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On certain sequences of Fourier-Young coefficients of a function of Wiener's class

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Abstract

In this paper we study the summability of certain sequences of Fourier-Young coefficients of a function of Wiener's class by Nörland method. From which we deduce under what conditions a function of Wiener's class is continuous.

Key words : Sequences, Fourier-Young coefficients, Wiener class.

Sobre ciertas sucesiones de coeficientes de Fourier-Young de una Función de Clase Wiener

Resumen

En este trabajo estudiamos la sumabilidad de ciertas sucesiones de coeficientes de Fourier-Young de una función de clase Wiener por el método Nörland. A partir de esto se deduce bajo qué condiciones una función de clase Wiener es continua.

Palabras claves : Sucesión, coeficientes de Fourier-Young, clase Wiener.

Introduction

Let f be a real or complex valued 2π-periodic function defined on $[0,2\pi]$ and let $P\varepsilon$: $0 = t_0 < t_1 < \ldots < t_n = 2\pi$ be a partition of $[0,2\pi]$ such that the norm

$$\mu(P\varepsilon) = \max_{1\le i\le n} |t_i - t_{i-1}| < \varepsilon,$$

where ε is an arbitrary positive number. For $1 \le p < \infty$, we define p- th variation of f by

$$V_p(f) = \lim_{\varepsilon\to 0} \sup_{P\varepsilon} \left\{\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|^p\right\}^{1/p}$$

where the suprəmum has been taken with respect to all partitions of the type $P\varepsilon$ of $[0,2\pi]$. Now we define Wiener's class simply by

$$V_p = \{f : V_p(f) < \infty\}$$

We note [5] that

$$V_{p_1} \subset V_{p_2} \quad (1 \le p_1 \le p_2 < \infty)$$

is a strict inclusion. Hence Wiener's class V_p $(1 < p < \infty)$ is a strictly larger class than the class V_1 of functions of bounded variation in ordinary sense. The class V_p was first introduced by Wiener [6]. Then Young [7] proved the following theorem in connection with the existence of Riemann-Stieltjes integral of a function of V_p.

Theorem A

If an $f \in V_p$ and $g \in V_q$ where $p,q > 0$, $\frac{1}{p} + \frac{1}{q} > 1$, have no common points of discontinuity, their Stieltjes integral

$$\int_0^{2\pi} f\, dg$$

exists in the Riemann sense.

From Theorem A, the sequence defined by

$$\hat{f}(k) = (2\pi)^{-1}\int_0^{2\pi} e^{ikt} df(t), (k = 0, \pm 1, \pm 2, ...)$$

exists for every $f \in Vp(1 < p < \infty)$. We shall call $\{\hat{f}(k)\}$ the sequence of Fourier-Young coefficients and the series

$$\sum_{-\infty}^{\infty} \hat{f}(k) \quad e^{ikx}$$

will be called the Fourier-Young series of f.

In terms of Young [7], a sequence $\{f_n\}$ of the class V_p converges densely in $[0,2\pi]$ to a function f if $f_n(x)$ tends to f(x) for each x of an everyhere dense set in $[0,2\pi]$.

And a sequence $\{f_n(x)\}$ converges uniformly to f(x) at x_0, if given an $\in > 0$, there is an η and a $\delta > 0$, such that, for $n > \eta$ and for all x distant less than δ from x_0, $|fn(x) - f(x)| < \in$.

Now we are able to state the following theorem due to Young [7].

Theorem B

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions such that $\{V_p(f_n)\}_{n=1}^{\infty}$ is uniformly bounded in n and p and let $g \in V_q$, where $p,q > 0$, $\frac{1}{p}+\frac{1}{q} > 1$. Suppose that $\{f_n\}_{n=1}^{\infty}$ converges densely in $[0,2\pi]$ to a function f of V_p and $\{f_n(x)\}_{n=1}^{\infty}$ converges uniformly at each point of discontinuity of f(x) in $[0,2\pi]$. Then

$$\lim_{n\to\infty} \int_0^{2\pi} f_n \quad dg = \int_0^{2\pi} f\, dg$$

There is also a well known theorem due to Wiener [6].

Theorem C

If $f \in V_p(1 \le p < \infty)$ is continuous, then $V_q(f) = 0$ for all $q > p$.

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of real or complex numbers such that $P_n = \sum_{k=0}^{n} p_k \neq 0$. A sequence $\{s_n\}_{n=1}^{\infty}$ is said to be summable by Nörlund method of summability defined by $\{p_n\}$, or simply summable (N,p) if

$$\lim_{n\to\infty} N_n^p(s) = \lim_{n\to\infty} P_n^{-1} \sum_{k=0}^{n} p_{n-k} \quad s_k$$

exists. The conditions for regularity are

(i) $p_n = O(P_n) \quad (n\to\infty)$ and

(ii) $\sum_{k=0}^{n} |p_k| = O(P_n) \quad (n\to\infty)$,

of which the latter is automatically satisfied when the sequence $\{p_n\}_{n=0}^{\infty}$ is positive (cf. Bari [1]) p.12). We write throughout:

$\Delta p_k = p_k - p_{k-1}$, $\Delta^r p_k = \Delta(\Delta^{r-1} p_k) \quad (r>1)$

with $p_k = 0$ for $k < 0$.

The main aim of this paper is to study the summability of the sequence $\{\hat{f}(k)\, e^{ikx}\}_{k=0}^{\infty}$ of Fourier-Young coefficients of a function of Wiener's class by Nörlund method. This will enable us to obtain a criterion, answering that under what conditions, a function of Wiener's class is continuous. More precisely, first we prove the following theorem.

Theorem 1

If $f \in V_p$ $(1 < p < \infty)$, then the sequence $\{f(k)\, e^{ikx}\}_{k=0}^{\infty}$ is summable (N,p) to π^{-1} D(x) = π^{-1} {f(x+0) — f(x-0)} for every $x \in [0,2\pi]$ provided that (N,p) is regular and

(i) $\sum_{k=0}^{n} (n-k) \; |\Delta p_k| = O(P_n) \quad (n\to\infty)$

(ii) $\sum_{k=0}^{n} k|\Delta p_k| = O(P_n) \quad (n\to\infty)$.

where O means capital order O in ordinary sense.

Proof :

Consider

$$N_n^p\,[\hat{f}(k)\;e^{ikx}] = P_n^{-1}\sum_{k=0}^{n} p_{n-k}\,\hat{f}(k)\,e^{ikx} =$$

$$\pi^{-1}P_n^{-1}\sum_{k=0}^{n} p_{n-k}\int_0^{2\pi} e^{ik(x-t)}\,dh(t) + \pi^{-1}D(x)$$

where

$$h(t) = f(t) - \pi^{-1}\sum_{j=0}^{\infty} D(x_j)\,g(t-x_j)\;, \quad (1)$$

$x_0, x_1, x_2, \ldots$ are points of discontinuity of f in $[0,2\pi]$ and

$$g(t) = \frac{\pi-t}{2} \text{ for } 0 < t < 2\pi$$

$g(0) = g(2\pi) = 0$ and outside of $[0,2\pi]$, $g(t)$ is defined by periodicity. It is evident that $h \in V_p$ and is continuous everywhere in $[0,2\pi]$. $D(x) = f(x+0) - f(x-0)$ denotes the jump of f at x. Hence we can write

$$N_n^p\,[\hat{f}(k)e^{ikx}] = \pi^{-1}\int_0^{2\pi} K_n(x-t)dh(t) + \pi^{-1}D(x) \quad (2)$$

where

$$K_n(x-t) = P_n^{-1}\sum_{k=0}^{n} p_{n-k}\,e^{ik(x-t)}$$

It is sufficient to show that the integral on the right hand side of (2) tends to zero as $n\to\infty$. Consider

$$\int_0^{2\pi} K_n(x-t)\,dh(t) = \int_{|x-t|\le\delta} + \int_{|x-t|>\delta} = I_1 + I_2$$

But the integral

$$\int_{|x-t|\le\delta} K_n(x-t)\;dh(t)$$

$$= \sup\sum_{i=1}^{N} K_n(x-t_i)\;(h(t_i) - h(t_{i-1}))$$

where the supremum has been taken with respect to all partitions P:$x-\delta = t_0 < t_1 < t_2 \ldots < t_n = x+\delta$ of $[x-\delta,\; x+\delta]$. But we can write

$$\sum_{i=1}^{N} K_n(x-t_i)\;(h(t_i) - h(t_{i-1})) =$$

$$\sum_{0<r<i<N}\sum \Delta_r\,(K_n)\;\Delta_i(h) \;+$$

$$K_n(x-\delta)\;[h(x+\delta) - h(x-\delta)]$$

where

$\Delta_i(h) = h(t_i) - h(t_{i-1})$ and $\Delta_r(K_n) = K_n(t_r) - K_n(t_{r-1})$.

Now using Hölder's inequality (cf. Young [7] p.254), we obtain for p,q > 0 satisfying $\frac{1}{p}+\frac{1}{q} > 1$,

$$\left|\sum_{i=1}^{N} K_n(x-t_i)\;(h(t_i) - h(t_{i-1}))\right| \le$$

$$\left\{1+\sum_{n=1}^{\infty} n^{-(\frac{1}{p}+\frac{1}{q})}\right\}$$

$$V_p(K_n)\,V_q(h) + |K_n(x-\delta)|\,|h(x+\delta) - h(x-\delta)|.$$

Since h is continuous, $V_q(h) = 0$ from Theorem C and $V_p(K_n)$ is bounded, hence by the definitions of $V_p(K_n)$ and $V_q(h)$, we obtain

$$I_1 = \left|\pi^{-1}\int_{|x-t|\le\delta} K_n(x-t)\;dh(t)\right| < \epsilon$$

for a given arbitrary small positive ϵ. Now we proceed to show that $\{V_p(K_n)\}$ is uniformly bounded in n. Since, for $0 = t_0 < \ldots t_n = 2\pi$,

$$K_n(t_i) = P_n^{-1}\sum_{k=0}^{n} p_{n-k}\;e^{ikt_i} =$$

$$P_n^{-1}\sum_{k=0}^{n} p_{n-k}\cos kt_i + iP_n^{-1}\sum_{k=0}^{n} p_{n-k}\sin kt_i$$

can be written into real and imaginary parts of $K_n(t_i)$. By using Abel's transformation on real part of $K_n(t_i)$, we can write

$$P_n^{-1}\sum_{k=0}^{n} p_{n-k}\cos kt_i =$$

$$P_n^{-1}\sum_{k=0}^{n-1} \Delta_{p_{n-k}}\,D_k(t_i) + P_n^{-1}\,D_n(t_i)$$

where $D_n(t_i) = \sum_{k=0}^{n}\cos kt_i$ denotes Dirichlet's kernel. It will be sufficient to show that the real part of $K_n(t)$ belongs to V_p uniformly in n. Since the real part of

$$[K_n(t_i) - K_n(t_{i-1})] =$$

$$P_n^{-1}\sum_{k=0}^{n-1}\Delta p_{n-k}\left[D_k(t_i) + D_k(t_{i-1})\right]$$

$$+ P_n^{-1}p_0[D_n(t_i) + D_n(t_{i-1})]$$

And by using Mean Value Theorem, we can write the real part of $K_n(t_i) - K_n(t_{i-1})$ is equal to

$$P_n^{-1}\sum_{k=0}^{n-1}\Delta p_{n-k}\, D'_k(\xi_i)(t_i-t_{i-1}) +$$

$$P_n^{-1}p_0 D'_k(\xi_i)(t_i-t_{i-1}) \qquad (3)$$

where $D'_k(\xi_i)$ denotes derivative of $D_k(t)$ at the point $\xi_i \in (t_{i-1}, t_i)$. Hence (3) is majorised by

$$P_n^{-1}\sum_{k=0}^{n} k|\Delta p_{n-k}|\,|t_i-t_{i-1}|M \le M(t_i-t_{i-1})$$

where M is an absolute constant. Hence, for $1 \le p < \infty$, the real part of

$$\sup\left\{\sum_{i=1}^{N}|K_n(t_i) - K_n(t_{i-1})|^p\right\}^{1/p} \le 2\pi M,$$

and, therefore, the real part of $K_n(t)$ belongs to V_p uniformly in n. Similarly, we can show that the imaginary part of $K_n(t)$ belongs to V_p uniformly in n. Hence $\{V_p(K_n)\}$ is uniformly bounded in n and

$$\lim_{n\to\infty} K_n(t_i) = 0 \text{ for all } t_i \ne 0 \pmod{2\pi}$$

Further h(t) is continuous part of f, hence using Young convergence theorem B, we obtain,

$$\lim_{n\to\infty}\int_{|x-t|>\delta} K_n(x-t)\, dh(t) = 0$$

which implies $I_2 \to 0$ as $n\to\infty$. This completes the proof of theorem 1.

A Nörlund method of summability (N,p) which is regular and satisfies the conditions I and II of Theorem 1 will be called an admissible method. Hence we deduce the following theorem from Theorem 1.

Theorem 2

For every $f \in V_p$ ($1\le p < 2$), the followings are equivalent.

(i) f is continuous

(ii) $\{|\hat{f}(k)|^2\}$ is summable (N,P) to zero by an admissible method.

(iii) $\{|\hat{f}(k)|\}$ is summable (N,P) to zero by an admissible method.

Proof:

Suppose that f is continuous in $[0,2\pi)]$. Hence D(x) = 0 for all $x \in [0,2\pi]$. Since the convolution of $f \in V_p$, defined by

$$f^*(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+t)\,\overline{dF(t)},$$

exists only for $1 \le p < 2$ by Theorem A for every x which does not belong to a countable set D of numbers $\{x_j\}$, where $\{x_j\}$ are the points of discontinuities of f. We define for x' in D.

$$f^*(x') = \lim_{x\to x'+0} f^*(x).$$

It is easily seen that $f^* \in V_p$ ($1 \le p < 2$) from Theorem A and its Fourier-Young series is $\sum_{-\infty}^{\infty}|f^*(k)|^2 e^{ikx}$. Since (cf. Zygmund [8] p.8-10).

$$f^*(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x+t)\,\overline{dh(t)} +$$

$$\frac{1}{2\pi}\sum_{j=0}^{\infty} f(x+x_j)\,\overline{D}(x_j),$$

it follows that

$$f^*(+0) - f^*(-0) = \frac{1}{2\pi}\sum_{j=0}^{\infty}|D(x)|^2$$

where summation is taken over all the points of discontinuity of f in $[0,2\pi]$ and h is continuous part of f defined in (1). Applying theorem 1 on f^* at x = 0, we obtain that $\{|\hat{f}(k)|^2\}$ is summable (N,p) to $\frac{1}{2\pi}\sum_{j=0}^{\infty}|D(x_j)|^2$ which is zero by hypothesis.

Hence (i) ⇒ (ii)

If $\{|\hat{f}(k)|^2\}$ is summable (N,p) to zero then by Schwarz's inequality, we obtain that $\{|\hat{f}(k)|\}$ is summable (N,p) to zero and hence (ii) ⇒ (iii).

Further, if $\{|\hat{f}(k)|\}$ is summable (N,p) to zero then $D(x) = 0 \ \forall x \in [0,2\pi]$ by applying theorem 1. Hence f is continuous in $[0,2\pi]$. This completes the proof of Theorem 2.

Theorem 2 extends to various theorems on continuity to Wiener's class V_p including those given by Wiener [6], Lozinskii[3], Matveev [4] (cf. Bari [1] page 256) and Golubov [2].

We also like to remark that Theorem 2 is no more true for $p \geq 2$. For we have the following functions:

$$f(x)\sum_{k=1}^{\infty}\frac{\sin kx}{k}; \quad g(x) = \sum_{k=1}^{\infty}\frac{\sin k(x+\ln k)}{k} \tag{4}$$

It is easy to verify (cf. Zygmund [8] p.241-243) that both series in (4) converges for all x. We also note [8] that f(x) is a discontinues function belonging to V_1 and g(x) belongs to $Lip_{1/2}$ and hence belongs to V_2. We can compute the Fourier-Young coefficients $\hat{f}(k) = \hat{g}(k) = 1$ for $k = 1,2,3,...$and $\hat{f}(0) = \hat{g}(0) = 0$. Hence we obtain two functions f and g belonging to $V_p(2 \leq p < \infty)$; one is discontinues and other is continuous such that $\hat{f}(k) = \hat{g}(k)$ for $k = 0,1,2,...$. Hence Theorem 2 cannot be extended for $p \geq 2$ in terms of Fourier-Young coefficients.

Applying Theorem 1 and Theorem 2, we can prove the following theorem.

Theorem 3

Let (N,p) be an admissible Nörlund method of summability

(i) If $1 \leq p < 2$, then the condition that $\{|\hat{f}(k)|\}$ is summable (N,p) to zero, is necessary and sufficient for f of V_p to be continuous.

(ii) If $p \geq 2$, then the condition that $\{|\hat{f}(k)|\}$ is summable (N, p) to zero, remains sufficient but is no longer necessary for f of V_p to be continuous.

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