar.

The three subcases that we distinguish are the following (see Figure 6)

**Proof of subcase 1.**

Notice that for any vertex z ($z \neq v$) adjacent to x we have that $N_z = \{u, z_1, x\}$, so that u is not in $H_k$, then the following situations can occur:

**i.** u is a vertex not substituted in $M_k$.

By restitution of a vertex $v_k$ in $M_k$, we obtain a graph $M_{k-1}$; note that the vertex y is the projection of the vertex v in $H_1$; now consider another vertex w, projection of some vertex $v_i$, such that $v_i$ is different from u,v and $y_k$ by theorem 2, there is a y-w path, which contains all vertices of cycles $H_j$, $1 \leq j \leq k-1$ (Figure 7.a).

Now in $M_k$ (Figure 7.b), the path y-w, also edge $wv_i$, $v_i$-u path which contains all vertices not

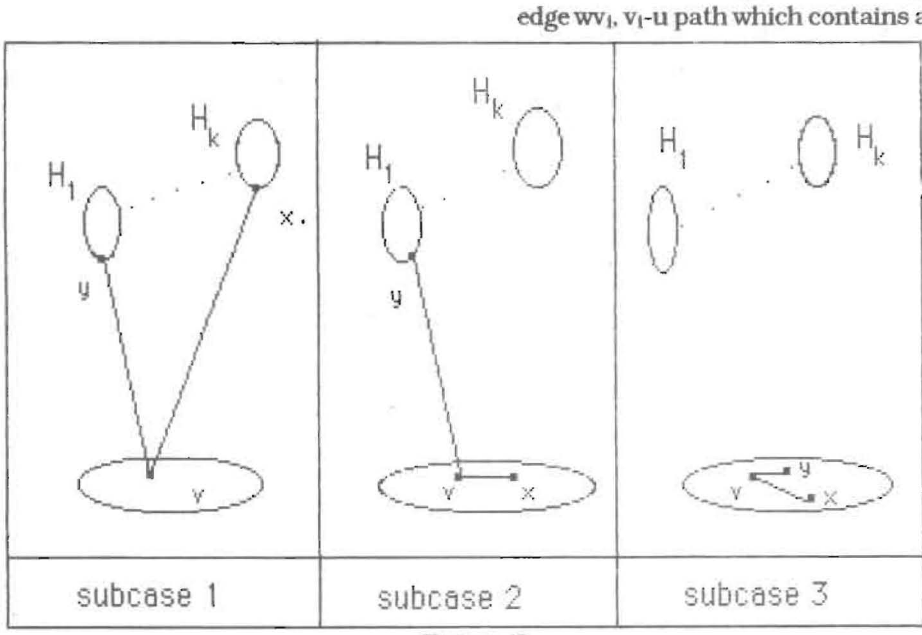


Figure 6.

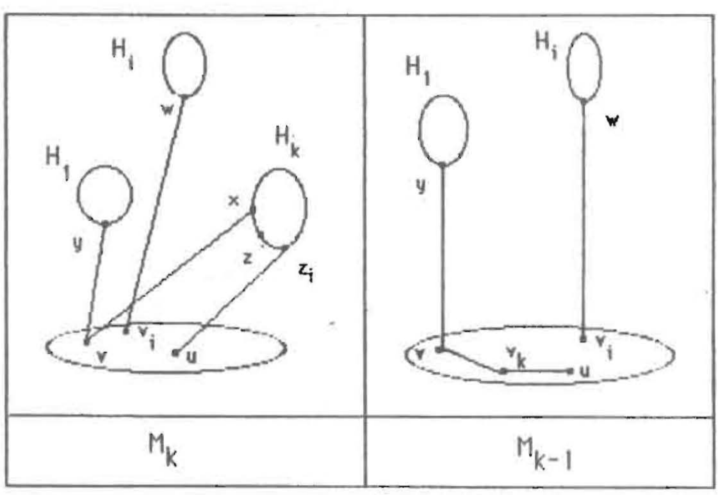


Figura 7.a

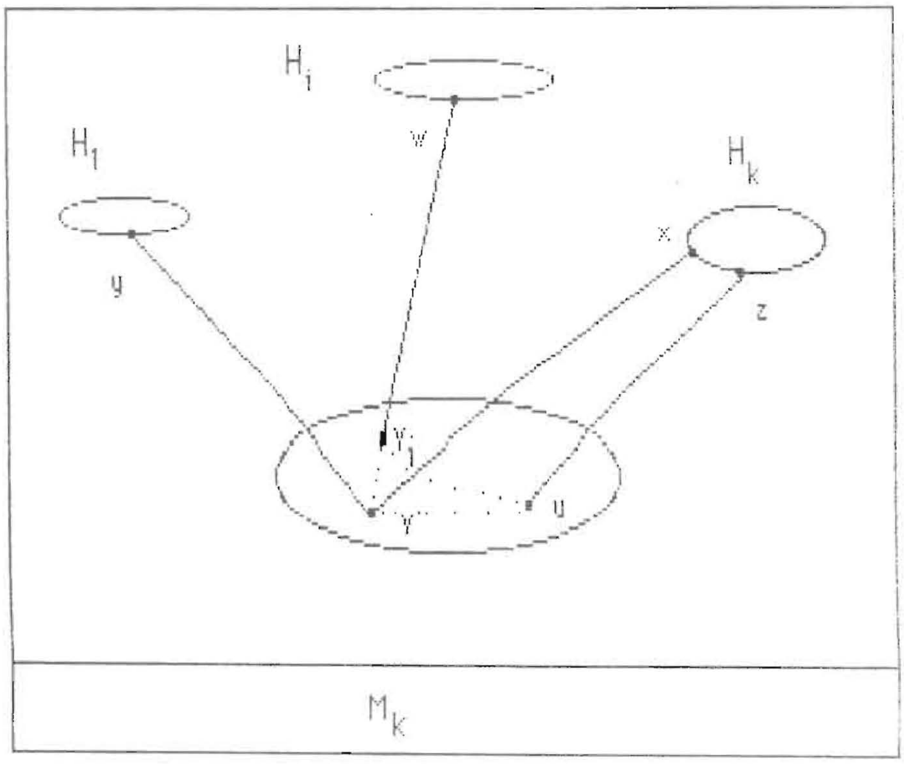


Figura 7.b

substituted excepting v, edge uz and the x-z path of length n-2 which contains n-2 edges of $H_k$, form a hamiltonian path of a $M_k$ - v with extremes x-y.

ii. u belongs to $H_i$, for some i, $i \neq k$.

In this situation we apply the former procedure to $N_u$, until we get subcase i., after an appropriate number of restitutions.

## Acknowledgement

This work was supported by Fondecyt Project 89-0164 and Project 124-702/91 of the Dirección General de Investigación de la Universidad Católica de Valparaíso.

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Recibido el 23 de Junio de 1988
En forma revisada en 28 de Octubre de 1991