Rev. Téc. Ing. Univ. Zulia. Vol. 16, No. 3, 247-251, 1993

Mikusinski's operational calculus for an operator containing a fractional derivative

Jorge J. Betancor y Javier A. Barrios*

Departamento de Análisis Matemático. Universidad de La Laguna. La Laguna-Tenerife Islas Canarias-España

**Departamento de Economía Aplicada. Universidad de La Laguna. La Laguna-Tenerife Islas Canarias-España*

Abstract

In this paper we develop an operational calculus for the operator $K_{\rho,\alpha}= t^{-\alpha}D^{\rho-1}t^{\alpha}D$ with $\rho>1$ and $\alpha> -1$ according to the ideas of J. Mikusinski. Here $D^{\rho-1}$ represents the Riemann-Liouville fractional derivative of $(\rho-1)$-th order.

Key words: Mikusinski calculus, fractional derivative, Riemann-Liouville integral.

Cálculo operacional de Mikusinski para un operador que contiene una derivada fraccionaria

Resumen

En este trabajo desarrollamos un cálculo operacional para el operador $K_{\rho,\alpha}= t^{-\alpha}D^{\rho-1}t^{\alpha}D$ cuando $\rho>1$ y $\alpha> -1$ siguiendo las ideas de J. Mikusinski. $D^{\rho-1}$ representa aquí la derivada fraccionaria de Riemann-Liouville de orden $\rho-1$.

Palabras claves : Cálculo de Mikusinski, derivada fraccional, integral de Riemann-Liouville.

Introduction

The foundations of the operational calculus were developed by J. Mikusinski [1]. V.A. Ditkin and A.P. Proudnikov [2] constructed an operational calculus for the operator DtD. N.A. Meller [3] and [4] considered the operator $B_\mu= t^{-\mu} Dt^{\mu+1}D$ with $-1 < \mu < 1$. She developed the corresponding calculus through the convolution operation

$$(f*g)(t) = \frac{1}{\Gamma(1+\mu)\Gamma(1-\mu)} \frac{d}{dt} \int_0^t (t-\xi)^{-\mu}$$

$$\frac{d}{d\xi}\xi\frac{d}{d\xi}\int_0^{\xi}\int_0^1 \eta^{\mu}(1-x)^{\mu}f(x\eta)\ g((1-x)(\xi-\eta))\,dx\,d\eta\,d\xi$$

for $f,g \in C^2(0,\infty)$. This calculus reduces to Ditkin's calculus for DtD when $\mu= 0$. Recently E.L. Koh [5], [6] and [7] has improved the results of N.A. Meller removing the restriction $\mu<1$ and obtaining an operational calculus for B_μ when $\mu>-1$. The study of the operator B_μ has been completed by J. Rodriguez [8] who extended the operational calculus to values of $\mu\leq-1$. I.H. Dimovski, in a series of papers (see [9], [10], [11] and [12]), investigated operational calculus for the generalized Bessel-type operator $t^{\alpha_0}Dt^{\alpha_1}\ldots Dt^{\alpha_n}$ along the lines of J. Mikusinski. The works of V.S. Kiryakova [13], I.H. Dimovski and V.S. Kiryakova [14], [15] and I.H. Dimovski [16] are important in this area, providing further development and new points of view.

In this paper we develop an operational calculus for the operator $K_{\rho,\alpha} = t^{-\alpha}D^{\rho-1}t^{\alpha}D$ when $\rho>1$ and $\alpha>-1$. Here $D^{\rho-1}$ is understood as the Riemann-Liouville fractional derivative (see B. Ross [17]) of $(\rho - 1)$-th order. The operator $K_{\rho,\alpha}$ for $\rho \in \mathbf{N}$, appears of investigations of E. Kraitzel [18] on a generalization of the Laplace integral transform. Recently, J. Rodriguez [19] constructed an operational calculus for $K_{2,2\mu+1}$ with $\mu \in \mathbf{R}$, through the field extension of a commutative ring without divisors of zero.

An operational calculus for $K_{\rho,\alpha}=t^{-\alpha}D^{\rho-1}t^{\alpha}D$.

Further, $\rho>1$ and $\alpha>-1$. We introduce the set of functions

$$A_\rho = \left\{ f(t) = \sum_{n=0}^{\infty} a_n t^{\rho n} : \{a_n\}_{n=0}^{\infty} \subset \mathbf{C} \text{ and } \lim_{n\to\infty} |a_n|^{1/n} = 0 \right\}$$

By defining on A_ρ the usual operations of addition and multiplication by complex numbers, A_ρ is a linear space over $\mathbf{C}$.

In order to develop an operational calculus for $K_{\rho,\alpha}$, we define a convolution operation $*$ on A_ρ as follows

$$f*g = T^{-1}(Tf\#Tg), \quad \text{for } f,g \in A_\rho$$

where:

$$(f\#g)(t) = \frac{\Gamma((\alpha/\rho)+1)}{\rho\Gamma(\alpha+1)} t^{1-\rho} D t^{\rho-\alpha} D^{\alpha+1} \int_0^t \int_0^1 u^{\alpha}(t-u)^{\alpha} f(uv^{1/\rho}) g((t-u)(1-v)^{1/\rho})\, dv\, du$$

for $f,g \in A_\rho$;

$$(Tf)(t) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1+(\alpha/\rho))} a_n t^{\rho n}$$

$$\text{for } f(t) = \sum_{n=0}^{\infty} a_n t^{\rho n} \in A_\rho$$

and $D^{\alpha+1}$ denotes the Riemann-Liouville fractional derivative of $(\alpha+1)$-th order.

It is not dificult to prove that

$$t^{\rho k} * t^{\rho m} = \frac{\Gamma((\alpha/\rho)+1)\ \Gamma(\rho k+\alpha+1)\Gamma(\rho m+\alpha+1)\Gamma(k+1)}{\Gamma(\alpha+1)\Gamma(\rho(k+m)+\alpha+1)\Gamma(k+m+1)} \frac{\Gamma(m+1)\Gamma(k+m+1+(\alpha/\rho))}{\Gamma(k+1+(\alpha/\rho))\Gamma(m+1+(\alpha/\rho))} x^{\rho(k+m)} \tag{1}$$

for every $m,k \in \mathbf{N}$. Hence, in virtue of the Stirling formula, $*$ is a closed operation in A_ρ.

Moreover, from (1) we can easily derive the following properties for $*$.

Proposition 1 : Let f, g and h be in A_ρ and $a \in \mathbf{C}$. Then,

i) $f*(g*h) = (f*g)*h$,

ii) $f*g = g*h$,

iii) $a*f = af$,

iv) $f*(g+h) = f*g+f*h$,

v) $f*g=0$ if, and only if, $f=0$ or $g=0$.

In virtue of Proposition 1, A_ρ endowed with the operations + and $*$ is a domain of integrity. Therefore, we can define the quotient field $M_\rho=A_\rho x(A_\rho-\{0\})/\sim$, where $\sim$ denotes as usual the equivalence relation

$$(f,g)\sim(F,G) \Leftrightarrow f*G=g*F, \text{ for } f,F\in A_\rho \text{ and } g,G \in A_\rho-\{0\}.$$

In the sequel, for simplicity's sake we shall denote by f/g the member of M_ρ corresponding to $(f,g) \in A_\rho x(A_\rho-\{0\})$. If $f\in A_\rho$, we will write f also to denote the element $f/1$ of M_ρ. Moreover, if α and $\beta \in M_\rho$, then α/β will represent $\alpha*\beta^{-1}$, where

$*$ is now understood as the multiplication in M_ρ. Finally, if $\alpha \in M\rho$ and $k \in \mathbf{N}$, we will denote $\alpha^k=\alpha*..^k.*\alpha$.

On the other hand, we have

$$K_{\rho,\alpha}t^{\rho n} = \begin{cases} 0 & , \text{if } n=0 \\ t^{\rho(n-1)} \dfrac{\Gamma(\rho n+\alpha)\rho n}{\Gamma(\rho(n-1)+\alpha+1)} & , \text{if } n \in \mathbf{N}-\{0\} \end{cases} \quad (2)$$

Then $K_{\rho,\alpha}$ is an homomorphism from A_ρ onto itself.

We now introduce the operator $W_{\rho,\alpha}$ defined by

$$W_{\rho,\alpha}\ f(t) = \int_0^t u^{-\alpha} I^{\rho-1} u^{\alpha} f(u)du, \text{ for } f \in A_\rho,$$

where $I^{\rho-1}$ denotes the Riemann-Liouville integral of (ρ-1)-th order (see B. Ross [17]). It is easy to see that

$$W_{\rho,\alpha}\ t^{\rho n} = t^{\rho(n+1)} \frac{\Gamma(\rho n+\alpha+1)}{\Gamma(\rho(n+1)+\alpha)\ \rho(n+1)},$$

for every $n \in \mathbf{N}$. (3)

Hence $W_{\rho,\alpha}$ is one to one homomorphism from A_ρ into itself. Moreover from (2) and (3) one deduces

$$W_{\rho,\alpha}\ K_{\rho,\alpha}\ f(t)=f(t)-f(0) \quad (4)$$

and $K_{\rho,\alpha}W_{\rho,\alpha}f(t) = f(t)$, for every $f \in A_\rho$.

In virtue of (1) and (3) we can obtain that

$$W^k_{\rho,\alpha}f(t) =$$

$$f(t)*\frac{(1+(\alpha/\rho))\Gamma(\alpha+1)}{\Gamma(\rho+\alpha+1)}t^\rho *.^k.* \frac{(1+(\alpha/\rho))\Gamma(\alpha+1)}{\Gamma(\rho+\alpha+1)}t^\rho \quad (5)$$

for every $f \in A_\rho$ and $k \in \mathbf{N}$.

By combining (4) and (5) we get

$$K_{\rho,\alpha}f(t) = V_{\rho,\alpha}*(f(t)-f(0)), \text{ for } f \in A_\rho, \quad (6)$$

where $V_{\rho,\alpha}= \dfrac{\Gamma(\rho+\alpha+1)}{(1+(\alpha/\rho))\Gamma(\alpha+1)\,t^\rho}$

Also, for every $f \in A_\rho$ and $k \in \mathbf{N}$ one has

$$K^k_{\rho,\alpha}f(t) = V^k_{\rho,\alpha}*f(t) - \sum_{l=0}^{k-1} V^{k-l}_{\rho,\alpha}K^l_{\rho,\alpha}f(t)\,|_{t=0}\,.$$

The above results can be summarized in the following

Proposition 2 : Let $f \in A_\rho$. Then, for every $k \in \mathbf{N}$

$$W^k_{\rho,\alpha}f(t)=f(t)*\frac{(1+(\alpha/\rho))\Gamma(\alpha+1)}{\Gamma(\rho+\alpha+1)}t^\rho *.^k.* \frac{(1+(\alpha/\rho))\Gamma(\alpha+1)}{\Gamma(\rho+\alpha+1)}t^\rho$$

and

$$K^k_{\rho,\alpha}f(t) = V^k_{\rho,\alpha}*f(t) - \sum_{l=0}^{k-1} V^{k-l}_{\rho,\alpha}K^l_{\rho,\alpha}f(t)\,|_{t=0}$$

We now consider the differential equation

$$K_{\rho,\alpha}\ f(t) = af(t) \quad (7)$$

with $a \in \mathbf{R}$. It is not difficult to see that the function

$$e_{\rho,\alpha,a}(t) = \sum_{n=0}^{\infty} a^n \prod_{j=1}^{n} \frac{\Gamma(\rho(j-1)+\alpha+1)}{\Gamma(\rho j+\alpha)\rho j} t^{\rho n}$$

where $\prod_{j=1}^{0}$ is understood as 1, is a solution of (7) on A_ρ, as a special case of the hyper-Bessel functions of Delerue (see [14]). Moreover, if $f \in A_\rho$ solves (7), then $f(t)=f(0)e_{\rho,\alpha,a}(t)$.

Therefore by invoking (6) it follows

$$a e_{\rho,\alpha,a}(t) = V_{\rho,\alpha}*(e_{\rho,\alpha,a}(t)-1)$$

and

$$e_{\rho,\alpha,a}(t) = \frac{V_{\rho,\alpha}}{V_{\rho,\alpha}-a} \quad (8)$$

By proceeding as in [5], p.296, we can obtain new operational rules invoving $e_{\rho,\alpha,a}$ and $V_{\rho,\alpha}$.

Applications

In this section we analyze some applications of the operational calculus developed in the previous paragraph. We solve some functional equations involving the operator $K_{\rho,\alpha}$ via the above operational calculus.

Let $P(s) = a_0(s-a_1)^{m_1}\ldots(s-a_k)^{m_k}$, where $a_0 \neq 0$, $a_i \neq a_j$, for $i,j = 1,2,\ldots,k$ and $i \neq j$, and $m_i \in N-\{0\}$, for $i=1,2,\ldots,k$. We wish to find functions y in Λ_ρ such that

$$P(K_{\rho,\alpha})y = f \quad , \quad (9)$$

where $f \in \Lambda_\rho$ is given and $P(K_{\rho,\alpha})$ is understood as usual by

$a_0(K_{\rho,\alpha}-a_1)^{m_1} \cdot \ldots \cdot (K_{\rho,\alpha}-a_k)^{m_k}$.

According to (6) we can write

$$a_0(V_{\rho,\alpha}-a_1)*(K_{\rho,\alpha}-a_1)^{m_1-1}\cdot(K_{\rho,\alpha}-a_2)^{m_2} \cdot \ldots \cdot (K_{\rho,\alpha}-a_k)^{m_k}y(t) = f(t) + H_{1,1}V_{\rho,\alpha},$$

where $H_{1,1} = a_0(K_{\rho,\alpha}a-a_1)^{m_1-1} \cdot(K_{\rho,\alpha}-a_2)^{m_2}\cdot \ldots \cdot(K_{\rho,\alpha}-a_k)^{m_k}y(t)|_{t=0}$.

By repeating the argument we get

$$a_0\prod_{j=1}^{k}(V_{\rho,\alpha}-a_j)^{m_j}*y(t) =$$

$$f(t) + \sum_{j=1}^{k}\prod_{\ell=1}^{j-1}(V_{\rho,\alpha}-a_\ell)^{m_\ell}*\sum_{i=1}^{m_j} H_{i,j}V_{\rho,\alpha}*V_{\rho,\alpha}-a_j)^{i-1}$$

with

$$H_{i,j} = a_0(K_{\rho,\alpha}-a_j)^{m_j-1}\cdot(K_{\rho,\alpha}-a_{j+1})^{m} \ldots (K_{\rho,\alpha}-a_k)^{m_k}y(t)|_{t=0}$$

for i=1, ..., m_j and j=1, 2, ..., k. Here $\prod_{\ell=1}^{0}$ is understood as 1.

Therefore

$$y(t) = \frac{1}{a_0}\frac{1}{\prod_{j=1}^{k}(V_{\rho,\alpha}-a_j)^{m_j}}*f(t)$$

$$+\sum_{j=1}^{k}\sum_{i=1}^{m_j}\frac{H_{i,j}}{a_0}\frac{V_{\rho,\alpha}}{(V_{\rho,\alpha}-a_j)^{m_j-i+1}*\prod_{\ell=j+1}^{k}(V_{\rho,\alpha}-a_\ell)^{m_\ell}}$$

Here $\prod_{\ell=k+1}^{k}$ is also understood as 1.

We can find suitable constants $C_{w,\ell}$, for $w=1,\ldots,m_\ell$ and $\ell=1,\ldots,k$, and $C_{w,j,i}$, for $w=1,\ldots,m_j-i+1$, $i=1,\ldots,m_j$ and $j=1,\ldots,k$, such that

$$\frac{1}{(V_{\rho,\alpha}-a_j)^{m_j-i+1}*\prod_{\ell=j+1}^{k}(V_{\rho,\alpha}-a_\ell)^{m_\ell}} =$$

$$\sum_{\ell=j+1}^{k}\sum_{w=1}^{m_\ell}\frac{C_{w,\ell}}{(V_{\rho,\alpha}-a_\ell)^{w}}+\sum_{w=1}^{m_j-i+1}\frac{C_{w,j,i}}{(V_{\rho,\alpha}-a_j)^{w}}$$

Now $\sum_{\ell=k+1}^{k}$ is understood as 0.

Hence the general solution of (9) on Λ_ρ is

$$y(t)=\frac{1}{a_0}f(t)*\sum_{j=1}^{k}\sum_{w=1}^{m_j}C_{w,j}\left[\frac{(1+(\alpha/\rho))\Gamma(\alpha+1)}{\Gamma(\rho+\alpha+1)}t^{\rho}*e_{\rho,\alpha,a_j}(t)\right.$$

$$+\sum_{j=1}^{k}\sum_{w=1}^{m_j}B_{w,j}\left[e_{\rho,\alpha,a_j}(t)\right]^{w},$$

where $B_{w,j}$, for $w=1,\ldots,m_j$ and $j=1,\ldots,k$, are arbitrary constants.

We can employ a similar procedure to solve the systems of equations

$$P(K_{\rho,\alpha})\ y = f$$

where $n \in \mathbf{N}$, $P(K_{\rho,\alpha})=(P_{i,j}(K_{\rho,\alpha}))_{i,j=1}^{n}$ and $P_{i,j}$, $i,j=1,\dots,n$, denote

polynomials, $f=\begin{Bmatrix} f_1 \\ . \\ . \\ f_n \end{Bmatrix}$ with given $f_i \in A_\rho$, $i=1,\dots,n$,

$y=\begin{Bmatrix} y_1 \\ . \\ . \\ y_n \end{Bmatrix}$ with unknown $y_i \in A_\rho$, $i=1,\dots,n$.

References

1. J. Mikusinski, Operational Calculus, Pergamon Press, Oxford, 1959.
2. V.A. Ditkin and A.P. Proudnikov, Integral Transforms and Operational Calculus, Pergamon Press, New York, 1965.
3. N.A. Meller, On an operational calculus for the operator $B_\alpha=t^{-\alpha}Dt^{\alpha+1}D$ (In Russian), Vychiss. Mat., 6 (1960), 161-168.
4. N.A. Meller, Some applications of operational calculus to problems in analysis (In Russian), Zh. Vychiss. Math. i Math. Fiz., 3 (1) (1963), 71-89.
5. E.L. Koh, A Mikusinski calculus for the Bessel operator B_μ, Proc. Diff. Eq., Lecture Notes 564, Springer Verlag, (1976), 291-300.
6. E.L. Koh, Application of an operational calculus to certain time-varying systems, Int. J. Systems Sc., 9 (1978), 595-601.
7. E.L. Koh, A direct extension of Meller's Calculus, Internat. J. Math. & Math. Sci., 5 (4) (1982), 785-791.
8. J. Rodriguez, Associated operational calculus for a Bessel Clifford operator, Jñānābha, 20 (1990), 79-90.
9. I.H. Dimovski, Operational calculus for a class of differential operators, Comptes Rendus Acad. Bulg. Sci., 19 (1966), No.12, 1111-1114.
10. I.H. Dimovski, On an operational calculus for a differential operator, Comptes Rendus Acad. Bulg. Sci., 21 (1968), No.6, 513-516.
11. I.H. Dimovski, On the uniqueness of the operational field of the operator $t^{\alpha_0}\prod_{k=1}^{n}\frac{d}{dt}t^{\alpha_k}$", Comptes Rend. Acad. Bulg. Sci., 21 (1968), No.12, 1255-1258.
12. I.H. Dimovski, Foundations of operational calculi for the Bessel type differential operators, Serdica Bulg. Math. Publ., I (1975), 51-63.
13. V.S. Kiryakiva, An application of Meijer's G-function to Bessel type differential operators, Constr. Theory of Funct.'84, Sofia (1984), 457-462.
14. I.H. Dimovski and V.S. Kiryakova, Generalized Poisson transmutations and corresponding representations of hyper-Bessel functions, Comptes Rendus Acad. Bulg. Sci., 39 (1986), No.10, 29-32.
15. I.H. Dimovski and V.S. Kiryakova, Transmutations, convolutions and fractional powers of Bessel type operators via Meijer's G-function, Proc. Complex Anal. and Appl.'83 Sofia, (1985), 45-66.
16. I.H. Dimovski, On an operational calculus for vector valued functions, Mathematika Balcanica, 4 (23) (1972), 129-135.
17. B. Ross, Fractional Calculus and Its Applications, Lecture Notes in Math. 457, Springer Verlag, New York-Berlin, 1975.
18. E. Kraltzel, Eine verallgemeinerung der Laplace- und Meijer- transformations, Wiss. Z. Univ. Jena Math. Naturw. Reihe, Heft 5, (1965) 369-381.
19. J. Rodriguez, Operational calculus for the generalized Bessel operator, Serdica Bulg. Math. Publ., 15 (1989), 179-186.

Recibido el 16 de Abril de 1991
Recibido en forma revisada el 03 de Septiembre de 1993